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Research Article

A-Harmonic Equations and the Dirac Operator

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We show how A-harmonic equations arise as components of Dirac systems. We generalize A-harmonic equations to A-Dirac equations. Removability theorems are proved for solutions to A-Dirac equations.

1. Introduction

This paper explains how A-harmonic equations arise from Dirac systems. Indeed the main purpose of this paper is to elucidate the connection between the theories of A-harmonic functions and Dirac analysis. An A-harmonic equation $\operatorname{div} A(x, \nabla u) = 0$ is a component of a Dirac system

$$D\widetilde{A}(x, Du) = 0. \tag{1.1}$$

This component is the scalar (real) part of the Dirac system, under appropriate identifications. Hence any real-valued solution to the Dirac system is an A-harmonic function. As such, the class of A-harmonic functions which are also solutions of the Dirac system are a special class of A-harmonic functions. See Section 3 for a detailed discussion. As an application, we show that a result concerning removable singularities for A-harmonic functions satisfying a Lipschitz condition or of bounded mean oscillation extends to Clifford valued solutions to corresponding Dirac equations. The result also holds for functions of a certain order of growth. This seems to be new even in the case of an A-harmonic function.

In Section 2, we present preliminaries about Clifford algebra along with definitions and notations. In Section 3, we introduce A-Dirac equations and show the correspondence with A-harmonic equations. The Caccioppoli estimate for solutions to A-Dirac equations appears in Section 4 and the removability theorems are in Section 5 along with references. For other recent work on nonlinear Dirac equations, see [1–6].

2. Preliminaries

We write $\mathcal{U}_n$ for the real universal Clifford algebra over $\mathbb{R}^n$. The Clifford algebra is generated over $\mathbb{R}$ by the basis of reduced products

$$\{e_1, e_2, \dots, e_n, e_1e_2, \dots, e_1 \cdots e_n\}, \tag{2.1}$$

where $\{e_1, e_2, \dots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ with the relation $e_ie_j + e_je_i = -2\delta_{ij}$. We write e_0 for the identity. The dimension of $\mathcal{U}_n$ is $\mathbb{R}^{2^n}$. We have an increasing tower $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathcal{U}_3 \subset \cdots$. The Clifford algebra $\mathcal{U}_n$ is a graded algebra as $\mathcal{U}_n = \bigoplus_l \mathcal{U}_n^l$, where $\mathcal{U}_n^l$ are those elements whose reduced Clifford products have length l.

For $A \in \mathcal{U}_n$, $\mathrm{Sc}(A)$ denotes the scalar part of A, that is, the coefficient of the element e_0.

Throughout, $\Omega \subset \mathbb{R}^n$ is a connected and open set with boundary $\partial\Omega$. A Clifford-valued function $u : \Omega \to \mathcal{U}_n$ can be written as $u = \Sigma_\alpha u_\alpha e_\alpha$ where each u_α is real-valued and e_α are reduced products. The norm used here is given by $|\Sigma_\alpha u_\alpha e_\alpha| = (\Sigma_\alpha u_\alpha^2)^{1/2}$. This norm is submultiplicative, $|AB| \le C|A||B|$.

The Dirac operator used here is as follows:

$$D = \Sigma_{j=1}^n e_j \frac{\partial}{\partial x_j}. \tag{2.2}$$

Also $D^2 = -\Delta$. Here Δ is the Laplace operator which operates only on coefficients. A function is monogenic when $Du = 0$.

Throughout, Q is a cube in Ω with volume $|Q|$. We write σQ for the cube with the same center as Q and with sidelength σ times that of Q. For $q > 0$, we write $L^q(\Omega, \mathcal{U}_n)$ for the space of Clifford-valued functions in Ω whose coefficients belong to the usual $L^q(\Omega)$ space. Also, $W^{1,q}(\Omega, \mathcal{U}_n)$ is the space of Clifford valued functions in Ω whose coefficients as well as their first distributional derivatives are in $L^q(\Omega)$. We also write $L^q_{\mathrm{loc}}(\Omega, \mathcal{U}_n)$ for $\cap L^q(\Omega', \mathcal{U}_n)$, where the intersection is over all Ω' compactly contained in Ω. We similarly write $W^{1,q}_{\mathrm{loc}}(\Omega, \mathcal{U}_n)$. Moreover, we write $\mathcal{M}_\Omega = \{u : \Omega \to \mathcal{U}_n \mid Du = 0\}$ for the space of monogenic functions in Ω.

Furthermore, we define the Dirac Sobolev space

$$W^{D,p}(\Omega) = \left\{ u \in \mathcal{U}_n \mid \int_\Omega |u|^p + \int_\Omega |Du|^p < \infty \right\}. \tag{2.3}$$

The local space $W^{D,p}_{\mathrm{loc}}$ is similarly defined. Notice that if u is monogenic, then $u \in L^p(\Omega)$ if and only if $u \in W^{D,p}(\Omega)$. Also it is immediate that $W^{1,p}(\Omega) \subset W^{D,p}(\Omega)$.

3. Correspondence and the A-Dirac Equation

We first define the A-harmonic equation. We define operators

$$A(x, \xi) : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^n. \tag{3.1}$$

Here $x \to A(x, \xi)$ is measurable for all ξ, and $\xi \to A(x, \xi)$ is continuous for a.e. $x \in \Omega$. We assume the structure conditions with $p > 1$:

$$\langle A(x,\xi),\xi\rangle \geq |\xi|^p, \qquad |A(x,\xi)| \leq a|\xi|^{p-1}, \tag{3.2}$$

for some $a > 0$. The exponent p will represent this exponent throughout.

An A-harmonic function, $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, is a weak solution to div $A(x, \nabla u) = 0$, when

$$\int_\Omega \langle A(x,\nabla u), \nabla\phi\rangle = 0 \tag{3.3}$$

for all $\phi \in W^{1,p}(\Omega)$ with compact support.

See [7] for the theory of A-harmonic equations.

To connect these equations with Dirac systems, we define the linear isomorphism $\theta : \mathbb{R}^n \to \mathcal{U}_n^1$ by

$$\theta(w_1, \ldots, w_n) = \Sigma_{i=1}^n w_i e_i. \tag{3.4}$$

It follows that for a real-valued function ϕ, we have $\theta(\nabla\phi) = D\phi$, and for $x, y \in \mathbb{R}^n$, we have

$$-\mathrm{Sc}\big(\theta(x)\theta(y)\big) = \langle x, y\rangle. \tag{3.5}$$

$$|\theta(x)| = |x|. \tag{3.6}$$

Next we define $\widetilde{A}(x,\xi) : \Omega \times \mathcal{U}_1 \to \mathcal{U}_1$ by

$$\widetilde{A}(x,\xi) = \theta A\left(x, \theta^{-1}\xi\right). \tag{3.7}$$

In this way, we see that (3.3) is equivalent to

$$\int_\Omega \mathrm{Sc}(\theta A(x,\nabla u)\theta(\nabla u)) = \int_\Omega \mathrm{Sc}\left(\widetilde{A}(x,Du)D\phi\right) = 0. \tag{3.8}$$

This motivates the following definition for Dirac systems of higher-order Clifford valued functions. We use the Clifford conjugation $\overline{(e_{j1}\cdots e_{jl})} = (-1)^l e_{jl}\cdots e_{j1}$. The product $\overline{\alpha}\beta$ defines a Clifford-valued inner product.

Moreover, the scalar part of this Clifford inner product $\mathrm{Sc}(\overline{\alpha}\beta)$ is the usual inner product in $\mathbb{R}^{2^n}$, $\langle\alpha, \beta\rangle$, when α and β are identified as vectors. We continue to use the Clifford notation for this scalar product. This conjugation compensates for the minus sign in (3.5) and in higher-order Clifford products. Moreover, the conjugation can be incorporated into an integration by parts formula.

To this purpose (we are replacing $\widetilde{A}$ with A for convenience) we recast the structure equations above and define operators:

$$A(x,\xi) : \Omega \times \mathcal{U}_n \longrightarrow \mathcal{U}_n. \tag{3.9}$$

We assume that A preserves the grading of the Clifford algebra. Here $x \to A(x,\xi)$ is measurable for all ξ, and $\xi \to A(x,\xi)$ is continuous for a.e. $x \in \Omega$. We assume the structure conditions with $p > 1$:

$$\mathrm{Sc}\left(\overline{A(x,\xi)}\xi\right) \geq |\xi|^p, \tag{3.10}$$

$$|A(x,\xi)| \leq a|\xi|^{p-1}, \tag{3.11}$$

for some $a > 0$.

Definition 3.1. A Clifford valued function $u \in W^{D,p}_{\mathrm{loc}}(\Omega, \mathcal{U}_n^k)$, for $k = 0, 1, \ldots, n$, is a weak solution to

$$DA(x, Du) = 0 \tag{3.12}$$

if for all $\phi \in W^{1,p}(\Omega, \mathcal{U}_n^k)$ with compact support we have

$$\int_\Omega \overline{A(x,Du)} D\phi = 0. \tag{3.13}$$

Notice that when A is the identity, then (3.13) is the Clifford Laplacian. Moreover these equations generalize the important case of the p-Dirac equation:

$$D\left(|Du|^{p-2} Du\right) = 0. \tag{3.14}$$

Here $A(x,\xi) = |\xi|^{p-2}\xi$.

These equations were introduced and their conformal invariance was studied in [8].

In the case of the p-Dirac equation, $|x|^{(p-n)/(p-1)}$, when $p \neq n$, and $\log|x|$, $p \neq n$, are solutions to (3.13). Notice that a monogenic function $u_\Omega \in \mathcal{M}_\Omega$ is trivially a solution to (3.13) and if u is a solution to (3.13), then so is $u + u_\Omega$ for any monogenic function u_Ω.

In the case that u is a real-valued function $DA(x, Du) = 0$ also implies that

$$\Sigma_{i<j} \int_\Omega \left(A_i(x, Du) \frac{\partial \phi}{\partial x_j} - A_j(x, Du) \frac{\partial \phi}{\partial x_i} \right) e_{ij} = 0, \tag{3.15}$$

where $A = (A_1, \ldots, A_n)$. So in this case (3.13) can be identified with the two equations

$$\begin{aligned} \operatorname{div} A(x, \nabla u) &= 0, \\ \operatorname{curl} A(x, \nabla u) &= 0. \end{aligned} \tag{3.16}$$

Hence when u is a function, (3.13) implies that $A(x, \nabla u)$ is a harmonic field and locally there exists a harmonic function H such that $A(x, \nabla u) = \nabla H$. If $A(x, \xi)$ is invertible, then $\nabla u = A^{-1}(x, \nabla H)$. Hence regularity of A implies regularity of the solution u. Notice that if $A(x, \xi) = |\xi|^{p-2}\xi$, then $A^{-1}(x, \xi) = |\xi|^{q-2}\xi$, where $1/p + 1/q = 1$.

In general, A-harmonic functions do not have such regularity. This suggests the study of the scalar part of the system equation (3.13) in general. Indeed a Caccioppoli estimate holds for solutions to the scalar part of (3.13). This is the topic of the next section.

Hence we see a special class of A-harmonic functions, namely, real-valued solutions to the system (3.13). This class should have special properties.

On the other hand, results about A-harmonic functions suggest possible properties of general solutions to (3.13). In this paper, we present one such extension, that of removable sets. The essential ingredient in the proof is the Caccioppoli estimate.

4. A Caccioppoli Estimate

Next is a Caccioppoli estimate for solutions to (3.13). This result appears in [9]. We give the short proof here for completeness.

Theorem 4.1. *Let u be a solution to scalar part of* (3.13) *and $\eta \in C_0^\infty(\Omega)$, $\eta > 0$. Then*

$$\left(\int_\Omega |Du|^p \eta^p\right)^{1/p} \le pa\left(\int_\Omega |u|^p |\nabla \eta|^p\right)^{1/p}. \tag{4.1}$$

Proof. Choose $\phi = -u\eta^p$. Then $D\phi = -p\eta^{p-1}(D\eta)u - \eta^p Du$. Hence using (3.13) and (3.10),

$$\begin{aligned} 0 &= \int_\Omega \mathrm{Sc}\left(\overline{A(x, Du)} D\phi\right) = \int_\Omega \mathrm{Sc}\left(\overline{A(x, Du)}\left(-p\eta^{p-1}(D\eta)u - \eta^p Du\right)\right) \\ &\le -\int_\Omega |Du|^p \eta^p + p\int_\Omega \left|\overline{A(x, Du)}\right| |u| \left|D\eta\right| \left|\eta\right|^{p-1}. \end{aligned} \tag{4.2}$$

Using Hölder's inequality and (3.11), we have

$$\int_\Omega |Du|^p \eta^p \le pa \int_\Omega |u| |\nabla \eta| |Du|^{p-1} |\eta|^{p-1} \le C\left(\int_\Omega |u|^p |\nabla \eta|^p\right)^{1/p} \left(\int_\Omega |Du|^p \eta^p\right)^{(p-1)/p}. \tag{4.3}$$

□

Corollary 4.2. *Suppose that u is a solution to* (3.13)*. Let Q be a cube with $\sigma Q \subset \Omega$ where $\sigma > 1$. Then there is a constant C, independent of u, such that*

$$\left(\int_Q |Du|^p\right)^{1/p} \le paC|Q|^{-1/n}\left(\int_{\sigma Q} |u|^p\right)^{1/p}. \tag{4.4}$$

Proof. Choose $\eta \in C_0^\infty(\sigma Q)$, $\eta > 0$, $\eta = 1$ in Q and $|\nabla \eta| \le C|Q|^{-1/n}$. □

5. Removability

For monogenic functions with modulus of continuity $\omega(r)$, sets of $r^n\omega(r)$-Hausdorff measure are removable [1]. For Hölder continuous analytic functions, see [10]. Sets satisfying a certain geometric condition related to Minkowski dimension are shown to be removable for A-harmonic functions in Hölder and bounded mean oscillation classes in [11]. In the case of Hölder continuity, this was sharped in [12] to a precise condition for removable sets for A-harmonic functions in terms of Hausdorff dimension.

We show here that sets satisfying a generalized Minkowski-type inequality, similar to that in [11], are removable for solutions to the A-Dirac equation which satisfy a certain oscillation condition. The following definition is motivated by the fact that real-valued functions satisfying various regularity properties are characterized by this definition. We explain below.

Definition 5.1. Assume that $u \in L^1_{\text{loc}}(\Omega, \mathcal{U}_n)$, $q > 0$ and that $-\infty < k \leq 1$.

We say that u is of q, k-oscillation in Ω when

$$\sup_{2Q\subset\Omega} |Q|^{-(qk+n)/qn} \inf_{u_Q\in\mathcal{M}_Q} \left(\int_Q \left|u - u_Q\right|^q\right)^{1/q} < \infty. \tag{5.1}$$

The infimum over monogenic functions is natural since they are trivially solutions to an A-Dirac equation just as constants are solutions to an A-harmonic equation. If u is a function and $q = 1$, then (5.1) is equivalent to the usual definition of the bounded mean oscillation when $k = 0$ and (5.1) is equivalent to the usual local Lipschitz condition when $0 < k \leq 1$ [13]. Moreover, at least when u is a solution to an A-harmonic equation, (5.1) is equivalent to a local order of growth condition when $-\infty < k < 0$; see [9, 14]. In these cases, the supremum is finite if we choose u_Q to be the average value of the function u over the cube Q. It is easy to see that in condition (5.1) the expansion factor "2" can be replaced by any factor greater than 1.

If the coefficients of an A-Dirac solution u are of bounded mean oscillation, local Hölder continuous, or of a certain local order of growth, then u is in an appropriate oscillation class; see [9].

Notice that monogenic functions satisfy (5.1) just as the space of constants is a subspace of the bounded mean oscillation and Lipschitz spaces of real-valued functions.

We remark that it follows from Hölder's inequality that if $s \leq q$ and if u is of q, k-oscillation, then u is of s, k-oscillation.

The following lemma shows that Definition 5.1 is independent of the expansion factor of the cube.

Lemma 5.2. *Suppose that $F \in L^1_{\text{loc}}(\Omega, \mathbb{R})$, $F > 0$ a.e., $\gamma \in \mathbb{R}$, and $\sigma_1, \sigma_2 > 1$. If*

$$\sup_{\sigma_1 Q\subset\Omega} |Q|^{\gamma} \int_Q F < \infty, \tag{5.2}$$

then

$$\sup_{\sigma_2 Q\subset\Omega} |Q|^{\gamma} \int_Q F < \infty. \tag{5.3}$$

Proof. If $\sigma_1 \leq \sigma_2$, then the implication is immediate. Assume $\sigma_1 > \sigma_2$. Let Q be a cube with $\sigma_2 Q \subset \Omega$. Dyadically subdivide Q into a finite number of subcubes $\{Q_i\}$ with $l(Q_i) \leq ((\sigma_2 - 1)/\sigma_1) l(Q)$. Then $\sigma_1 Q_i \subset \Omega$ for all i. Moreover,

$$|Q|^{\gamma} \int_{Q} F \leq |Q|^{\gamma} \Sigma_i \int_{Q_i} F = C(\sigma_1, \sigma_2, \gamma, n) \Sigma_i |Q_i|^{\gamma} \int_{Q_i} F. \tag{5.4}$$

We use a Whitney decomposition $\mathcal{W} = \{Q\}$ of Ω. The decomposition consists of closed dyadic cubes with disjoint interiors which satisfy

(a) $\Omega = \bigcup_{Q \in \mathcal{W}} Q$,

(b) $|Q|^{1/n} \leq d(Q, \partial\Omega) \leq 4|Q|^{1/n}$,

(c) $(1/4)|Q_1|^{1/n} \leq |Q_2|^{1/n} \leq 4|Q_1|^{1/n}$ when $Q_1 \cap Q_2$ is not empty.

Here $d(Q, \partial\Omega)$ is the Euclidean distance between Q and the boundary of Ω; see [15].

We use the following definition. If $A \subset \mathbb{R}^n$ and $r > 0$, then we define the r-inflation of A as

$$A(r) = \bigcup_{x \in A} B(x, r). \tag{5.5}$$

We now state the removability result. □

Theorem 5.3. *Let E be a relatively closed subset of Ω. Suppose that $u \in L^p_{\mathrm{loc}}(\Omega)$ has distributional first derivatives in Ω, u is a solution to the scalar part of the A-Dirac equation* (3.13) *in $\Omega \setminus E$, and u is of p, k-oscillation in $\Omega \setminus E$. If for each compact subset K of E*

$$\int_{K(1)\setminus K} d(x, K)^{p(k-1)-k} < \infty, \tag{5.6}$$

then u extends to a solution of the A-Dirac equation in Ω.

Proof. Let Q be a cube in the Whitney decomposition of $\Omega \setminus E$. Using the Caccioppoli estimate and the p, k-oscillation condition, we have

$$\int_{Q} |Du|^p \leq C \inf_{u_Q \in \mathcal{M}_{\sigma Q}} |Q|^{-p/n} \int_{\sigma Q} \left| u - u_{\sigma Q} \right|^p \leq C|Q|^a. \tag{5.7}$$

Here $a = (n + pk - p)/n$. Since the problem is local (use a partition of unity), we show that (3.13) holds whenever $\phi \in W_0^{1,p}(B(x_0, r))$ with $x_0 \in E$ and $r > 0$ sufficiently small. Choose $r = (1/5\sqrt{n}) \min\{1, d(x_0, \partial\Omega)\}$ and let $K = E \cap \overline{B}(x_0, 4r)$. Then K is a compact subset of E. Also let W_0 be those cubes in the Whitney decomposition of $\Omega \setminus E$ which meet $B = B(x_0, r)$.

Notice that each cube $Q \in W_0$ lies in $K(1) \setminus K$. Let $\gamma = p(k-1) - k$. First, since $\gamma \geq -1$, it follows that $m(K) = m(E) = 0$; see [11]. Also since $a - n \geq \gamma$ using (5.6) and (5.7), we obtain

$$\begin{aligned}\int_{B(x_0,r)} |Du|^p &\leq C\Sigma_{Q\in W_0} |Q|^{a/n} \leq C\Sigma_{Q\in W_0} d(Q,K)^a \\ &\leq C\Sigma_{Q\in W_0} \int_Q d(x,K)^{a-n} \leq C\int_{K(1)\setminus K} d(x,K)^{a-n} \\ &\leq C\int_{K(1)\setminus K} d(x,K)^{\gamma} < \infty.\end{aligned} \tag{5.8}$$

Hence $u \in W^{D,p}_{\mathrm{loc}}(\Omega)$.

Next let $B = B(x_0, r)$ and assume that $\psi \in C_0^\infty(B)$. Also let W_j, $j = 1, 2, \ldots$, be those cubes $Q \in W_0$ with $l(Q) \leq 2^{-j}$.

Consider the scalar functions

$$\phi_j = \max\left\{\left(2^{-j} - d(x,K)\right)2^j, 0\right\}. \tag{5.9}$$

Thus each ϕ_j, $j = 1, 2, \ldots$, is Lipschitz, equal to 1 on K and as such $\psi(1-\phi_j) \in W^{1,p}(B \setminus E)$ with compact support. Hence

$$\int_B \overline{A(x,Du)} D\psi = \int_{B\setminus E} \overline{A(x,Du)} D\left(\psi\left(1-\phi_j\right)\right) + \int_B \overline{A(x,Du)} D\left(\psi\phi_j\right) = I' + I''. \tag{5.10}$$

Since u is a solution in $B \setminus E$, $I' = 0$.

Also we have

$$I'' = \int_B A(x,Du)\psi D\phi_j + \int_B \phi_j A(x,Du) D\psi = I_1 + I_2. \tag{5.11}$$

Now there exists a constant c such that $|\psi| \leq c < \infty$. Hence using Hölder's inequality,

$$\begin{aligned}|I_1| &\leq C\Sigma_{Q\in W_j} \int_Q |A(x,Du)| \left|D\phi_j\right| \leq C\Sigma_{Q\in W_j} \int_Q |Du|^{p-1} \left|D\phi_j\right| \\ &\leq C\Sigma_{Q\in W_j} \left(\int_Q |Du|^p\right)^{(p-1)/p} \left(\int_Q \left|D\phi_j\right|^p\right)^{1/p}.\end{aligned} \tag{5.12}$$

Next using (4.4), the above is

$$\leq C\Sigma_{Q\in W_j} |Q|^{(p(k-1)+n)(p-1)/np} 2^j |Q|^{1/p}. \tag{5.13}$$

Now for $x \in Q \in W_j, d(x, K)$ is bounded above and below by a multiple of $|Q|^{1/n}$ and for $Q \in W_j$, $|Q|^{1/n} \le 2^{-j}$. Hence

$$|I_1| \le C\Sigma_{Q\in W_j}|Q|^{-1/n+1/p+(p(k-1)+n)(p-1)/np} \le C\int_{\cup W_j} d(x,K)^{p(k-1)-k}. \tag{5.14}$$

Since $\cup W_j \subset K(1) \setminus K$ and $|\cup W_j| \to 0$ as $j \to \infty$, it follows that $I_1 \to 0$ as $j \to \infty$.

Next again using Hölder's inequality,

$$\begin{aligned} |I_2| &\le C\sup_B |D\psi| \left(\int_{\cup W_j} |Du|^p\right)^{(p-1)/p} \left|\cup W_j\right|^{1/p} \\ &\le C\left(\int_{K\setminus K(1)} |Du|^p\right)^{(p-1)/p} \left|\cup W_j\right|^{1/p}. \end{aligned} \tag{5.15}$$

Since $u \in W^{1,D}_{\mathrm{loc}}(\Omega)$ and $|\cup W_j| \to 0$ as $j \to \infty$, we have that $I_2 \to 0$ as $j \to \infty$. Hence $I'' \to 0$. □

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