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Research Article

General System of A-Monotone Nonlinear Variational Inclusions Problems with Applications

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We introduce and study a new system of nonlinear variational inclusions involving a combination of A-Monotone operators and relaxed cocoercive mappings. By using the resolvent technique of the A-monotone operators, we prove the existence and uniqueness of solution and the convergence of a new multistep iterative algorithm for this system of variational inclusions. The results in this paper unify, extend, and improve some known results in literature.

1. Introduction

Recently, Fang and Huang [1] introduced a new class of H-monotone mappings in the context of solving a system of variational inclusions involving a combianation of H-monotone and strongly monotone mappings based on the resolvent operator techniques. The notion of the H-monotonicity has revitalized the theory of maximal monotone mappings in several directions, especially in the domain of applications. Verma [2] introduced the notion of A-monotone mappings and its applications to the solvability of a system of variational inclusions involving a combination of A-monotone and strongly monotone mappings. As Verma point out "the class of A-monotone mappings generalizes H-monotone mappings. On the top of that, A-monotonicity originates from hemivariational inequalities, and emerges as a major contributor to the solvability of nonlinear variational problems on nonconvex settings." and as a matter of fact, some nice examples on A-monotone (or generalized maximal monotone) mappings can be found in Naniewicz and Panagiotopoulos [3] and Verma [4]. Hemivariational inequalities—initiated and developed by Panagiotopoulos [5]—are connected with nonconvex energy functions and turned out to be useful tools proving the existence of solutions of nonconvex constrained problems. It is worthy noting that

A-monotonicity is defined in terms of relaxed monotone mappings—a more general notion than the monotonicity or strong monotonocity—which gives a significant edge over the H-monotonocity. Very recently, Verma [6] studied the solvability of a system of variational inclusions involving a combination of A-monotone and relaxed cocoercive mappings using resolvent operator techniques of A-monotone mappings. Since relaxed cocoercive mapping is a generalization of strong monotone mappings, the main result in [6] is more general than the corresponding results in [1, 2].

Inspired and motivated by recent works in [1, 2, 6], the purpose of this paper is to introduce a new mathematical model, which is called a general system of A-monotone nonlinear variational inclusion problems, that is, a family of A-monotone nonlinear variational inclusion problems defined on a product set. This new mathematical model contains the system of inclusions in [1, 2, 6], the variational inclusions in [7, 8], and some variational inequalities in literature as special cases. By using the resolvent technique for the A-monotone operators, we prove the existence and uniqueness of solution for this system of variational inclusions. We also prove the convergence of a multistep iterative algorithm approximating the solution for this system of variational inclusions. The result in this paper unifies, extends, and improves some results in [1, 2, 6–8] and the references therein.

2. Preliminaries

We suppose that $\mathscr{H}$ is a real Hilbert space with norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively, $2^{\mathscr{H}}$ denotes the family of all the nonempty subsets of $\mathscr{H}$. If $M:\mathscr{H}\to 2^{\mathscr{H}}$ be a set-valued operator, then we denote the effective domain $D(M)$ of M as follows:

$$D(M)=\{x\in\mathscr{H}:M(x)\neq\emptyset\}. \tag{2.1}$$

Now we recall some definitions needed later.

Definition 2.1 (see [2, 6, 7]). Let $A:\mathscr{H}\to\mathscr{H}$ be a single-valued operator and let $M:\mathscr{H}\to 2^{\mathscr{H}}$ be a set-valued operator. M is said to be

(i) m-relaxed monotone, if there exists a constant $m>0$ such that

$$\langle x-y,u-v\rangle\geq -m\|u-v\|^2,\quad \forall u,v\in D(M),\ x\in Mu,\ y\in Mv, \tag{2.2}$$

(ii) A-monotone with a constant m if

(a) M is m-relaxed monotone,

(b) $A+\lambda M$ is maximal monotone for $\lambda>0$ (i.e., $(A+\lambda M)(\mathscr{H})=\mathscr{H}$, for all $\lambda>0$).

Remark 2.2. If $m=0$, $A=H:\mathscr{H}\to\mathscr{H}$, then the definition of A-monotonicity is that of H-monotonicity in [1, 8]. It is easy to know that if $H=I$ (the identity map on $\mathscr{H}$), then the definition of I-monotone operators is that of maximal monotone operators. Hence, the class of A-monotone operators provides a unifying frameworks for classes of maximal monotone operators, H-monotone operators. For more details about the above definitions, please refer to [1–8] and the references therein.

It follow from [3, Lemma 7.11] we know that if X is a reflexive Banach space with X^* its dual, and $A : X \to X^*$ be m-strongly monotone and $f : X \to R$ is a locally Lipschitz such that ∂f is α-relaxed monotone, then ∂f is A-monotone with a constant $m - \alpha$.

Definition 2.3 (see [1, 7, 8]). Let $A, T : \mathcal{H} \to \mathcal{H}$, be two single-valued operators. T is said to be

(i) monotone if

$$\langle Tu - Tv, u - v \rangle \geq 0, \quad \forall u,\ v \in \mathcal{H}; \tag{2.3}$$

(ii) strictly monotone if T is monotone and

$$\langle Tu - Tv, u - v \rangle = 0, \quad \text{iff } u = v; \tag{2.4}$$

(iii) γ-strongly monotone if there exists a constant $\gamma > 0$ such that

$$\langle Tu - Tv, u - v \rangle \geq \gamma \|u - v\|^2, \quad \forall u, v \in \mathcal{H}; \tag{2.5}$$

(iv) s-Lipschitz continuous if there exists a constant $s > 0$ such that

$$\|T(u) - T(v)\| \leq s\|u - v\|, \quad \forall u, v \in \mathcal{H}; \tag{2.6}$$

(v) r-strongly monotone with respect to A if there exists a constant $\gamma > 0$ such that

$$\langle Tu - Tv, Au - Av \rangle \geq r\|u - v\|^2, \quad u, v \in \mathcal{H}. \tag{2.7}$$

Definition 2.4 (see [2]). Let $A : \mathcal{H} \to \mathcal{H}$ be a γ-strongly monotone operator and let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an A-monotone operator. Then the resolvent operator $R^A_{M,\lambda} : \mathcal{H} \to \mathcal{H}$ is defined by

$$R^A_{M,\lambda}(x) = (A + \lambda M)^{-1}(x), \quad \forall x \in \mathcal{H}. \tag{2.8}$$

We also need the following result obtained by Verma [2].

Lemma 2.5. *Let $A : \mathcal{H} \to \mathcal{H}$ be a γ-strongly monotone operator and let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an A-monotone operator. Then, the resolvent operator $R^A_{M,\lambda} : \mathcal{H} \to \mathcal{H}$ is Lipschitz continuous with constant $1/(\gamma - m\lambda)$ for $0 < \lambda < \gamma/m$, that is,*

$$\left\| R^A_{M,\lambda}(x) - R^A_{M,\lambda}(y) \right\| \leq \frac{1}{\gamma - m\lambda} \|x - y\|, \quad \forall x, y \in H. \tag{2.9}$$

One needs the following new notions.

Definition 2.6. Let $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_p$ be Hilbert spaces and $\|\cdot\|_1$ denote the norm of $\mathcal{H}_1$, also let $A_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $N_1 : \prod_{j=1}^{p} \mathcal{H}_j \to \mathcal{H}_1$ be two single-valued mappings:

(i) N_1 is said to be ξ-Lipschitz continuous in the first argument if there exists a constant $\xi > 0$ such that

$$\begin{aligned}\left\|N_1(x_1, x_2, \dots, x_p) - N_1(y_1, x_2, \dots, x_p)\right\|_1 \le \xi \left\|x_1 - y_1\right\|_1, \\ \forall x_1, y_1 \in \mathcal{H}_1,\ x_j \in \mathcal{H}_j\ (j = 2, 3, \dots, p);\end{aligned} \tag{2.10}$$

(ii) N_1 is said to be monotone with respect to A_1 in the first argument if

$$\begin{aligned}\left\langle N_1(x_1, x_2, \dots, x_p) - N_1(y_1, x_2, \dots, x_p), A_1(x_1) - A_1(y_1)\right\rangle \ge 0, \\ \forall x_1, y_1 \in \mathcal{H}_1,\ x_j \in \mathcal{H}_j\ (j = 2, 3, \dots, p);\end{aligned} \tag{2.11}$$

(iii) N_1 is said to be β-strongly monotone with respect to A_1 in the first argument if there exists a constant $\beta > 0$ such that

$$\begin{aligned}\left\langle N_1(x_1, x_2, \dots, x_p) - N_1(y_1, x_2, \dots, x_p), A_1(x_1) - A_1(y_1)\right\rangle \ge \beta \left\|x_1 - y_1\right\|_1^2, \\ \forall x_1, y_1 \in \mathcal{H}_1,\ x_j \in \mathcal{H}_j (j = 2, 3, \dots, p);\end{aligned} \tag{2.12}$$

(iv) N_1 is said to be γ-cocoercive with respect to A_1 in the first argument if there exists a constant $\gamma > 0$ such that

$$\begin{aligned}&\left\langle N_1(x_1, x_2, \dots, x_p) - N_1(y_1, x_2, \dots, x_p), A_1(x_1) - A_1(y_1)\right\rangle \\ &\quad \ge \gamma \left\|N_1(x_1, x_2, \dots, x_p) - N_1(y_1, x_2, \dots, x_p)\right\|_1^2, \quad \forall x_1, y_1 \in \mathcal{H}_1,\ x_j \in \mathcal{H}_j\ (j = 2, 3, \dots, p);\end{aligned} \tag{2.13}$$

(v) N_1 is said to be γ-relaxed cocoercive with respect to A_1 in the first argument if there exists a constant $\gamma > 0$ such that

$$\begin{aligned}&\left\langle N_1(x_1, x_2, \dots, x_p) - N_1(y_1, x_2, \dots, x_p), A_1(x_1) - A_1(y_1)\right\rangle \\ &\quad \ge -\gamma \left\|N_1(x_1, x_2, \dots, x_p) - N_1(y_1, x_2, \dots, x_p)\right\|_1^2, \quad \forall x_1, y_1 \in \mathcal{H}_1,\ x_j \in \mathcal{H}_j\ (j = 2, 3, \dots, p);\end{aligned} \tag{2.14}$$

(vi) N_1 is said to be (γ, r)-relaxed cocoercive with respect to A_1 in the first argument if there exists a constant $\gamma > 0$ such that

$$\begin{aligned}&\left\langle N_1(x_1, x_2, \dots, x_p) - N_1(y_1, x_2, \dots, x_p), A_1(x_1) - A_1(y_1)\right\rangle \\ &\quad \ge -\gamma \left\|N_1(x_1, x_2, \dots, x_p) - N_1(y_1, x_2, \dots, x_p)\right\|_1^2 + r\left\|x_1 - y_1\right\|_1^2, \\ &\qquad \forall x_1, y_1 \in \mathcal{H}_1,\ x_j \in \mathcal{H}_j\ (j = 2, 3, \dots, p).\end{aligned} \tag{2.15}$$

In a similar way, we can define the Lipschitz continuity and the strong monotonicity (monotonicity), relaxed cocoercivity (cocoercivity) of $N_i : \prod_{j=1}^{p}\mathcal{H}_j \to \mathcal{H}_i$ with respect to $A_i : \mathcal{H}_i \to \mathcal{H}_i$ in the ith argument ($i = 2, 3, \ldots, p$).

3. A System of Set-Valued Variational Inclusions

In this section, we will introduce a new system of nonlinear variational inclusions in Hilbert spaces. In what follows, unless other specified, for each $i = 1, 2, \ldots, p$, we always suppose that $\mathcal{H}_i$ is a Hilbert space with norm denoted by $\|\cdot\|_i$, $A_i : \mathcal{H}_i \to \mathcal{H}_i$, $F_i : \prod_{j=1}^{p}\mathcal{H}_j \to \mathcal{H}_i$ are single-valued mappings, and $M_i : \mathcal{H}_i \to 2^{\mathcal{H}_i}$ is a nonlinear mapping. We consider the following problem of finding $(x_1, x_2, \ldots, x_p) \in \prod_{i=1}^{p}\mathcal{H}_i$ such that for each $i = 1, 2, \ldots, p$,

$$0 \in F_i(x_1, x_2, \ldots, x_p) + M_i(x_i). \tag{3.1}$$

Below are some special cases of (3.1).

If $p = 2$, then (3.1) becomes the following problem of finding $(x_1, x_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that

$$\begin{gathered} 0 \in F_1(x_1, x_2) + M_1(x_1), \\ 0 \in F_2(x_1, x_2) + M_2(x_1). \end{gathered} \tag{3.2}$$

However, (3.2) is called a system of set-valued variational inclusions introduced and researched by Fang and Huang [1, 9] and Verma [2, 6].

If $p = 1$, then (3.1) becomes the following variational inclusion with an A-monotone operator, which is to find $x_1 \in \mathcal{H}_1$ such that

$$0 \in F_1(x_1) + M_1(x_1), \tag{3.3}$$

problem (3.3) is introduced and studied by Fang and Huang [8]. It is easy to see that the mathematical model (2) studied by Verma [7] is a variant of (3.3).

4. Existence of Solutions and Convergence of an Iterative Algorithm

In this section, we will prove existence and uniqueness of solution for (3.1). For our main results, we give a characterization of the solution of (3.1) as follows.

Lemma 4.1. *For $i = 1, 2, \ldots, p$, let $A_i : \mathcal{H}_i \to \mathcal{H}_i$ be a strictly monotone operator and let $M_i : \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be an A_i-monotone operator. Then $(x_1, x_2, \ldots, x_p) \in \prod_{i=1}^{p}\mathcal{H}_i$ is a solution of* (3.1) *if and only if for each $i = 1, 2, \ldots, p$,*

$$x_i = R_{M_i,\lambda_i}^{A_i}\left(A_i(x_i) - \lambda_i F_i(x_1, x_2, \ldots, x_p)\right), \tag{4.1}$$

where $\lambda_i > 0$ is a constant.

Proof. It holds that $(x_1, x_2, \ldots, x_p) \in \prod_{i=1}^{p} \mathcal{H}_i$ is a solution of (3.1)

$$\begin{aligned}
&\Longleftrightarrow \theta_i \in F_i(x_1, x_2, \ldots, x_p) + M_i(x_i), \quad i = 1, 2, \ldots, p, \\
&\Longleftrightarrow A_i(x_i) - \lambda_i F_i(x_1, x_2, \ldots, x_p) \in (A_i + \lambda_i M_i)(x_i), \quad i = 1, 2, \ldots, p, \\
&\Longleftrightarrow x_i = R_{M_i,\lambda_i}^{A_i}(A_i(x_i) - \lambda_i F_i(x_1, x_2, \ldots, x_p)), \quad i = 1, 2, \ldots, p.
\end{aligned} \tag{4.2}$$

Let $\Gamma = \{1, 2, \ldots, p\}$. □

Theorem 4.2. *For $i = 1, 2, \ldots, p$, let $A_i : \mathcal{H}_i \to \mathcal{H}_i$ be γ_i-strongly monotone and let τ_i-Lipschitz continuous, $M_i : \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be an A_i-monotone operator with a constant m_i, let $F_i : \prod_{j=1}^{p} \mathcal{H}_j \to \mathcal{H}_i$ be a single-valued mapping such that F_i is (θ_i, r_i)-relaxed cocoercive monotone with respect to A_i and s_i-Lipschitz continuous in the ith argument, F_i is l_{ij}-Lipschitz continuous in the jth arguments for each $j \in \Gamma$, $j \neq i$. Suppose that there exist constants $\lambda_i > 0$ $(i = 1, 2, \ldots, p)$ such that*

$$\begin{gathered}
\frac{1}{\gamma_1 - m_1\lambda_1}\sqrt{\tau_1^2\theta_1^2 - 2\lambda_1 r_1 + 2\lambda_1\theta_1 s_1^2 + {\lambda_1}^2 s_1^2} + \sum_{k=2}^{p} \frac{l_{k1}\lambda_k}{\gamma_k - m_k\lambda_k} < 1, \\
\frac{1}{\gamma_2 - m_2\lambda_2}\sqrt{\tau_2^2\theta_2^2 - 2\lambda_2 r_2 + 2\lambda_2\theta_2 s_2^2 + {\lambda_2}^2 s_2^2} + \sum_{k\in\Gamma,\, k\neq 2} \frac{l_{k2}\lambda_k}{\gamma_k - m_k\lambda_k} < 1, \\
\ldots, \\
\frac{1}{\gamma_p - m_p\lambda_p}\sqrt{\tau_p^2\theta_p^2 - 2\lambda_p r_p + 2\lambda_p\theta_p s_p^2 + {\lambda_p}^2 s_p^2} + \sum_{k=1}^{p-1} \frac{l_{k,p}\lambda_k}{\gamma_k - m_k\lambda_k} < 1.
\end{gathered} \tag{4.3}$$

Then, (3.1) *admits a unique solution.*

Proof. For $i = 1, 2, \ldots, p$ and for any given $\lambda_i > 0$, define a single-valued mapping $T_{i,\lambda_i} : \prod_{j=1}^{p} \mathcal{H}_j \to \mathcal{H}_i$ by

$$T_{i,\lambda_i}(x_1, x_2, \ldots, x_p) = R_{M_i,\lambda_i}^{A_i}(A_i(x_i) - \lambda_i F_i(x_1, x_2, \ldots, x_p)), \tag{4.4}$$

for any $(x_1, x_2, \ldots, x_p) \in \prod_{i=1}^{p} \mathcal{H}_i$.

For any $(x_1, x_2, \ldots, x_p), (y_1, y_2, \ldots, y_p) \in \prod_{i=1}^{p} \mathscr{H}_i$, it follows from (4.4) and Lemma 2.5 that for $i = 1, 2, \ldots, p$,

$$\begin{aligned}
&\left\| T_{i,\lambda_i}(x_1, x_2, \ldots, x_p) - T_{i,\lambda_i}(y_1, y_2, \ldots, y_p) \right\|_i \\
&\quad = \left\| R^{A_i}_{M_i,\lambda_i}(A_i(x_i) - \lambda_i F_i(x_1, x_2, \ldots, x_p)) - R^{A_i}_{M_i,\lambda_i}(A_i(y_i) - \lambda_i F_i(y_1, y_2, \ldots, y_p)) \right\|_i \\
&\quad \le \frac{1}{\gamma_i - m_i\lambda_i} \left\| A_i(x_i) - A_i(y_i) - \lambda_i (F_i(x_1, x_2, \ldots, x_p) - F_i(y_1, y_2, \ldots, y_p)) \right\|_i \\
&\quad \le \frac{1}{\gamma_i - m_i\lambda_i} \| A_i(x_i) - A_i(y_i) \\
&\qquad\qquad -\lambda_i (F_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) - F_i(x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_p)) \|_i \\
&\qquad + \frac{\lambda_i}{\gamma_i - m_i\lambda_i} \left(\sum_{j\in\Gamma, j\neq i} \| F_i(x_1, x_2, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_p) \right. \\
&\qquad\qquad\qquad \left. - F_i(x_1, x_2, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_p) \|_i \right).
\end{aligned} \tag{4.5}$$

For $i = 1, 2, \ldots, p$, since A_i is τ_i-Lipschitz continuous, F_i is (θ_i, r_i)-relaxed cocoercive with respected to A_i and s_i-Lipschitz continuous in the ith argument, we have

$$\begin{aligned}
&\| A_i(x_i) - A_i(y_i) \\
&\qquad -\lambda_i (F_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) - F_i(x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_p)) \|_i^2 \\
&\quad \le \| A_i(x_i) - A_i(y_i) \|_i^2 \\
&\qquad - 2\lambda_i \langle F_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) \\
&\qquad\qquad -F_i(x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_p), A_i(x_i) - A_i(y_i) \rangle \\
&\qquad + {\lambda_i}^2 \| F_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) - F_i(x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_p) \|_i^2 \\
&\quad \le \tau_i^2 \| x_i - y_i \|_i^2 - 2\lambda_i r_i \| x_i - y_i \|_i^2 + 2\lambda_i \theta_i s_i^2 \| x_i - y_i \|_i^2 + {\lambda_i}^2 s_i^2 \| x_i - y_i \|_i^2 \\
&\quad \le \left(\tau_i^2 - 2\lambda_i r_i + 2\lambda_i \theta_i s_i^2 + {\lambda_i}^2 s_i^2 \right) \| x_i - y_i \|_i^2.
\end{aligned} \tag{4.6}$$

For $i = 1, 2, \ldots, p$, since F_i is l_{ij}-Lipschitz continuous in the jth arguments $(j \in \Gamma, j \neq i)$, we have

$$\| F_i(x_1, x_2, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_p) - F_i(x_1, x_2, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_p) \|_i \le l_{ij} \| x_j - y_j \|_j, \tag{4.7}$$

It follows from (4.5)–(4.7) that for each $i = 1, 2, \dots, p$,

$$\begin{aligned}
&\left\|T_{i,\lambda_i}(x_1, x_2, \dots, x_p) - T_{i,\lambda_i}(y_1, y_2, \dots, y_p)\right\|_i \\
&\quad \le \frac{1}{\gamma_i - m_i\lambda_i}\sqrt{\tau_i^2 - 2\lambda_i r_i + 2\lambda_i\theta_i s_i^2 + {\lambda_i}^2 s_i^2}\left\|x_i - y_i\right\|_i + \frac{\lambda_i}{\gamma_i - m_i\lambda_i}\left(\sum_{j\in\Gamma, j\neq i} l_{ij}\left\|x_j - y_j\right\|_j\right).
\end{aligned} \tag{4.8}$$

Hence,

$$\begin{aligned}
&\sum_{i=1}^{p}\left\|T_{i,\lambda_i}(x_1, x_2, \dots, x_p) - T_{i,\lambda_i}(y_1, y_2, \dots, y_p)\right\|_i \\
&\quad \le \sum_{i=1}^{p}\left[\frac{1}{\gamma_i - m_i\lambda_i}\sqrt{\tau_i^2 - 2\lambda_i r_i + 2\lambda_i\theta_i s_i^2 + {\lambda_i}^2 s_i^2}\left\|x_i - y_i\right\|_i + \frac{\lambda_i}{\gamma_i - m_i\lambda_i}\left(\sum_{j\in\Gamma, j\neq i} l_{ij}\left\|x_j - y_j\right\|_j\right)\right] \\
&\quad = \left(\frac{1}{\gamma_1 - m_1\lambda_1}\sqrt{\tau_1^2\theta_1^2 - 2\lambda_1 r_1 + 2\lambda_1\theta_1 s_1^2 + {\lambda_1}^2 s_1^2} + \sum_{k=2}^{p}\frac{l_{k1}\lambda_k}{\gamma_k - m_k\lambda_k}\right)\left\|x_1 - y_1\right\|_1 \\
&\qquad + \left(\frac{1}{\gamma_2 - m_2\lambda_2}\sqrt{\tau_2^2\theta_2^2 - 2\lambda_2 r_2 + 2\lambda_2\theta_2 s_2^2 + {\lambda_2}^2 s_2^2} + \sum_{k\in\Gamma, k\neq 2}\frac{l_{k2}\lambda_k}{\gamma_k - m_k\lambda_k}\right)\left\|x_2 - y_2\right\|_2 \\
&\qquad + \cdots + \left(\frac{1}{\gamma_p - m_p\lambda_p}\sqrt{\tau_p^2\theta_p^2 - 2\lambda_p r_p + 2\lambda_p\theta_p s_p^2 + {\lambda_p}^2 s_p^2} + \sum_{k=1}^{p-1}\frac{l_{k,p}\lambda_k}{\gamma_k - m_k\lambda_k}\right)\left\|x_p - y_p\right\|_p \\
&\quad \le \xi\left(\sum_{k=1}^{p}\left\|x_k - y_k\right\|_k\right),
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
\xi = \max\Bigg\{&\frac{1}{\gamma_1 - m_1\lambda_1}\sqrt{\tau_1^2\theta_1^2 - 2\lambda_1 r_1 + 2\lambda_1\theta_1 s_1^2 + {\lambda_1}^2 s_1^2} + \sum_{k=2}^{p}\frac{l_{k1}\lambda_k}{\gamma_k - m_k\lambda_k}, \\
&\frac{1}{\gamma_2 - m_2\lambda_2}\sqrt{\tau_2^2\theta_2^2 - 2\lambda_2 r_2 + 2\lambda_2\theta_2 s_2^2 + {\lambda_2}^2 s_2^2} + \sum_{k\in\Gamma, k\neq 2}\frac{l_{k2}\lambda_k}{\gamma_k - m_k\lambda_k}, \\
&\dots, \\
&\frac{1}{\gamma_p - m_p\lambda_p}\sqrt{\tau_p^2\theta_p^2 - 2\lambda_p r_p + 2\lambda_p\theta_p s_p^2 + {\lambda_p}^2 s_p^2} + \sum_{k=1}^{p-1}\frac{l_{k,p}\lambda_k}{\gamma_k - m_k\lambda_k}\Bigg\}.
\end{aligned} \tag{4.10}$$

Define $\|\cdot\|_\Gamma$ on $\prod_{i=1}^{p}\mathcal{H}_i$ by $\|(x_1,x_2,\ldots,x_p)\|_\Gamma = \|x_1\|_1 + \|x_2\|_2 + \cdots + \|x_p\|_p$, for all $(x_1,x_2,\ldots,x_p) \in \prod_{i=1}^{p}\mathcal{H}_i$. It is easy to see that $\prod_{i=1}^{p}\mathcal{H}_i$ is a Banach space. For any given $\lambda_i > 0$ $(i \in \Gamma)$, define $W_{\Gamma,\lambda_1,\lambda_2,\ldots,\lambda_p} : \prod_{i=1}^{p}\mathcal{H}_i \to \prod_{i=1}^{p}\mathcal{H}_i$ by

$$\begin{aligned} &W_{\Gamma,\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1,x_2,\ldots,x_p) \\ &\quad = \Big(T_{1,\lambda_1}(x_1,x_2,\ldots,x_p), T_{2,\lambda_2}(x_1,x_2,\ldots,x_p),\ldots,T_{p,\lambda_p}(x_1,x_2,\ldots,x_p)\Big), \end{aligned} \tag{4.11}$$

for all $(x_1,x_2,\ldots,x_p) \in \prod_{i=1}^{p}\mathcal{H}_i$.

By (4.3), we know that $0 < \xi < 1$, it follows from (4.9) that

$$\begin{aligned} &\left\|W_{\Gamma,\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1,x_2,\ldots,x_p) - W_{\Gamma,\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1,x_2,\ldots,x_p)\right\|_\Gamma \\ &\quad \le \xi\left\|(x_1,x_2,\ldots,x_p) - (y_1,y_2,\ldots,y_p)\right\|_\Gamma. \end{aligned} \tag{4.12}$$

This shows that $W_{\Gamma,\lambda_1,\lambda_2,\ldots,\lambda_p}$ is a contraction operator. Hence, there exists a unique $(x_1,x_2,\ldots,x_p) \in \prod_{i=1}^{p}\mathcal{H}_i$, such that

$$W_{\Gamma,\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1,x_2,\ldots,x_p) = (x_1,x_2,\ldots,x_p), \tag{4.13}$$

that is, for $i = 1,2,\ldots,p$,

$$x_i = R_{M_i,\lambda_i}^{A_i}\left(A_i(x_i) - \lambda_i F_i(x_1,x_2,\ldots,x_p)\right). \tag{4.14}$$

By Lemma 4.1, $(x_1,x_2,\ldots,x_p)$ is the unique solution of (3.1). This completes this proof. □

Corollary 4.3. *For $i = 1,2,\ldots,p$, let $H_i : \mathcal{H}_i \to \mathcal{H}_i$ be γ_i-strongly monotone and τ_i-Lipschitz continuous, let $M_i : \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be an H_i-monotone operator, let $F_i : \prod_{j=1}^{p}\mathcal{H}_j \to \mathcal{H}_i$ be a single-valued mapping such that F_i is r_i-strongly monotone with respect to H_i and s_i-Lipschitz continuous in the ith argument, F_i is l_{ij}-Lipschitz continuous in the jth arguments for each $j \in \Gamma$, $j \ne i$. Suppose that there exist constants $\lambda_i > 0$ $(i = 1,2,\ldots,p)$ such that*

$$\begin{gathered} \frac{1}{\gamma_1}\sqrt{\tau_1^2 - 2\lambda_1 r_1 + {\lambda_1}^2 s_1^2} + \sum_{k=2}^{p}\frac{l_{k1}\lambda_k}{\gamma_k} < 1, \\ \frac{1}{\gamma_2}\sqrt{\tau_2^2 - 2\lambda_2 r_2 + {\lambda_2}^2 s_2^2} + \sum_{k\in\Gamma,k\ne 2}\frac{l_{k2}\lambda_k}{\gamma_k} < 1, \\ \vdots \\ \frac{1}{\gamma_p}\sqrt{\tau_p^2 - 2\lambda_p r_p + {\lambda_p}^2 s_p^2} + \sum_{k=1}^{p-1}\frac{l_{k,p}\lambda_k}{\gamma_k} < 1. \end{gathered} \tag{4.15}$$

Then, problem (3.1) *admits a unique solution.*

Remark 4.4. Theorem 4.2 and Corollary 4.3 unify, extend, and generalize the main results in [1, 2, 6–8].

5. Iterative Algorithm and Convergence

In this section, we will construct some multistep iterative algorithm for approximating the unique solution of (3.1) and discuss the convergence analysis of these Algorithms.

Lemma 5.1 (see [8, 9]). *Let $\{c_n\}$ and $\{k_n\}$ be two real sequences of nonnegative numbers that satisfy the following conditions:*

(1) $0 \le k_n < 1$, $n = 0, 1, 2, \ldots$ *and* $\limsup_n k_n < 1$,

(2) $c_{n+1} \le k_n c_n$, $n = 0, 1, 2, \ldots$,

then c_n converges to 0 as $n \to \infty$.

Algorithm 5.2. For $i = 1, 2, \ldots, p$, let A_i, M_i, F_i be the same as in Theorem 4.2. For any given $(x_1^0, x_2^0, \ldots, x_p^0) \in \prod_{j=1}^{p} \mathscr{H}_j$, define a multistep iterative sequence $\{(x_1^n, x_2^n, \ldots, x_p^n)\}$ by

$$x_i^{n+1} = \alpha_n x_i^n + (1 - \alpha_n)\left[R_{M_i,\lambda_i}^{A_i}\left(A_i(x_i^n) - \lambda_i F_i\left(x_1^n, x_2^n, \ldots, x_p^n\right)\right)\right], \tag{5.1}$$

where

$$0 \le \alpha_n < 1, \quad \limsup_n \alpha_n < 1. \tag{5.2}$$

Theorem 5.3. *For $i = 1, 2, \ldots, p$, let A_i, M_i, F_i be the same as in Theorem 4.2. Assume that all the conditions of theorem 4.1 hold. Then $\{(x_1^n, x_2^n, \ldots, x_p^n)\}$ generated by Algorithm 5.2 converges strongly to the unique solution $(x_1, x_2, \ldots, x_p)$ of* (3.1).

Proof. By Theorem 4.2, problem (3.1) admits a unique solution $(x_1, x_2, \ldots, x_p)$, it follows from Lemma 4.1 that for each $i = 1, 2, \ldots, p$,

$$x_i = R_{M_i,\lambda_i}^{A_i}(A_i(x_i) - \lambda_i F_i(x_1, x_2, \ldots, x_p)). \tag{5.3}$$

It follows from (4.3), (5.1) and (5.3) that for each $i = 1, 2, \ldots, p$,

$$\begin{aligned}\left\|x_i^{n+1} - x_i\right\|_i = \Big\|\alpha_n(x_i^n - x_i) + (1 - \alpha_n)\Big[&R_{M_i,\lambda_i}^{A_i}\left(A_i(x_i^n) - \lambda_i F_i\left(x_1^n, x_2^n, \ldots, x_p^n\right)\right) \\ &- R_{M_i,\lambda_i}^{A_i}(A_i(x_i) - \lambda_i F_i(x_1, x_2, \ldots, x_p))\Big]\Big\|_i\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n \left\| x_i^n - x_i \right\|_i + (1-\alpha_n) \frac{1}{\gamma_i - m_i\lambda_i} \\
&\quad \times \Big\| A_i(x_i^n) - A_i(x_i) - \lambda_i \Big(F_i\Big(x_1^n, x_2^n, \dots, x_{i-1}^n, x_i^n, x_{i+1}^n, \dots, x_p^n\Big) \\
&\qquad\qquad -F_i\Big(x_1^n, x_2^n, \dots, x_{i-1}^n, x_i, x_{i+1}^n, \dots, x_p^n\Big)\Big)\Big\|_i \\
&\quad + \frac{\lambda_i}{\gamma_i - m_i\lambda_i} \left(\sum_{j\in\Gamma, j\neq i} \Big\| F_i\Big(x_1^n, x_2^n, \dots, x_{j-1}^n, x_j^n, x_{j+1}^n, \dots, x_p^n\Big) \right. \\
&\qquad\qquad \left. -F_i\Big(x_1^n, x_2^n, \dots, x_{j-1}^n, x_j, x_{j+1}^n, \dots, x_p^n\Big)\Big\|_i \right).
\end{aligned}
\tag{5.4}
$$

For $i = 1, 2, \dots, p$, since A_i is τ_i-Lipschitz continuous, F_i is (θ_i, r_i)-relaxed cocoercive with respected to A_i, and s_i-Lipschitz is continuous in the ith argument, we have

$$
\begin{aligned}
&\Big\| A_i(x_i^n) - A_i(x_i) \\
&\quad -\lambda_i\Big(F_i\Big(x_1^n, x_2^n, \dots, x_{i-1}^n, x_i^n, x_{i+1}^n, \dots, x_p^n\Big) - F_i\Big(x_1^n, x_2^n, \dots, x_{i-1}^n, x_i, x_{i+1}^n, \dots, x_p^n\Big)\Big)\Big\|_i^2 \\
&\quad \le \Big(\tau_i^2 - 2\lambda_i r_i + 2\lambda_i\theta_i s_i^2 + {\lambda_i}^2 s_i^2\Big) \left\| x_i^n - x_i \right\|^2.
\end{aligned}
\tag{5.5}
$$

For $i = 1, 2, \dots, p$, since F_i is l_{ij}-Lipschitz continuous in the jth arguments $(j \in \Gamma, j \neq i)$, we have

$$
\Big\| F_i\Big(x_1^n, x_2^n, \dots, x_{j-1}^n, x_j^n, x_{j+1}^n, \dots, x_p^n\Big) - F_i\Big(x_1^n, x_2^n, \dots, x_{j-1}^n, x_j, x_{j+1}^n, \dots, x_p^n\Big)\Big\|_i \le l_{ij} \Big\| x_j^n - x_j \Big\|_j.
\tag{5.6}
$$

It follows from (5.4)–(5.6) that for $i = 1, 2, \dots, p$,

$$
\begin{aligned}
\Big\| x_i^{n+1} - x_i \Big\|_i &\le \alpha_n \left\| x_i^n - x_i \right\|_i + (1-\alpha_n) \frac{1}{\gamma_i - m_i\lambda_i} \sqrt{\tau_i^2 - 2\lambda_i r_i + 2\lambda_i\theta_i s_i^2 + {\lambda_i}^2 s_i^2} \left\| x_i^n - x_i \right\|_i \\
&\quad + (1-\alpha_n) \frac{\lambda_i}{\gamma_i - m_i\lambda_i} \left(\sum_{j\in\Gamma, j\neq i} l_{ij} \Big\| x_j^n - x_j \Big\|_j \right).
\end{aligned}
\tag{5.7}
$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{p}\left\|x_i^{n+1}-x_i\right\|_i &\le \sum_{i=1}^{p}\Bigg[\alpha_n\left\|x_i^n-x_i\right\|_i+(1-\alpha_n)\frac{1}{\gamma_i-m_i\lambda_i}\sqrt{\tau_i^2-2\lambda_i r_i+2\lambda_i\theta_i s_i^2+{\lambda_i}^2 s_i^2}\left\|x_i^n-x_i\right\|_i \\
&\qquad +(1-\alpha_n)\frac{\lambda_i}{\gamma_i-m_i\lambda_i}\Bigg(\sum_{j\in\Gamma, j\neq i} l_{ij}\left\|x_j^n-x_j\right\|_j\Bigg)\Bigg] \\
&\le \alpha_n\left(\sum_{i=1}^{p}\left\|x_i^n-x_i\right\|_i\right)+(1-\alpha_n)\xi\left(\sum_{i=1}^{p}\left\|x_i^n-x_i\right\|_i\right) \\
&= (\xi+(1-\xi)\alpha_n)\left(\sum_{i=1}^{p}\left\|x_i^n-x_i\right\|_i\right),
\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}
\xi = \max\Bigg\{ &\frac{1}{\gamma_1-m_1\lambda_1}\sqrt{\tau_1^2\theta_1^2-2\lambda_1 r_1+2\lambda_1\theta_1 s_1^2+{\lambda_1}^2 s_1^2}+\sum_{k=2}^{p}\frac{l_{k1}\lambda_k}{\gamma_k-m_k\lambda_k}, \\
&\frac{1}{\gamma_2-m_2\lambda_2}\sqrt{\tau_2^2\theta_2^2-2\lambda_2 r_2+2\lambda_2\theta_2 s_2^2+{\lambda_2}^2 s_2^2}+\sum_{k\in\Gamma, k\neq 2}\frac{l_{k2}\lambda_k}{\gamma_k-m_k\lambda_k}, \\
&\dots, \\
&\frac{1}{\gamma_p-m_p\lambda_p}\sqrt{\tau_p^2\theta_p^2-2\lambda_p r_p+2\lambda_p\theta_p s_p^2+{\lambda_p}^2 s_p^2}+\sum_{k=1}^{p-1}\frac{l_{k,p}\lambda_k}{\gamma_k-m_k\lambda_k}\Bigg\}.
\end{aligned} \tag{5.9}$$

It follows from hypothesis (4.3) that $0<\xi<1$.

Let $a_n=\sum_{i=1}^{p}\|x_i^n-x_i\|_i$, $\xi_n=\xi+(1-\xi)\alpha_n$. Then, (5.8) can be rewritten as $a_{n+1}\le\xi_n a_n$, $n=0,1,2,\dots$. By (5.2), we know that $\limsup_n \xi_n<1$, it follows from Lemma 5.1 that

$$a_n=\sum_{i=1}^{p}\left\|x_i^n-x_i\right\|_i \text{ converges to } 0 \text{ as } \quad n\longrightarrow\infty. \tag{5.10}$$

Therefore, $\{(x_1^n, x_2^n, \dots, x_p^n)\}$ converges to the unique solution $(x_1, x_2, \dots, x_p)$ of (3.1). This completes the proof. □

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