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Research Article

Some Inequalities for the L_p-Curvature Image

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Lutwak introduced the notion of L_p-curvature image and proved an inequality for the volumes of convex body and its L_p-curvature image. In this paper, we first give an monotonic property of L_p-curvature image. Further, we establish two inequalities for the L_p-curvature image and its polar, respectively. Finally, an inequality for the volumes of L_p-projection body and L_p-curvature image is obtained.

1. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$, for the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we, respectively, write $\mathcal{K}_o^n$ and $\mathcal{K}_s^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and denote by $V(K)$ the n-dimensional volume of body K, for the standard unit ball B in $\mathbb{R}^n$, and denote $\omega_n = V(B)$. The groups of nonsingular linear transformations and the group of special linear transformations are denoted by $GL(n)$ and $SL(n)$, respectively.

Suppose that $\mathbb{R}$ is the set of real numbers. If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K,\cdot)\colon \mathbb{R}^n \to \mathbb{R}$, is defined by (see [1, page 16])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $x \cdot y$ denotes the standard inner product of x and y.

A convex body $K \in \mathcal{K}^n$ is said to have a curvature function $f(K,\cdot) : S^{n-1} \to \mathbb{R}$, if its surface area measure $S(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{dS(K,\cdot)}{dS} = f(K,\cdot). \tag{1.2}$$

For $K \in \mathcal{K}_o^n$, and real $p \geq 1$, the L_p-surface area measure, $S_p(K,\cdot)$, of K is defined by (see [2, 3])

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)} = h(K,\cdot)^{1-p}. \tag{1.3}$$

Equation (1.3) is also called Radon-Nikodym derivative, and the measure $S_p(K,\cdot)$ is absolutely continuous with respect to surface area measure $S(K,\cdot)$.

A convex body $K \in \mathcal{K}_o^n$ is said to have a L_p-curvature function (see [2]) $f_p(K,\cdot) : S^{n-1} \to \mathbb{R}$, if its L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{dS_p(K,\cdot)}{dS} = f_p(K,\cdot). \tag{1.4}$$

If K is a compact star shaped (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,+\infty)$, is defined by (see [1, page 18])

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{1.5}$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

For the radial function, if $c > 0$, then (see [1, page 18])

$$\rho_K(cx) = \frac{1}{c}\rho_K(x). \tag{1.6}$$

From (1.6), we have that, for $\mu > 0$,

$$\rho_{\mu K}(x) = \max\{\lambda \geq 0 : \lambda x \in \mu K\} = \max\left\{\lambda \geq 0 : \lambda \frac{x}{\mu} \in K\right\} = \rho_K\left(\frac{x}{\mu}\right) = \mu \rho_K(x). \tag{1.7}$$

Let $\mathcal{F}_o^n$, $\mathcal{F}_s^n$ denote the set of all bodies in $\mathcal{K}_o^n$, $\mathcal{K}_s^n$, respectively, that have a positive continuous curvature function.

Lutwak in [2] showed the notion of L_p-curvature image as follows. For each $K \in \mathcal{F}_o^n$ and real $p \geq 1$, define $\Lambda_p K \in \mathcal{S}_o^n$, the L_p-curvature image of K, by

$$f_p(K,\cdot) = \frac{\omega_n}{V(\Lambda_p K)} \rho(\Lambda_p K,\cdot)^{n+p}. \tag{1.8}$$

Note that, for $p = 1$, this definition differs from the definition of classical curvature image (see [2]). For the study of classical curvature image [1, 4–7].

Further, he proved that if $K \in \mathcal{F}_s^n$ *and* $p \geq 1$, *then*

$$V(\Lambda_p K) \leq \omega_n^{(2p-n)/p} V(K)^{(n-p)/p} \tag{1.9}$$

with equality if and only if K *is an ellipsoid centered at the origin.*

In this paper, we continuously study the L_p-curvature image for convex bodies. First, we give a monotonic property of L_p-curvature image as follows.

Theorem 1.1. *If* $K, L \in \mathcal{F}_o^n$, $p \geq 1$, *and* $\Lambda_p K \subseteq \Lambda_p L$, *then*

$$V(\Lambda_p K) V(K)^{(n-p)/n} \leq V(\Lambda_p L) V(L)^{(n-p)/n} \tag{1.10}$$

with equality for $n = p > 1$ *if and only if* K *and* L *are dilates, for* $n \neq p > 1$ *if and only if* $K = L$, *and for* $n \neq p = 1$ *if and only if* K *and* L *are translation.*

Next, we establish an inequality for the L_p-curvature image as follows.

Theorem 1.2. *If* $K \in \mathcal{F}_s^n$, *and* $p \geq 1$, *then*

$$V(\Lambda_p K) \leq \omega_n^{n/p} V(K^*)^{(p-n)/p} \tag{1.11}$$

with equality if and only if K *is an ellipsoid.*

Further, we get the following inequality for the polar of the L_p-curvature image.

Theorem 1.3. *If* $K \in \mathcal{F}_o^n$, $\Lambda_p K \in \mathcal{K}_o^n$, *and* $p \geq 1$, *then*

$$V\left(\Lambda_p^* K\right) \leq \omega_n^{n/p} V(K)^{(p-n)/p} \tag{1.12}$$

with equality for $p > 1$ *if and only if* $\Lambda_p^* K$ *and* K *are dilates, and for* $p = 1$ *if and only if* $\Lambda_p^* K$ *and* K *are homothetic.*

Here $\Lambda_p^* K$ denote the polar of $\Lambda_p K$, rather than $(\Lambda_p K)^*$. Compare with inequality (1.9), we see that inequality (1.12) may be regarded as a dual form of inequality (1.9).

Finally, we obtain an interesting inequality for the L_p-curvature image and L_p-projection body $\Pi_p K$ as follows.

Theorem 1.4. *If* $K \in \mathcal{F}_o^n$, $p \geq 1$, *then*

$$V(\Pi_p K) \geq V(\Lambda_p K) \tag{1.13}$$

with equality if and only if K *is an ellipsoid centered at the origin.*

2. Preliminaries

2.1. Polar of Convex Body

If $K \in \mathcal{K}_o^n$, the polar body of K, K^*, is defined by (see [1, page 20])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \forall y \in K\}. \tag{2.1}$$

From the definition (2.1), we know that if $K \in \mathcal{K}_o^n$, then the support and radial functions of K^*, the polar body of K, are defined, respectively, by (see [1])

$$h_{K^*} = \frac{1}{\rho_K}, \qquad \rho_{K^*} = \frac{1}{h_K}. \tag{2.2}$$

The Blaschke-Santaló inequality can be stated that (see [1] or [7]): *If $K \in \mathcal{K}_s^n$, then*

$$V(K)V(K^*) \leq \omega_n^2 \tag{2.3}$$

with equality if and only if K is an ellipsoid.

2.2. L_p-Mixed Volume

For $K, L \in \mathcal{K}_o^n$ and $\varepsilon > 0$, the Firey L_p-combination $K +_p \varepsilon \cdot L \in \mathcal{K}_o^n$ is defined by (see [8])

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p \tag{2.4}$$

where "$\cdot$" in $\varepsilon \cdot L$ denotes the Firey scalar multiplication.

If $K, L \in \mathcal{K}_o^n$ in $\mathbb{R}^n$, then for $p \geq 1$, the L_p-mixed volume, $V_p(K, L)$, of the K and L is defined by (see [9])

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.5}$$

Corresponding to each $K \in \mathcal{K}_o^n$, there is a positive Borel measure, $S_p(K, \cdot)$, on S^{n-1} such that (see [9])

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h(Q, u)^p dS_p(K, u) \tag{2.6}$$

for each $Q \in \mathcal{K}_o^n$. The measure $S_p(K, \cdot)$ is just the L_p-surface area measure of K.

From the formula (2.6) and definition (1.3), we immediately get that

$$V_p(K, K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K, u) = V(K). \tag{2.7}$$

The L_p-Minkowski inequality states that (see [9]) if $K, L \in \mathcal{K}_o^n$ and $p \geq 1$, then

$$V_p(K,L) \geq V(K)^{(n-p)/n} V(L)^{p/n} \tag{2.8}$$

with equality for $p > 1$ if and only if K and L are dilates, and for $p = 1$ if and only if K and L are homothetic.

2.3. L_p-Dual Mixed Volume

For $K, L \in \mathcal{S}_o^n$, and $\varepsilon > 0$, the L_p-harmonic radial combination $K \tilde{+}_{-p} \varepsilon \cdot L$ is the star body whose radial function is defined by (see [2])

$$\rho\left(K \tilde{+}_{-p} \varepsilon \cdot L, \cdot\right)^{-p} = \rho(K,\cdot)^{-p} + \varepsilon \rho(L,\cdot)^{-p}. \tag{2.9}$$

Note that here "$\varepsilon \cdot L$" and the Firey scalar multiplication "$\varepsilon \cdot L$" are different.

If $K, L \in \mathcal{S}_o^n$, for $p \geq 1$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K,L)$, of the K and L is defined by (see [2])

$$\frac{n}{-p} \widetilde{V}_{-p}(K,L) = \lim_{\varepsilon \to 0^+} \frac{V\left(K \tilde{+}_{-p} \varepsilon \cdot L\right) - V(K)}{\varepsilon}. \tag{2.10}$$

The definition above and the polar coordinate formula for volume give the following integral representation of the L_p-dual mixed volume $\widetilde{V}_{-p}(K,L)$ of $K, L \in \mathcal{S}_o^n$:

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n+p} \rho(L,u)^{-p} dS(u), \tag{2.11}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From the formula (2.11), it follows immediately that, for each $K \in \mathcal{S}_o^n$ and $p \geq 1$,

$$\widetilde{V}_{-p}(K,K) = V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^n dS(u). \tag{2.12}$$

The Minkowski inequality for the L_p-dual mixed volume $\widetilde{V}_{-p}$ is that if $K, L \in \mathcal{S}_o^n$ and $p \geq 1$ (see [2]), then

$$\widetilde{V}_{-p}(K,L) \geq V(K)^{(n+p)/n} V(L)^{-p/n} \tag{2.13}$$

with equality if and only if K and L are dilates.

2.4. L_p-Affine Surface Area

Lutwak in [2] showed that for each $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-affine surface area, $\Omega_p(K)$, of K can be defined by

$$n^{-p/n}\Omega_p(K)^{(n+p)/n} = \inf\left\{nV_p(K, Q^*)V(Q)^{p/n} : Q \in S_o^n\right\}. \tag{2.14}$$

For $p = 1$, $\Omega_p(K)$ is just classical affine surface area $\Omega(K)$ by Leichtweiβ (see [4]). Further, Lutwak proved that if $K \in \mathcal{F}_o^n$ and $p \geq 1$, then the L_p-affine surface area of K has the integral representation

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K, u)^{n/(n+p)} dS(u). \tag{2.15}$$

2.5. L_p-Projection Body

The notion of L_p-projection body is shown by Lutwak et al. (see [10]). For $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-projection body, $\Pi_p K$, of K is the origin-symmetric convex body whose support function is given by

$$h_{\Pi_p K}^p(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v) \tag{2.16}$$

for all $u \in S^{n-1}$. Here $S_p(K, \cdot)$ is just the L_p-surface area measure of K, and

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}. \tag{2.17}$$

2.6. L_p-Centroid Body

Lutwak and Zhang in [11] introduced the notion of L_p-centroid body. For each compact star-shaped body about the origin $K \subset \mathbb{R}^n$ and for real number $p \geq 1$, the polar of L_p-centroid body, $\Gamma_p^* K$ (rather than $(\Gamma_p K)^*$), of K is the origin-symmetric convex body, whose radial function is defined by [11]

$$\rho_{\Gamma_p^* K}^{-p}(u) = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx \tag{2.18}$$

for all $u \in S^{n-1}$, where $c_{n,p}$ satisfy (2.17).

From definition (2.18) and equality (2.2), if $K \in \mathcal{S}_o^n$, then the L_p-centroid body $\Gamma_p K$ of K is the origin-symmetric convex body whose support function is given by

$$\begin{aligned} h_{\Gamma_p K}^p(u) &= \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx \\ &= \frac{1}{(n+p) c_{n,p} V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_K^{n+p}(v) dS(v) \end{aligned} \tag{2.19}$$

for all $u \in S^{n-1}$.

3. The Proof of Theorems

In order to prove our theorems, the following lemmas are essential.

Lemma 3.1. *If $K \in F_o^n$, $p \geq 1$ and the constant $c > 0$, then*

$$\Lambda_p c K = c^{(n-p)/p} \Lambda_p K. \tag{3.1}$$

Proof. For $c > 0$, from (1.3) and (1.4), then

$$f_p(cK, \cdot) = c^{n-p} f_p(K, \cdot), \tag{3.2}$$

this together with (1.7) and (1.8), and notice that $V(\lambda Q) = \lambda^n V(Q)$ for $\lambda > 0$, we get that

$$\frac{\rho(\Lambda_p cK, \cdot)^{n+p}}{V(\Lambda_p cK)} = \frac{f_p(cK, \cdot)}{\omega_n} = c^{n-p} \frac{f_p(K, \cdot)}{\omega_n} = c^{n-p} \frac{\rho(\Lambda_p K, \cdot)^{n+p}}{V(\Lambda_p K)} = \frac{\rho\left(c^{(n-p)/p} \Lambda_p K, \cdot\right)^{n+p}}{V\left(c^{(n-p)/p} \Lambda_p K\right)}, \tag{3.3}$$

that is,

$$\rho(\Lambda_p cK, \cdot) = \left[\frac{V(\Lambda_p cK)}{V\left(c^{(n-p)/p} \Lambda_p K\right)}\right]^{1/(n+p)} \rho\left(c^{(n-p)/p} \Lambda_p K, \cdot\right), \tag{3.4}$$

and this together with formula (2.12), we have that

$$V(\Lambda_p cK) = V\left(c^{(n-p)/p} \Lambda_p K\right). \tag{3.5}$$

Hence, from (3.4), then

$$\rho(\Lambda_p cK, \cdot) = \rho\left(c^{(n-p)/p} \Lambda_p K, \cdot\right), \tag{3.6}$$

and this yields (3.1). □

If $\phi \in SL(n)$, Lutwak (see [2]) proved that, for $p \geq 1$,

$$\Lambda_p \phi K = \phi^{-t} \Lambda_p K, \tag{3.7}$$

where ϕ^{-t} denotes the inverse of the transpose of ϕ.

Now we rewrite (3.1) as follows:

$$\Lambda_p cK = c^{(n-p)/p} \Lambda_p K = (c^n)^{1/p} c^{-1} \Lambda_p K, \tag{3.8}$$

this together with (3.7) and the fact $\Lambda_p(-K) = -\Lambda_p K$, we easily get the following result.

Proposition 3.2. *If $K \in \mathcal{F}_o^n$, $p \geq 1$, then for $\phi \in GL(n)$,*

$$\Lambda_p \phi K = \left|\det \phi\right|^{1/p} \phi^{-t} \Lambda_p K. \tag{3.9}$$

Lemma 3.3 (see [2]). *If $K \in F_o^n$, $p \geq 1$, then*

$$V_p(K, Q^*) = \frac{\omega_n}{V(\Lambda_p K)} \widetilde{V}_{-p}(\Lambda_p K, Q) \tag{3.10}$$

for all $Q \in S_o^n$.

Lemma 3.4. *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, then for all $Q \in \mathcal{S}_o^n$,*

$$\widetilde{V}_{-p}(K, Q) = \widetilde{V}_{-p}(L, Q) \Longleftrightarrow K = L. \tag{3.11}$$

Proof. Taking $Q = K$ in (3.11), and using (2.12), we have that $V(K) = \widetilde{V}_{-p}(L, K)$. Now inequality (2.13) gives $V(K) \geq V(L)$, with equality if and only if K and L are dilates. Let $Q = L$ in (3.11), and get $V(L) \geq V(K)$. Hence $V(K) = V(L)$, and K and L must be dilates. Thus $K = L$. In turn, when $K = L$, the result obviously is true. □

Proof of Theorem 1.1. Since $\Lambda_p K \subseteq \Lambda_p L$, then from formula (2.11), we know

$$\widetilde{V}_{-p}(\Lambda_p K, Q) \leq \widetilde{V}_{-p}(\Lambda_p L, Q) \tag{3.12}$$

for all $Q \in \mathcal{S}_o^n$, with equality in (3.12) if and only if $\Lambda_p K = \Lambda_p L$ by (3.11). Using equality (3.10), then inequality (3.12) can be rewritten

$$V(\Lambda_p K) V_p(K, Q^*) \leq V(\Lambda_p L) V_p(L, Q^*), \tag{3.13}$$

for all $Q \in \mathcal{S}_o^n$. Let $Q^* = L$, together with (2.7) and L_p-Minkowski inequality (2.8), we have

$$V(\Lambda_p L) V(L) \geq V(\Lambda_p K) V_p(K, L) \geq V(\Lambda_p K) V(K)^{(n-p)/n} V(L)^{p/n}. \tag{3.14}$$

Thus

$$V(\Lambda_p K)V(K)^{(n-p)/n} \leq V(\Lambda_p L)V(L)^{(n-p)/n} \tag{3.15}$$

and this is just inequality (1.10).

According to the conditions of equality that hold in inequalities (3.12) and (2.8), we know that equality holds in inequality (1.10) for $p > 1$ if and only if K and L are dilates and $\Lambda_p K = \Lambda_p L$, and for $p = 1$ if and only if K and L are homothetic and $\Lambda_p K = \Lambda_p L$.

For the case $p > 1$ of equality that holds in (1.10), we may suppose $L = cK$ $(c > 0)$, and together with $\Lambda_p K = \Lambda_p L$, then $\Lambda_p K = \Lambda_p cK$. Thus, from (3.1), we have $\Lambda_p K = c^{(n-p)/p}\Lambda_p K$. Hence $c = 1$ when $n \neq p$, this means that if $n \neq p$, then $K = L$. For $n = p > 1$, we easily see that K and L are dilates that impliy $\Lambda_p K = \Lambda_p L$. So we know that equality holds in inequality (1.10) for $n = p > 1$ if and only if K and L are dilates, and for $n \neq p > 1$ if and only if $K = L$.

For the case $p = 1$ of equality that holds in (1.10), we may take $L = x + cK$ $(c > 0, x \in \mathbb{R}^n)$, then

$$\Lambda_1 K = \Lambda_1 L = \Lambda_1(x + cK). \tag{3.16}$$

But $S_1(x + K, \cdot) = S(x + K, \cdot) = S(K, \cdot)$, then $f_1(x + K, \cdot) = f(x + K, \cdot) = f(K, \cdot)$ by (1.2). By this together with (2.15) and (3.10), we have $\Omega(x + K) = \Omega(K)$ and $V(\Lambda_1(x + K)) = V(\Lambda_1 K)$, respectively. Thus, from the definition (1.8), we obtain that

$$\begin{aligned} \rho(\Lambda_1(x + K), \cdot)^{n+1} &= \frac{V(\Lambda_1(x + K))}{\omega_n} f(x + K, \cdot) \\ &= \frac{V(\Lambda_1 K)}{\omega_n} f(K, \cdot) = \rho(\Lambda_1 K, \cdot)^{n+1}, \end{aligned} \tag{3.17}$$

hence

$$\Lambda_1(x + K) = \Lambda_1 K. \tag{3.18}$$

From (3.18) and (3.1), equality (3.16) can be rewritten as follows:

$$\Lambda_1 K = \Lambda_1(x + cK) = c^{n-1}\Lambda_1 K, \tag{3.19}$$

and this gives $c = 1$, that is, $L = x + K$ when $n > 1$. Therefore, we see that equality holds in inequality (1.10) for $n \neq p = 1$ if and only if K and L are translation. □

Proof of Theorem 1.2. Let $Q = K^*$ in (3.10), together with (2.7) and (2.13), we have that

$$\begin{aligned} V(K) &= \frac{\omega_n}{V(\Lambda_p K)} \widetilde{V}_{-p}(\Lambda_p K, K^*) \\ &\geq \frac{\omega_n}{V(\Lambda_p K)} V(\Lambda_p K)^{(n+p)/n} V(K^*)^{-p/n} \\ &= \omega_n V(\Lambda_p K)^{p/n} V(K^*)^{-p/n} \end{aligned} \tag{3.20}$$

with equality in inequality (3.20) if and only if $\Lambda_p K$ and K^* are dilates.

From this, and using the Blaschke-Santaló inequality (2.3), then

$$\begin{aligned} V(\Lambda_p K)^{p/n} &\leq \frac{1}{\omega_n} V(K) V(K^*)^{p/n} \\ &\leq \omega_n V(K^*)^{(p-n)/n}, \end{aligned} \tag{3.21}$$

and equality holds in second inequality of (3.21) if and only if K is an ellipsoid.

From (3.21), we immediately obtain inequality (1.11). According to the conditions of equality that hold in (3.20) and second inequality of (3.21), we get equality in (1.11) if and only if K is an ellipsoid. □

Proof of Theorem 1.3. Taking $Q = \Lambda_p K$ in (3.10), and using (2.12), then

$$V_p\left(K, \Lambda_p^* K\right) = \omega_n. \tag{3.22}$$

From (3.22), and together with inequality (2.8), we have

$$\omega_n = V_p\left(K, \Lambda_p^* K\right) \geq V(K)^{(n-p)/n} V\left(\Lambda_p^* K\right)^{p/n}, \tag{3.23}$$

this inequality immediately gives (1.12). According to equality conditions of inequality (2.8), we get equality in (1.12) for $p > 1$ if and only if $\Lambda_p^* K$ and K are dilates, and for $p = 1$ if and only if $\Lambda_p^* K$ and K are homothetic. □

The proof of Theorem 1.4 requires the following two lemmas.

Lemma 3.5. *If $K \in \mathcal{F}_o^n$, $p \geq 1$, then*

$$\Pi_p K = \Gamma_p \Lambda_p K. \tag{3.24}$$

Note that the proof of Lemma 3.5 can be found in [12]. Here, for the sake of completeness, we present the proof as follows.

Proof. Using the definitions (2.16), (1.4), and (1.8), we have

$$\begin{aligned} h^p_{\Pi_p K}(u) &= \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p dS_p(K,v) \\ &= \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p f_p(K,v) dS(v) \\ &= \frac{1}{n c_{n-2,p} V(\Lambda_p K)} \int_{S^{n-1}} |u \cdot v|^p \rho(\Lambda_p K, v)^{n+p} dS(v) \end{aligned} \tag{3.25}$$

for all $u \in S^{n-1}$. According to (2.19), we also have that, for all $u \in S^{n-1}$,

$$h^p_{\Gamma_p \Lambda_p K}(u) = \frac{1}{(n+p) c_{n,p} V(\Lambda_p K)} \int_{S^{n-1}} |u \cdot v|^p \rho(\Lambda_p K, v)^{n+p} dS(v). \tag{3.26}$$

But (2.17) gives $n c_{n-2,p} = (n+p) c_{n,p}$; hence from (3.25) and (3.26), we obtain

$$h_{\Pi_p K}(u) = h_{\Gamma_p \Lambda_p K}(u) \tag{3.27}$$

for all $u \in S^{n-1}$. Thus $\Pi_p K = \Gamma_p \Lambda_p K$. □

Lemma 3.6 ([10] (L_p-Busemann-Petty centroid inequality)). *If $K \in \mathcal{S}^n_o$, $p \ge 1$, then*

$$V(\Gamma_p K) \ge V(K) \tag{3.28}$$

with equality if and only if K is an ellipsoid centered at the origin.

Proof of Theorem 1.4. From (3.28) and (3.24), we immediately get inequality (1.13). According to the case of equality that holds in (3.28), we see equality in (1.13) if and only if K is an ellipsoid centered at the origin. □

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