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Research Article

The Best Lower Bound Depended on Two Fixed Variables for Jensen's Inequality with Ordered Variables

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We give the best lower bound for the weighted Jensen's discrete inequality with ordered variables applied to a convex function f, in the case when the lower bound depends on f, weights, and two given variables. Furthermore, under the same conditions, we give some sharp lower bounds for the weighted AM-GM inequality and AM-HM inequality.

1. Introduction

Let $\tilde{x} = \{x_1, x_2, \ldots, x_n\}$ be a sequence of real numbers belonging to an interval I, and let $\tilde{p} = \{p_1, p_2, \ldots, p_n\}$ be a sequence of given positive weights associated to $\tilde{x}$ and satisfying $p_1 + p_2 + \cdots + p_n = 1$. If f is a convex function on I, then the well-known discrete Jensen's inequality [1] states that

$$\Delta(f, \tilde{p}, \tilde{x}) \geq 0, \tag{1.1}$$

where

$$\Delta(f, \tilde{p}, \tilde{x}) = p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) - f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n) \tag{1.2}$$

is the so-called Jensen's difference. The next refinement of Jensen's inequality was proven in [2], as a consequence of its Theorem 2.1, part (ii)

$$\Delta(f, \tilde{p}, \tilde{x}) \geq \max_{1 \leq i < k \leq n} \left[p_i f(x_i) + p_k f(x_k) - (p_i + p_k) f\left(\frac{p_i x_i + p_k x_k}{p_i + p_k} \right) \right] \geq 0. \tag{1.3}$$

By (1.3), for fixed x_i and x_k, we get

$$\Delta(f,\tilde{p},\tilde{x}) \geq p_i f(x_i) + p_k f(x_k) - (p_i + p_k) f\left(\frac{p_i x_i + p_k x_k}{p_i + p_k}\right) := S_{\tilde{p},f}(x_i, x_k). \tag{1.4}$$

In this paper, we will establish that the best lower bound $L_{\tilde{p},f}(x_i, x_k)$ of Jensen's difference $\Delta(f,\tilde{p},\tilde{x})$ for

$$x_1 \leq \cdots \leq x_i \leq \cdots \leq x_k \leq \cdots \leq x_n \tag{1.5}$$

has the expression

$$L_{\tilde{p},f}(x_i, x_k) = Q_i f(x_i) + R_k f(x_k) - (Q_i + R_k) f\left(\frac{Q_i x_i + R_k x_k}{Q_i + R_k}\right), \tag{1.6}$$

where

$$Q_i = p_1 + p_2 + \cdots + p_i, \qquad R_k = p_k + p_{k+1} + \cdots + p_n. \tag{1.7}$$

Logically, we need to have

$$L_{\tilde{p},f}(x_i, x_k) \geq S_{\tilde{p},f}(x_i, x_k). \tag{1.8}$$

Indeed, this inequality is equivalent to Jensen's inequality

$$\begin{aligned} &(Q_i - p_i) f(x_i) + (R_k - p_k) f(x_k) + (p_i + p_k) f\left(\frac{p_i x_i + p_k x_k}{p_i + p_k}\right) \\ &\quad \geq (Q_i + R_k) f\left(\frac{Q_i x_i + R_k x_k}{Q_i + R_k}\right). \end{aligned} \tag{1.9}$$

2. Main Results

Theorem 2.1. *Let f be a convex function on I, and let $x_1, x_2, \ldots, x_n \in I$ $(n \geq 3)$ such that*

$$x_1 \leq x_2 \leq \cdots \leq x_n. \tag{2.1}$$

For fixed x_i and x_k $(1 \leq i < k \leq n)$, Jensen's difference $\Delta(f,\tilde{p},\tilde{x})$ is minimal when

$$\begin{aligned} &x_1 = x_2 = \cdots = x_{i-1} = x_i, \qquad x_{k+1} = x_{k+2} = \cdots = x_n = x_k, \\ &x_{i+1} = x_{i+2} = \cdots = x_{k-1} = \frac{Q_i x_i + R_k x_k}{Q_i + R_k}, \end{aligned} \tag{2.2}$$

that is,

$$\Delta(f,\tilde{p},\tilde{x}) \geq Q_i f(x_i) + R_k f(x_k) - (Q_i + R_k) f\left(\frac{Q_i x_i + R_k x_k}{Q_i + R_k}\right) := L_{\tilde{p},f}(x_i, x_k). \tag{2.3}$$

For proving Theorem 2.1, we will need the following three lemmas.

Lemma 2.2. *Let p, q be nonnegative real numbers, and let f be a convex function on I. If $a, b, c, d \in I$ such that $c, d \in [a, b]$ and*

$$pa + qb = pc + qd, \tag{2.4}$$

then

$$pf(a) + qf(b) \geq pf(c) + qf(d). \tag{2.5}$$

Lemma 2.3. *Let f be a convex function on I, and let $x_1, x_2, \ldots, x_n \in I$ ($n \geq 3$) such that*

$$x_1 \leq x_2 \leq \cdots \leq x_n. \tag{2.6}$$

For fixed $x_i, x_{i+1}, \ldots, x_n$, where $i \in \{2, 3, \ldots, n-1\}$, Jensen's difference $\Delta(f,\tilde{p},\tilde{x})$ is minimal when

$$x_1 = x_2 = \cdots = x_{i-1} = x_i. \tag{2.7}$$

Lemma 2.4. *Let f be a convex function on I, and let $x_1, x_2, \ldots, x_n \in I$ ($n \geq 3$) such that*

$$x_1 \leq x_2 \leq \cdots \leq x_n. \tag{2.8}$$

For fixed $x_1, x_2, \ldots, x_k$, where $k \in \{2, 3, \ldots, n-1\}$, Jensen's difference $\Delta(f,\tilde{p},\tilde{x})$ is minimal when

$$x_{k+1} = x_{k+2} = \cdots = x_n = x_k. \tag{2.9}$$

Applying Theorem 2.1 for $f(x) = e^x$ and using the substitutions $a_1 = e^{x_1}$, $a_2 = e^{x_2}, \ldots, a_n = e^{x_n}$, we obtain

Corollary 2.5. *Let*

$$0 < a_1 \leq \cdots \leq a_i \leq \cdots \leq a_k \leq \cdots \leq a_n, \tag{2.10}$$

and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that $p_1 + p_2 + \cdots + p_n = 1$. Then,

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n - a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \geq Q_i a_i + R_k a_k - (Q_i + R_k) a_i^{Q_i/(Q_i+R_k)} a_k^{R_k/(Q_i+R_k)}, \tag{2.11}$$

with equality for

$$a_1 = a_2 = \cdots = a_i, \qquad a_k = a_{k+1} = \cdots = a_n,$$
$$a_{i+1} = a_{i+2} = \cdots = a_{k-1} = a_i^{Q_i/(Q_i+R_k)} a_k^{R_k/(Q_i+R_k)}. \tag{2.12}$$

Using Corollary 2.5, we can prove the propositions below.

Proposition 2.6. *Let*

$$0 < a_1 \le \cdots \le a_i \le \cdots \le a_k \le \cdots \le a_n, \tag{2.13}$$

and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that $p_1 + p_2 + \cdots + p_n = 1$. If

$$P = \begin{cases} \dfrac{2Q_i R_k}{Q_i + R_k}, & Q_i \le R_k, \\ R_k, & Q_i \ge R_k, \end{cases} \tag{2.14}$$

then

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n - a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \ge P\left(\sqrt{a_k} - \sqrt{a_i}\right)^2, \tag{2.15}$$

with equality for $a_1 = a_2 = \cdots = a_n$. When $Q_i = R_k$, equality holds again for $a_1 = a_2 = \cdots = a_i$, $a_{i+1} = \cdots = a_{k-1} = \sqrt{a_i a_k}$, $a_k = a_{k+1} = \cdots = a_n$.

Proposition 2.7. *Let*

$$0 < a_1 \le \cdots \le a_i \le \cdots \le a_k \le \cdots \le a_n, \tag{2.16}$$

and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that $p_1 + p_2 + \cdots + p_n = 1$. Then,

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n - a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \ge \frac{3 Q_i R_k (a_k - a_i)^2}{(4Q_i + 2R_k) a_k + (2Q_i + 4R_k) a_i}, \tag{2.17}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Remark 2.8. For $p_1 = p_2 = \cdots = p_n = 1/n$, from Proposition 2.6 we get the inequality

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \ge P\left(\sqrt{a_k} - \sqrt{a_i}\right)^2, \tag{2.18}$$

where

$$P = \begin{cases} \dfrac{2i(n-k+1)}{n+i-k+1}, & i + k \le n + 1, \\ n - k + 1, & i + k \ge n + 1. \end{cases} \tag{2.19}$$

Equality in (2.18) holds for $a_1 = a_2 = \cdots = a_n$. If $i + k = n + 1$, then equality holds again for $a_1 = a_2 = \cdots = a_i$, $a_{i+1} = \cdots = a_{k-1} = \sqrt{a_i a_k}$, $a_k = a_{k+1} = \cdots = a_n$.

Remark 2.9. For $p_1 = p_2 = \cdots = p_n = 1/n$, from Proposition 2.7, we get the inequality

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{3i(n-k+1)(a_k - a_i)^2}{2(n+2i-k+1)a_k + 2(2n+i-2k+2)a_i}, \tag{2.20}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Applying Theorem 2.1 for $f(x) = -\ln x$, we obtain

Corollary 2.10. *Let*

$$0 < a_1 \leq \cdots \leq a_i \leq \cdots \leq a_k \leq \cdots \leq a_n, \tag{2.21}$$

and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that $p_1 + p_2 + \cdots + p_n = 1$. Then,

$$\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}} \geq \frac{((Q_i a_i + R_k a_k)/(Q_i + R_k))^{Q_i + R_k}}{a_i^{Q_i} a_k^{R_k}}, \tag{2.22}$$

with equality for

$$\begin{gathered} a_1 = a_2 = \cdots = a_i, \qquad a_k = a_{k+1} = \cdots = a_n, \\ a_{i+1} = a_{i+2} = \cdots = a_{k-1} = \frac{Q_i a_i + R_k a_k}{Q_i + R_k}. \end{gathered} \tag{2.23}$$

Remark 2.11. For $p_1 = p_2 = \cdots = p_n = 1/n$, from Corollary 2.10, we get the inequality

$$\frac{a_1 + a_2 + \cdots + a_n}{n\sqrt[n]{a_1 a_2 \cdots a_n}} \geq \sqrt[n]{\frac{((ia_i + (n-k+1)a_k)/(n+i-k+1))^{n+i-k+1}}{a_i^i a_k^{n-k+1}}}, \tag{2.24}$$

with equality for

$$\begin{gathered} a_1 = a_2 = \cdots = a_i, \qquad a_k = a_{k+1} = \cdots = a_n, \\ a_{i+1} = a_{i+2} = \cdots = a_{k-1} = \frac{ia_i + (n-k+1)a_k}{n+i-k+1}. \end{gathered} \tag{2.25}$$

If $i \leq n/2$ and $k = n - i + 1$, then (2.24) becomes

$$\frac{a_1 + a_2 + \cdots + a_n}{n\sqrt[n]{a_1 a_2 \cdots a_n}} \geq \left(\frac{\sqrt{a_i/a_{n-i+1}} + \sqrt{a_{n-i+1}/a_i}}{2}\right)^{2i/n}, \tag{2.26}$$

with equality for

$$a_1 = a_2 = \cdots = a_i, \qquad a_{n-i+1} = a_{n-i+2} = \cdots = a_n,$$
$$a_{i+1} = a_{i+2} = \cdots = a_{n-i} = \frac{a_i + a_{n-i+1}}{2}. \tag{2.27}$$

In the case $i = 1$, from (2.26), we get

$$\frac{a_1 + a_2 + \cdots + a_n}{n\sqrt[n]{a_1 a_2 \cdots a_n}} \geq \left(\frac{\sqrt{a_1/a_n} + \sqrt{a_n/a_1}}{2}\right)^{2/n}, \tag{2.28}$$

with equality for

$$a_2 = a_3 = \cdots = a_{n-1} = \frac{a_1 + a_n}{2}. \tag{2.29}$$

Applying Theorem 2.1 for $f(x) = 1/x$, we obtain the following.

Corollary 2.12. *Let*

$$0 < a_1 \leq \cdots \leq a_i \leq \cdots \leq a_k \leq \cdots \leq a_n, \tag{2.30}$$

and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that $p_1 + p_2 + \cdots + p_n = 1$. Then,

$$\frac{p_1}{a_1} + \frac{p_2}{a_2} + \cdots + \frac{p_n}{a_n} - \frac{1}{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n} \geq \frac{Q_i R_k (a_k - a_i)^2}{a_i a_k (Q_i a_i + R_k a_k)}, \tag{2.31}$$

with equality for

$$a_1 = a_2 = \cdots = a_i, \qquad a_k = a_{k+1} = \cdots = a_n,$$
$$a_{i+1} = a_{i+2} = \cdots = a_{k-1} = \frac{Q_i a_i + R_k a_k}{Q_i + R_k}. \tag{2.32}$$

Using Corollary 2.12, we can prove the following proposition.

Proposition 2.13. *Let*

$$0 < a_1 \leq \cdots \leq a_i \leq \cdots \leq a_k \leq \cdots \leq a_n, \tag{2.33}$$

and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that $p_1 + p_2 + \cdots + p_n = 1$. If

$$P = \begin{cases} Q_i, & Q_i \leq 3R_k, \\ \dfrac{4Q_i R_k}{Q_i + R_k}, & Q_i \geq 3R_k, \end{cases} \tag{2.34}$$

then

$$\frac{p_1}{a_1} + \frac{p_2}{a_2} + \cdots + \frac{p_n}{a_n} - \frac{1}{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n} \geq P\left(\frac{1}{\sqrt{a_i}} - \frac{1}{\sqrt{a_k}}\right)^2, \tag{2.35}$$

with equality for $a_1 = a_2 = \cdots = a_n$.

3. Proof of Lemmas

Proof of Lemma 2.2. Since $c, d \in [a, b]$, there exist $\lambda_1, \lambda_2 \in [0, 1]$ such that

$$c = \lambda_1 a + (1 - \lambda_1)b, \qquad d = \lambda_2 a + (1 - \lambda_2)b. \tag{3.1}$$

In addition, from $pa + qb = pc + qd$, we get

$$q\lambda_2 = (1 - \lambda_1)p. \tag{3.2}$$

Applying Jensen's inequality twice, we obtain

$$\begin{aligned} f(c) &= f(\lambda_1 a + (1 - \lambda_1)b) \leq \lambda_1 f(a) + (1 - \lambda_1) f(b), \\ f(d) &= f(\lambda_2 a + (1 - \lambda_2)b) \leq \lambda_2 f(a) + (1 - \lambda_2) f(b), \end{aligned} \tag{3.3}$$

and hence

$$\begin{aligned} pf(c) + qf(d) &\leq p\left[\lambda_1 f(a) + (1 - \lambda_1) f(b)\right] + q\left[\lambda_2 f(a) + (1 - \lambda_2) f(b)\right] \\ &= pf(a) + qf(b). \end{aligned} \tag{3.4}$$

□

Proof of Lemma 2.3. We need to show that

$$\begin{aligned} &p_1 f(x_1) + p_2 f(x_2) + \cdots + p_i f(x_i) - f\left(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n\right) \\ &\quad \geq Q_i f(x_i) - f\left(Q_i x_i + p_{i+1} x_{i+1} + \cdots + p_n x_n\right). \end{aligned} \tag{3.5}$$

Using Jensen's inequality

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_i f(x_i) \geq Q_i f\left(\frac{p_1 x_1 + p_2 x_2 + \cdots + p_i x_i}{Q_i}\right), \tag{3.6}$$

it suffices to prove that

$$Q_i f\left(\frac{p_1x_1 + p_2x_2 + \cdots + p_ix_i}{Q_i}\right) + f(Q_ix_i + p_{i+1}x_{i+1} + \cdots + p_nx_n) \\ \geq Q_i f(x_i) + f(p_1x_1 + p_2x_2 + \cdots + p_nx_n), \tag{3.7}$$

which can be written as

$$Q_i f(X_i) + f(Y_i) \geq Q_i f(x_i) + f(X), \tag{3.8}$$

where

$$X_i = \frac{p_1x_1 + p_2x_2 + \cdots + p_ix_i}{Q_i}, \\ Y_i = Q_ix_i + p_{i+1}x_{i+1} + \cdots + p_nx_n, \\ X = p_1x_1 + p_2x_2 + \cdots + p_nx_n. \tag{3.9}$$

Since $x_i, X \in [X_i, Y_i]$ and

$$Q_iX_i + Y_i = Q_ix_i + X, \tag{3.10}$$

by Lemma 2.2, the conclusion follows. □

Proof of Lemma 2.4. We need to prove that

$$p_k f(x_k) + p_{k+1} f(x_{k+1}) + \cdots + p_n f(x_n) - f(p_1x_1 + p_2x_2 + \cdots + p_nx_n) \\ \geq R_k f(x_k) - f(p_1x_1 + \cdots + p_{k-1}x_{k-1} + R_kx_k). \tag{3.11}$$

By Jensen's inequality, we have

$$p_k f(x_k) + p_{k+1} f(x_{k+1}) + \cdots + p_n f(x_n) \geq R_k f\left(\frac{p_kx_k + p_{k+1}x_{k+1} + \cdots + p_nx_n}{R_k}\right). \tag{3.12}$$

Therefore, it suffices to prove that

$$R_k f\left(\frac{p_kx_k + p_{k+1}x_{k+1} + \cdots + p_nx_n}{R_k}\right) + f(p_1x_1 + \cdots + p_{k-1}x_{k-1} + R_kx_k) \\ \geq R_k f(x_k) + f(p_1x_1 + p_2x_2 + \cdots + p_nx_n), \tag{3.13}$$

or, equivalently,

$$R_k f(X_k) + f(Y_k) \geq R_k f(x_k) + f(X), \tag{3.14}$$

where

$$\begin{aligned} X_k &= \frac{p_k x_k + p_{k+1} x_{k+1} + \cdots + p_n x_n}{R_k}, \\ Y_k &= p_1 x_1 + \cdots + p_{k-1} x_{k-1} + R_k x_k, \\ X &= p_1 x_1 + p_2 x_2 + \cdots + p_n x_n. \end{aligned} \tag{3.15}$$

The inequality (3.14) follows from Lemma 2.2, since $x_k, X \in [Y_k, X_k]$ and

$$R_k X_k + Y_k = R_k x_k + X. \tag{3.16}$$

□

4. Proof of Theorem

Proof. By Lemmas 2.3 and 2.4, it follows that for fixed x_i, $x_{i+1}, \ldots, x_k$, Jensen's difference $\Delta(f, \tilde{p}, \tilde{x})$ is minimal when $x_1 = x_2 = \cdots = x_{i-1} = x_i$ and $x_{k+1} = x_{k+2} = \cdots = x_n = x_k$; that is,

$$\begin{aligned} \Delta(f, \tilde{p}, \tilde{x}) \geq{}& Q_i f(x_i) + p_{i+1} f(x_{i+1}) + \cdots + p_{k-1} f(x_{k-1}) + R_k f(x_k) \\ &- f(Q_i x_i + p_{i+1} x_{i+1} + \cdots + p_{k-1} x_{k-1} + R_k x_k). \end{aligned} \tag{4.1}$$

Therefore, towards proving (2.3), we only need to show that

$$\begin{aligned} p_{i+1} f(x_{i+1}) + \cdots + p_{k-1} f(x_{k-1}) + (Q_i + R_k) f\left(\frac{Q_i x_i + R_k x_k}{Q_i + R_k}\right) \\ \geq f(Q_i x_i + p_{i+1} x_{i+1} + \cdots + p_{k-1} x_{k-1} + R_k x_k). \end{aligned} \tag{4.2}$$

Since

$$Q_i + p_{i+1} + \cdots + p_{k-1} + R_k = 1, \tag{4.3}$$

this inequality is a consequence of Jensen's inequality. Thus, the proof is completed. □

5. Proof of Propositions

Proof of Proposition 2.6. Using Corollary 2.5, we need to prove that

$$Q_i a_i + R_k a_k - (Q_i + R_k) a_i^{Q_i/(Q_i+R_k)} a_k^{R_k/(Q_i+R_k)} \geq P(\sqrt{a_k} - \sqrt{a_i})^2. \tag{5.1}$$

Since this inequality is homogeneous in a_i and a_k, and also in Q_i and R_k, without loss of generality, assume that $a_i = 1$ and $Q_i = 1$. Using the notations $a_k = x^2$ and $R_k = p$, where $x \geq 1$ and $p > 0$, the inequality is equivalent to $g(x) \geq 0$, where

$$g(x) = 1 + px^2 - (1+p)x^{2p/(1+p)} - P(x-1)^2, \tag{5.2}$$

with

$$P = \begin{cases} \dfrac{2p}{p+1}, & p \geq 1, \\ p, & p \leq 1. \end{cases} \tag{5.3}$$

We have

$$\begin{aligned} g'(x) &= 2p\left(x - x^{(p-1)/(p+1)}\right) - 2P(x-1), \\ g''(x) &= 2(p-P) - \frac{2p(p-1)}{p+1} x^{-2/(p+1)}. \end{aligned} \tag{5.4}$$

If $p \geq 1$, then

$$g''(x) = \frac{2p(p-1)}{p+1}\left(1 - x^{-2/(p+1)}\right) \geq 0, \tag{5.5}$$

and if $p \leq 1$, then

$$g''(x) = \frac{2p(1-p)}{p+1} x^{-2/(p+1)} \geq 0. \tag{5.6}$$

Since $g''(x) \geq 0$ for $x \geq 1$, and $g'(x)$ is increasing, $g'(x) \geq g(1) = 0$, $g(x)$ is increasing, and hence $g(x) \geq g(1) = 0$ for $x \geq 1$. This concludes the proof. □

Proof of Proposition 2.7. Using Corollary 2.5, we need to prove that

$$Q_i a_i + R_k a_k - (Q_i + R_k) a_i^{Q_i/(Q_i+R_k)} a_k^{R_k/(Q_i+R_k)} \geq \frac{3Q_i R_k (a_k - a_i)^2}{(4Q_i + 2R_k)a_k + (2Q_i + 4R_k)a_i}. \tag{5.7}$$

Since this inequality is homogeneous in a_i and a_k, and also in Q_i and R_k, we may set $a_i = 1$ and $Q_i = 1$. Using the notations $a_k = x$ and $R_k = p$, where $x \geq 1$ and $p > 0$, the inequality is equivalent to $g(x) \geq 0$, where

$$g(x) = \left[(4+2p)x + 2 + 4p\right]\left[1 + px - (1+p)x^{p/(1+p)}\right] - 3p(x-1)^2. \tag{5.8}$$

We have

$$\begin{aligned}
\frac{1}{2(1+2p)}g'(x) &= p\left(x - x^{-1/(1+p)}\right) - (2+p)\left(x^{p/(1+p)} - 1\right),\\
\frac{1+p}{2p(1+2p)}g''(x) &= 1 + p + x^{(-2-p)/(1+p)} - (2+p)x^{-1/(1+p)},\\
\frac{(1+p)^2}{2p(1+2p)(2+p)}g'''(x) &= (x-1)x^{(-3-2p)/(1+p)}.
\end{aligned} \tag{5.9}$$

Since $g'''(x) \geq 0$ for $x \geq 1$, $g''(x)$ is strictly increasing, $g''(x) \geq g''(1) = 0$, and $g'(x)$ is strictly increasing, $g'(x) \geq g'(1) = 0$, $g(x)$ is strictly increasing, and hence $g(x) \geq g(1) = 0$ for $x \geq 1$. □

Proof of Proposition 2.13. Using Corollary 2.12, we need to prove that

$$\frac{Q_i R_k (a_k - a_i)^2}{a_i a_k (Q_i a_i + R_k a_k)} \geq P\left(\frac{1}{\sqrt{a_i}} - \frac{1}{\sqrt{a_k}}\right)^2. \tag{5.10}$$

This inequality is true if

$$Q_i R_k (\sqrt{a_i} + \sqrt{a_k})^2 \geq P(Q_i a_i + R_k a_k). \tag{5.11}$$

For $Q_i \leq 3R_k$, we have

$$Q_i R_k (\sqrt{a_i} + \sqrt{a_k})^2 - P(Q_i a_i + R_k a_k) = Q_i a_i \left(2R_k \sqrt{\frac{a_k}{a_i}} + R_k - Q_i\right) \geq Q_i a_i (3R_k - Q_i) \geq 0. \tag{5.12}$$

Also, for $Q_i \geq 3R_k$, we get

$$\begin{aligned}
&Q_i R_k (\sqrt{a_i} + \sqrt{a_k})^2 - P(Q_i a_i + R_k a_k)\\
&\quad = \frac{Q_i R_k}{Q_i + R_k}\left[(R_k - 3Q_i)a_i + (Q_i - 3R_k)a_k + 2(Q_i + R_k)\sqrt{a_i a_k}\right]\\
&\quad \geq \frac{Q_i R_k}{Q_i + R_k}\left[(R_k - 3Q_i)a_i + (Q_i - 3R_k)a_i + 2(Q_i + R_k)a_i\right] = 0.
\end{aligned} \tag{5.13}$$

The proposition is proved. □

References

[1] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, vol. 61 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[2] S. S. Dragomir, J. Pečarić, and L. E. Persson, "Properties of some functionals related to Jensen's inequality," *Acta Mathematica Hungarica*, vol. 70, no. 1-2, pp. 129–143, 1996.