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Research Article

On a Converse of Jensen's Discrete Inequality

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We give the best possible global bounds for a form of discrete Jensen's inequality. By some examples the fruitfulness of this result is shown.

1. Introduction

Throughout this paper $\mathbf{x} = \{x_i\}$ represents a finite sequence of real numbers belonging to a fixed closed interval $I = [a,b],\ a < b$, and $\mathbf{p} = \{p_i\},\ \sum p_i = 1$ is a positive weight sequence associated with $\mathbf{x}$.

If f is a convex function on I, then the well-known Jensen's inequality [1, 2] asserts that

$$0 \le \sum p_i f(x_i) - f\left(\sum p_i x_i\right). \tag{1.1}$$

There are many important inequalities which are particular cases of Jensen's inequality among which are the weighted $A - G - H$ inequality, Cauchy's inequality, the Ky Fan and Hölder's inequalities.

One can see that the lower bound zero is of global nature since it does not depend on $\mathbf{p}, \mathbf{x}$ but only on f and the interval I whereupon f is convex.

We give in [1] an upper global bound (i.e., depending on f and I only) which happens to be better than already existing ones. Namely, we prove that

$$(0 \le) \sum p_i f(x_i) - f\left(\sum p_i x_i\right) \le T_f(a,b), \tag{1.2}$$

with

$$T_f(a,b) := \max_p \left[pf(a) + (1-p)f(b) - f(pa + (1-p)b)\right]. \tag{1.3}$$

Note that, for a (strictly) positive convex function f, Jensen's inequality can also be stated in the form

$$1 \leq \frac{\sum p_i f(x_i)}{f\left(\sum p_i x_i\right)}. \tag{1.4}$$

It is not difficult to prove that 1 is the best possible global lower bound for Jensen's inequality written in the above form. Our aim in this paper is to find the best possible global upper bound for (1.4). We will show with examples that by following this approach one may consequently obtain converses of some important inequalities.

2. Results

Our main result is contained in what follows.

Theorem 2.1. *Let f be a (strictly) positive, twice continuously differentiable function on $I := [a,b]$, $x_i \in I$ and $0 \leq p,\ q \leq 1,\ p + q = 1$. One has that*

(i) *if f is (strictly) convex function on I, then*

$$1 \leq \frac{\sum p_i f(x_i)}{f\left(\sum p_i x_i\right)} \leq \max_p \left[\frac{pf(a) + qf(b)}{f(pa + qb)}\right] := S_f(a,b), \tag{2.1}$$

(ii) *if f is (strictly) concave function on I, then*

$$1 \leq \frac{f\left(\sum p_i x_i\right)}{\sum p_i f(x_i)} \leq \max_p \left[\frac{f(pa + qb)}{pf(a) + qf(b)}\right] := S'_f(a,b). \tag{2.2}$$

Both estimates are independent of $\mathbf{p}$.

The next assertion shows that $S_f(a,b)$ (resp., $S'_f(a,b)$) exists and is unique.

Theorem 2.2. *There is unique $p_0 \in (0,1)$ such that*

$$S_f(a,b) = \frac{p_0 f(a) + (1-p_0)f(b)}{f(p_0 a + (1-p_0)b)}. \tag{2.3}$$

Of particular importance is the following theorem.

Theorem 2.3. *The expression $S_f(a,b)$ represents the best possible global upper bound for Jensen's inequality written in the form* (1.4).

3. Proofs

We will give proofs of the previous assertions related to the first part of Theorem 2.1. Proofs concerning concave functions go along the same lines.

Proof of Theorem 2.1. We apply the method already shown in [1]. Namely, since $a \le x_i \le b$, there is a sequence $t_i \in [0,1]$ such that $x_i = t_i a + (1-t_i)b$.

Hence,

$$\frac{\sum p_i f(x_i)}{f(\sum p_i x_i)} = \frac{\sum p_i f(t_i a + (1-t_i)b)}{f(\sum p_i (t_i a + (1-t_i)b))} \le \frac{f(a)\sum p_i t_i + f(b)(1-\sum p_i t_i)}{f(a\sum p_i t_i + b(1-\sum p_i t_i))}. \tag{3.1}$$

Denoting $\sum p_i t_i := p,\ 1-\sum p_i t_i := q;\ p,q \in [0,1]$, we get

$$\frac{\sum p_i f(x_i)}{f(\sum p_i x_i)} \le \frac{pf(a)+qf(b)}{f(pa+qb)} \le \max_p \left[\frac{pf(a)+qf(b)}{f(pa+qb)}\right] := S_f(a,b). \tag{3.2}$$

□

Proof of Theorem 2.2. For fixed $a,b \in I$, denote

$$F(p) := \frac{pf(a)+qf(b)}{f(pa+qb)}. \tag{3.3}$$

We get $F'(p) = g(p)/f^2(pa+qb)$ with

$$g(p) := (f(a)-f(b))f(pa+qb) - (pf(a)+qf(b))f'(pa+qb)(a-b). \tag{3.4}$$

Also,

$$\begin{gathered} g'(p) = -(a-b)^2(pf(a)+qf(b))f''(pa+qb), \\ g(0) = f(b)(f(a)-f(b)-f'(b)(a-b)), \qquad g(1) = -f(a)(f(b)-f(a)-f'(a)(b-a)). \end{gathered} \tag{3.5}$$

Since f is strictly convex on I and $pa+qb \in I$, we conclude that $g(p)$ is monotone decreasing on $[0,1]$ with $g(0) > 0,\ g(1) < 0$. Since g is continuous, there exists unique $p_0 \in (0,1)$ such that $g(p_0) = F'(p_0) = 0$. Also $F''(p_0) = g'(p_0)/f^2(p_0 a + q_0 b) < 0$, showing that $\max_p F(p)$ is attained at the point p_0. The proof is completed. □

Proof of Theorem 2.3. Let $R_f(a,b)$ be an arbitrary global upper bound. By definition, the inequality

$$\frac{\sum p_i f(x_i)}{f(\sum p_i x_i)} \le R_f(a,b) \tag{3.6}$$

holds for arbitrary $\mathbf{p}$ and $x_i \in [a,b]$.

In particular, for $\mathbf{x} = \{x_1, x_2\},\ x_1 = a,\ x_2 = b,\ p_1 = p_0$ we obtain that $S_f(a,b) \leq R_f(a,b)$ as required. □

4. Applications

In the sequel we will give some examples to demonstrate the fruitfulness of the assertions from Theorem 2.1. Since all bounds will be given as a combination of means from the Stolarsky class, here is its definition.

Stolarsky (or extended) two-parametric mean values are defined for positive values of x, y as

$$E_{r,s}(x,y) := \left(\frac{r(x^s - y^s)}{s(x^r - y^r)}\right)^{1/(s-r)}, \qquad rs(r-s)(x-y) \neq 0. \tag{4.1}$$

E means can be continuously extended on the domain

$$\{(r,s;x,y) \mid r,s \in \mathbb{R}; x,y \in \mathbb{R}_+\} \tag{4.2}$$

by the following:

$$E_{r,s}(x,y) = \begin{cases} \left(\dfrac{r(x^s - y^s)}{s(x^r - y^r)}\right)^{1/(s-r)}, & rs(r-s) \neq 0, \\ \exp\left(-\dfrac{1}{s} + \dfrac{x^s \log x - y^s \log y}{x^s - y^s}\right), & r = s \neq 0, \\ \left(\dfrac{x^s - y^s}{s(\log x - \log y)}\right)^{1/s}, & s \neq 0,\ r = 0, \\ \sqrt{xy}, & r = s = 0, \\ x, & x = y > 0, \end{cases} \tag{4.3}$$

and this form is introduced by Stolarsky in [3].

Most of the classical two variable means are special cases of the class E. For example, $E_{1,2} = (x+y)/2$ is the arithmetic mean $A(x,y)$, $E_{0,0} = \sqrt{xy}$ is the geometric mean $G(x,y)$, $E_{0,1} = (x-y)/(\log x - \log y)$ is the logarithmic mean $L(x,y)$, $E_{1,1} = (x^x/y^y)^{1/(x-y)}/e$ is the identric mean $I(x,y)$, and so forth. More generally, the rth power mean $((x^r + y^r)/2)^{1/r}$ is equal to $E_{r,2r}$.

Example 4.1. Taking $f(x) = 1/x$, after an easy calculation it follows that $S_{1/x}(a,b) = (A(a,b)/G(a,b))^2$. Therefore we consequently obtain the result.

Proposition 4.2. *If $0 < a \le x_i \le b$, then the inequality*

$$1 \le \left(\sum p_i x_i\right)\left(\sum \frac{p_i}{x_i}\right) \le \frac{(a+b)^2}{4ab} \tag{4.4}$$

holds for an arbitrary weight sequence $\mathbf{p}$.

This is the extended form of Schweitzer inequality.

Example 4.3. For $f(x) = x^2$ we get that the maximum of $F(p)$ is attained at the point $p_0 = b/(a+b)$.

Hence, we have the following.

Proposition 4.4. *If $0 < a \le x_i \le b$, then the following means inequality*

$$1 \le \frac{\sqrt{\sum p_i x_i^2}}{\sum p_i x_i} \le \frac{A(a,b)}{G(a,b)} \tag{4.5}$$

holds for an arbitrary weight sequence $\mathbf{p}$.

As a special case of the above inequality, that is, by putting $p_i = u_i^2 / \sum_i u_i^2,\ x_i = v_i/u_i$ and noting that $0 < u \le u_i \le U,\ 0 < v \le v_i \le V$ imply $a = v/U \le x_i \le V/u = b$, we obtain a converse of the well-known Cauchy's inequality.

Proposition 4.5. *If $0 < u \le u_i \le U,\ 0 < v \le v_i \le V$, then*

$$1 \le \frac{\sum u_i^2 \sum v_i^2}{\left(\sum u_i v_i\right)^2} \le \left(\frac{\sqrt{UV/uv} + \sqrt{uv/UV}}{2}\right)^2. \tag{4.6}$$

In this form the Cauchy's inequality was stated in [2, page 80].

Note that the same result can be obtained from inequality (4.4) by taking $p_i = u_i v_i / \sum_i u_i v_i, x_i = u_i/v_i$.

Example 4.6. Let $f(x) = x^\alpha,\ 0 < \alpha < 1$. Since in this case f is a concave function, applying the second part of Theorem 2.1, we get the following.

Proposition 4.7. *If $0 < a \le x_i \le b$, then*

$$1 \le \frac{\left(\sum p_i x_i\right)^\alpha}{\sum p_i x_i^\alpha} \le \left(\frac{E_{\alpha,1}(a,b) E_{1-\alpha,1}(a,b)}{G^2(a,b)}\right)^{\alpha(1-\alpha)}, \tag{4.7}$$

independently of $\mathbf{p}$.

In the limiting cases we obtain two important converses. Namely, writing (4.7) as

$$1 \le \frac{\sum p_i x_i}{\left(\sum p_i x_i^{\alpha}\right)^{1/\alpha}} \le \left(\frac{E_{\alpha,1}(a,b)E_{1-\alpha,1}(a,b)}{G^2(a,b)}\right)^{1-\alpha}, \tag{4.8}$$

and, letting $\alpha \to 0^+$, the converse of generalized $A - G$ inequality arises.

Proposition 4.8. *If* $0 < a \le x_i \le b$, *then*

$$1 \le \frac{\sum p_i x_i}{\prod x_i^{p_i}} \le \frac{L(a,b)I(a,b)}{G^2(a,b)}. \tag{4.9}$$

Note that the right-hand side of (4.9) is exactly the Specht ratio (cf. [1]).

Analogously, writing (4.7) in the form

$$1 \le \left(\frac{\left(\sum p_i x_i\right)^{\alpha}}{\sum p_i x_i^{\alpha}}\right)^{1/(1-\alpha)} \le \left(\frac{E_{\alpha,1}(a,b)E_{1-\alpha,1}(a,b)}{G^2(a,b)}\right)^{\alpha}, \tag{4.10}$$

and taking the limit $\alpha \to 1^-$, one has the following.

Proposition 4.9. *If* $0 < a \le x_i \le b$, *then*

$$0 \le \frac{\sum p_i x_i \log x_i - \sum p_i x_i \log\left(\sum p_i x_i\right)}{\sum p_i x_i} \le \log \frac{L(a,b)I(a,b)}{G^2(a,b)}. \tag{4.11}$$

Finally, putting in (4.7) $p_i = v_i / \sum_i v_i, x_i = u_i/v_i, \alpha = 1/p, 1-\alpha = 1/q$, we obtain the converse of discrete Hölder's inequality.

Proposition 4.10. *If* $p, q > 1,\ 1/p + 1/q = 1;\ 0 < a \le u_i/v_i \le b$, *then*

$$1 \le \frac{\left(\sum u_i\right)^{1/p}\left(\sum v_i\right)^{1/q}}{\sum u_i^{1/p} v_i^{1/q}} \le \left(\frac{E_{1/p,1}(a,b)E_{1/q,1}(a,b)}{G^2(a,b)}\right)^{1/pq}. \tag{4.12}$$

It is interesting to compare (4.12) with the converse of Hölder's inequality for integral forms (cf. [4]).

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