Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 172059, 13 pages
doi:10.1155/2010/172059

Research Article

Optimality Conditions in Nondifferentiable G-Invex Multiobjective Programming

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Received 29 October 2009; Revised 10 March 2010; Accepted 14 March 2010

Academic Editor: Jong Kyu Kim

We consider a class of nondifferentiable multiobjective programs with inequality and equality constraints in which each component of the objective function contains a term involving the support function of a compact convex set. We introduce G-Karush-Kuhn-Tucker conditions and G-Fritz John conditions for our nondifferentiable multiobjective programs. By using suitable G-invex functions, we establish G-Karush-Kuhn-Tucker necessary and sufficient optimality conditions, and G-Fritz John necessary and sufficient optimality conditions of our nondifferentiable multiobjective programs. Our optimality conditions generalize and improve the results in Antczak (2009) to the nondifferentiable case.

1. Introduction and Preliminaries

A number of different forms of invexity have appeared. In [1], Martin defined Kuhn-Tucker invexity and weak duality invexity. In [2], Ben-Israel and Mond presented some new results for invex functions. Hanson [3] introduced the concepts of invex functions, and Type I, Type II functions were introduced by Hanson and Mond [4]. Craven and Glover [5] established Kuhn-Tucker type optimality conditions for cone invex programs, and Jeyakumar and Mond [6] introduced the class of the so-called V-invex functions to proved some optimality for a class of differentiable vector optimization problems than under invexity assumption. Egudo [7] established some duality results for differentiable multiobjective programming problems with invex functions. Kaul et al. [8] considered Wolfe-type and Mond-Weir-type duals and generalized the duality results of Weir [9] under weaker invexity assumptions.

Based on the paper by Mond and Schechter [10], Yang et al. [11] studied a class of nondifferentiable multiobjective programs. They replaced the objective function by the support function of a compact convex set, constructed a more general dual model for a class of nondifferentiable multiobjective programs, and established only weak duality theorems for efficient solutions under suitable weak convexity conditions. Subsequently, Kim et al.

[12] established necessary and sufficient optimality conditions and duality results for weakly efficient solutions of nondifferentiable multiobjective fractional programming problems.

Recently, Antczak [13, 14] studied the optimality and duality for G-multi-objective programming problems. They defined a new class of differentiable nonconvex vector valued functions, namely, the vector G-invex (G-incave) functions with respect to η. They used vector G-invexity to develop optimality conditions for differentiable multiobjective programming problems with both inequality and equality constraints. Considering the concept of a (weak) Pareto solution, they established the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable vector optimization problems under the Kuhn-Tucker constraint qualification.

In this paper, we obtain an extension of the results in [13],which were established in the differentiable to the nondifferentiable case. We proposed a class of nondifferentiable multiobjective programming problems in which each component of the objective function contains a term involving the support function of a compact convex set. We obtain G-Karush-Tucker necessary and sufficient conditions and G-Fritz John necessary and sufficient conditions for weak Pareto solution. Necessary optimal theorems are presented by using alternative theorem [15] and Mangasarian-Fromovitz constraint qualification [16]. In addition, we give sufficient optimal theorems under suitable G-invexity conditions.

We provide some definitions and some results that we shall use in the sequel. Throughout the paper, the following convention will be used.

For any $x = (x_1, x_2, \ldots, x_n)^T,\ y = (y_1, y_2, \ldots, y_n)^T$, we write

$$\begin{aligned} x = y, &\quad \text{iff } x_i = y_i, \forall i = 1, 2, \ldots, n, \\ x < y, &\quad \text{iff } x_i < y_i, \forall i = 1, 2, \ldots, n, \\ x \leqq y, &\quad \text{iff } x_i \leq y_i, \forall i = 1, 2, \ldots, n, \\ x \leq y, &\quad \text{iff } x_i \leqq y_i, x \neq y, n > 1. \end{aligned} \tag{1.1}$$

Throughout the paper, we will use the same notation for row and column vectors when the interpretation is obvious. We say that a vector $z \in \mathbb{R}^n$ is negative if $z \leqq 0$ and strictly negative if $z < 0$.

Definition 1.1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be strictly increasing if and only if

$$\forall x, y \in \mathbb{R}, \quad x < y \Longrightarrow f(x) < f(y). \tag{1.2}$$

Let $f = (f_1, \ldots, f_k) : X \to \mathbb{R}^k$ be a vector-valued differentiable function defined on a nonempty open set $X \subset \mathbb{R}^n$, and $I_{f_i}(X), i = 1, \ldots, k$, the range of f_i, that is, the image of X under f_i.

Definition 1.2 (see [11]). Let C be a compact convex set in $\mathbb{R}^n$. The support function $s(x \mid C)$ is defined by

$$s(x \mid C) := \max\left\{x^T y : y \in C\right\}. \tag{1.3}$$

The support function $s(x \mid C)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that

$$s(y \mid C) \geq s(x \mid C) + z^T(y - x), \quad \forall y \in D. \tag{1.4}$$

Equivalently,

$$z^T x = s(x \mid C). \tag{1.5}$$

The subdifferential of $s(x \mid C)$ at x is given by

$$\partial s(x \mid C) := \left\{ z \in C : z^T x = s(x \mid C) \right\}. \tag{1.6}$$

Now, in the natural way, we generalize the definition of a real-valued G-invex function. Let $f = (f_1, \ldots, f_k) : X \to \mathbb{R}^k$ be a vector-valued differentiable function defined on a nonempty open set $X \subset \mathbb{R}^n$, and $I_{f_i}(X), i = 1, \ldots, k$, the range of f_i, that is, the image of X under f_i.

Definition 1.3. Let $f : X \to \mathbb{R}^n$ be a vector-valued differentiable function defined on a nonempty set $X \subset \mathbb{R}^n$ and $u \in X$. If there exist a differentiable vector-valued function $G_f = (G_{f_1}, \ldots, G_{f_k}) : \mathbb{R} \to \mathbb{R}^k$ such that any of its component $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$ is a strictly increasing function on its domain and a vector-valued function $\eta : X \times X \to \mathbb{R}^n$ such that, for all $x \in X$ $(x \neq u)$ and for any $i = 1, \ldots, k$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geqq (>) G'_{f_i}(f_i(u)) \nabla f_i(u) \eta(x, u), \tag{1.7}$$

then f is said to be a (strictly) vector G_f-invex function at u on X (with respect to η) (or shortly, G-invex function at u on X). If (1.7) is satisfied for each $u \in X$, then f is vector G_f-invex on X with respect to η.

Lemma 1.4 (see [13]). *In order to define an analogous class of (strictly) vector G_f-incave functions with respect to η, the direction of the inequality in the definition of G_f-invex function should be changed to the opposite one.*

We consider the following multiobjective programming problem.

$$\begin{aligned} \text{(NMP) Minimize } & \left(G_{F_1}(f_1(x) + s(x \mid C_1)), \ldots, G_{F_k}(f_k(x) + s(x \mid C_k))\right) \\ \text{subject to } & \left(G_{g_1}(g_1(x)), \ldots, G_{g_m}(g_m(x))\right) \leqq 0, \\ & \left(G_{h_1}(h_1(x)), \ldots, G_{h_p}(h_p(x))\right) = 0, \end{aligned} \tag{1.8}$$

where $f_i : X \to \mathbb{R}$, $i \in I = \{1, \ldots, k\}$, $g_j : X \to \mathbb{R}$, $j \in J = \{1, \ldots, m\}$, $h_t : X \to \mathbb{R}$, $t \in T = \{1, \ldots, p\}$, are differentiable functions on a nonempty open set $X \subset \mathbb{R}^n$. Moreover, $G_{F_i}, i \in I$, are differentiable real-valued strictly increasing functions, $G_{g_j}, j \in J$, are differentiable real-valued strictly increasing functions, and $G_{h_t}, t \in T$, are differentiable real-valued strictly increasing functions. Let $D = \{x \in X : G_{g_j}(g_j(x)) \leqq 0,\ j \in J, G_{h_t}(h_t(x)) = 0, t \in T\}$ be the set of all feasible solutions for problem (NMP), and $F_i = f_i(\cdot) + (\cdot)^T w_i$. Further, we denote by $J(z) := \{j \in J : G_{g_j}(g_j(z)) = 0\}$ the set of inequality constraint functions active at $z \in D$ and by $I(z) := \{i \in I : \lambda_i > 0\}$ the objective functions indices set, for which the corresponding Lagrange multiplier is not equal 0. For such optimization problems, minimization means in general obtaining weak Pareto optimal solutions in the following sense.

Definition 1.5. A feasible point $\overline{x}$ is said to be a weak Pareto solution (a weakly efficient solution, a weak minimum) of (NMP) if there exists no other $x \in D$ such that

$$G_{f(x)+x^T w}\big(f(x) + s(x \mid C)\big) < G_{f(\overline{x})+\overline{x}^T w}\big(f(\overline{x}) + s(\overline{x} \mid C)\big). \tag{1.9}$$

Definition 1.6 (see [17]). Let W be a given set in $\mathbb{R}^n$ ordered by $\leqq$ or by $<$. Specifically, we call the minimal element of W defined by $\leq$ a minimal vector, and that defined by $<$ a weak minimal vector. Formally speaking, a vector $\overline{z} \in w$ is called a minimal vector in W if there exists no vector z in W such that $z \leq \overline{z}$; it is called a weak minimal vector if there exists no vector z in W such that $z < \overline{z}$.

By using the result of Antczak [13] and the definition of a weak minimal vector, we obtain the following proposition.

Proposition 1.7. *Let $\overline{x}$ be feasible solution in a multiobjective programming problem and let $G_{f_i(\cdot)+(\cdot)^T w_i}, i = 1,\ldots,k$, be a continuous real-valued strictly increasing function defined on $I_{f_i+(\cdot)^T w_i}(X)$. Further, we denote $W = \{G_{f_1(\cdot)+(\cdot)^T w_1}(f_1(x)+s(x \mid C_1)),\ldots,G_{f_k(\cdot)+(\cdot)^T w_k}(f_k(x)+s(x \mid C_k)) : x \in X\} \subset \mathbb{R}^k$ and $\overline{z} = (G_{f_1(\cdot)+(\cdot)^T w_1}(f_1(\overline{x}) + s(\overline{x} \mid C_1)),\ldots,G_{f_k(\cdot)+(\cdot)^T w_k}(f_k(\overline{x}) + s(\overline{x} \mid C_k)) \in W$. Then, $\overline{x}$ is a weak Pareto solution in the set of all feasible solutions X for a multiobjective programming problem if and only if the corresponding vector $\overline{z}$ is a weak minimal vector in the set W.*

Proof. Let $\overline{x}$ be a weak Pareto solution. Then there does not exist x^* such that

$$G_{f(\cdot)+(\cdot)^T w_i}\big(f_i(x^*) + s(x^* \mid C_i)\big) < G_{f(\cdot)+(\cdot)^T w_i}\big(f_i(\overline{x}) + s(\overline{x} \mid C_i)\big). \tag{1.10}$$

By the strict increase of $G_{f_i(\cdot)+(\cdot)^T w_i}$ involving the support function, we have

$$G_{f(\cdot)+(\cdot)^T w_i}\big(f_i(x^*) + x^{*w}_{i}\big) < G_{f(\cdot)+(\cdot)^T w_i}\big(f_i(x^*) + s(x^* \mid C_i)\big). \tag{1.11}$$

Therefore, $\overline{z} = (G_{f_1(\cdot)+(\cdot)^T w_1}(f_1(\overline{x}) + s(\overline{x} \mid C_1)),\ldots,G_{f_k(\cdot)+(\cdot)^T w_k}(f_k(\overline{x}) + s(\overline{x} \mid C_k)))$ is a weak minimal vector in the set W. The converse part is proved similarly. □

Lemma 1.8 (see [13]). *In the case when $G_{F_i}(a) \equiv a,\ i = 1,\ldots,k$, for any $a \in I_{F_i}(X)$, we obtain a definition of a vector-valued invex function.*

2. Optimality Conditions

In this section, we establish G-Fritz John and G-Karush-Kuhn-Tucker necessary and sufficient conditions for a weak Pareto optimal point of (NMP).

Theorem 2.1 (G-Fritz John Necessary Optimality Conditions). *Suppose that G_{F_i}, $i \in I$, are differentiable real-valued strictly increasing functions defined on $I_{F_i}(D)$, G_{g_j}, $j \in J$, are differentiable real-valued strictly increasing functions defined on $I_{g_j}(D)$, and G_{h_t}, $t \in T$, are differentiable real-valued strictly increasing functions defined on $I_{h_t}(D)$, and let $F_i = f_i(\cdot) + (\cdot)^T w_i$. Let $\overline{x} \in D$ be a weak Pareto optimal point in problem (NMP). Then there exist $\lambda \in \mathbb{R}^k_+$, $\xi \in \mathbb{R}^m_+$, $\mu \in \mathbb{R}^p$, and $w_i \in C_i$ such that*

$$\begin{aligned}
&\sum_{i=1}^{k} \lambda_i G'_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right)\left(\nabla f_i(\overline{x}) + w_i\right) + \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(\overline{x})\right)\nabla g_j(\overline{x}) \\
&\quad + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x}) = 0, \\
&\xi_j G_{g_j}\left(g_j(\overline{x})\right) = 0, \quad j \in J, \\
&\langle w_i, \overline{x}\rangle = s(\overline{x} \mid C_i), \quad i = 1, \ldots, k, \\
&\lambda \geqq 0, \xi \geqq 0, \quad (\lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_m, \mu_1, \ldots, \mu_p) \neq 0.
\end{aligned} \tag{2.1}$$

Proof. Let $b_i(\overline{x}) = s(\overline{x} \mid C_i)$, $i = 1, \ldots, k$. Since C_i is convex and compact,

$$b'_i(\overline{x}; d) = \frac{\lim_{\lambda \to 0+} b_i(\overline{x} + \lambda d) - b_i(\overline{x})}{\lambda} \tag{2.2}$$

is finite. Also, for all $d \in \mathbb{R}^n$,

$$\begin{aligned}
&\left(G_{F_i}\left(f_i + b_i\right)\right)'(\overline{x}; d) \\
&\quad = \frac{\lim_{\lambda \to 0+} G_{F_i}\left(f_i(\overline{x} + \lambda d) + b_i(\overline{x} + \lambda d)\right) - G_{F_i}\left(f_i(\overline{x}) + b_i(\overline{x})\right)}{\lambda} \\
&\quad = G'_{F_i}\left(f_i + b_i\right)\left(\nabla f_i + b'_i\right)(\overline{x}; d) \\
&\quad = \left\langle G'_{F_i}\left(f_i(\overline{x}) + b_i(\overline{x})\right)\left(\nabla f_i(\overline{x}) + b'_i(\overline{x})\right), d\right\rangle.
\end{aligned} \tag{2.3}$$

Since $\overline{x}$ is a weak Pareto optimal point in (NMP)

$$\begin{aligned}
&\left\langle G'_{F_i}\left(f_i(\overline{x}) + b_i(\overline{x})\right)\left(\nabla f_i(\overline{x}) + b'_i(\overline{x})\right), d\right\rangle < 0, \quad i = 1, \ldots, k, \\
&\left\langle G'_{g_j}\left(g_j(\overline{x})\right)\nabla g_j(\overline{x}), d\right\rangle \leqq 0, \quad j \in J(\overline{x}), \\
&\left\langle G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x}), d\right\rangle = 0, \quad t \in T,
\end{aligned} \tag{2.4}$$

has no solution $d \in \mathbb{R}^n$. By [15, Corollary 4.2.2], there exist $\lambda_i \geqq 0,\ i = 1,\dots,k,\ \xi_j \geqq 0,\ j \in J(\overline{x})$, and $\mu_t, t = 1,\dots,p$, not all zero, such that for any $d \in \mathbb{R}^n$,

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i\left\langle G'_{F_i}\left(f_i(\overline{x})+b_i(\overline{x})\right)\left(\nabla f_i(\overline{x})+b'_i(\overline{x})\right),d\right\rangle\\&\quad+\sum_{j\in J(\overline{x})}\xi_j\left\langle G'_{g_j}\left(g_j(\overline{x})\right)\nabla g_j(\overline{x}),d\right\rangle+\sum_{t=1}^{p}\mu_t\langle G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x}),d\rangle\geqq 0.\end{aligned}\tag{2.5}$$

Let $A = \{\sum_{i=1}^{k}\lambda_i[G'_{F_i}(f_i(\overline{x}) + b_i(\overline{x}))(\nabla f_i(\overline{x}) + w_i)] + \sum_{j\in J(\overline{x})}\xi_j G'_{g_j}(g_j(\overline{x}))\nabla g_j(\overline{x}) + \sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x}) \mid w_i \in \partial b_i(\overline{x}), i = 1,\dots,k\}$. Then $0 \in A$. Assume to the contrary that $0 \notin A$. By separation theorem, there exists $d^* \in \mathbb{R}^n$, $d^* \neq (0,\dots,0)$ such that for all $a \in A, \langle a, d^*\rangle < 0$, that is, for all $w_i \in b_i(\overline{x})$

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i\left\langle G'_{F_i}\left(f_i(\overline{x})+b_i(\overline{x})\right)\left(\nabla f_i(\overline{x})+b'_i(\overline{x})\right)d^*\right\rangle\\&\quad+\sum_{j\in J(\overline{x})}\xi_j\left\langle G'_{g_j}\left(g_j(\overline{x})\right)\nabla g_j(\overline{x}),d^*\right\rangle+\sum_{t=1}^{p}\mu_t\left\langle G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x}),d^*\right\rangle<0.\end{aligned}\tag{2.6}$$

This contradicts (2.5).

Letting $\xi_j = 0$, for all $j \notin J(\overline{x})$, we get

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i G'_{F_i}\left(f_i(\overline{x})+s(\overline{x}\mid C_i)\right)\left(\nabla f_i(\overline{x})+\partial b_i(\overline{x})\right)\\&\quad+\sum_{j=1}^{m}\xi_j G'_{g_j}\left(g_j(\overline{x})\right)\nabla g_j(\overline{x})+\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x})=0,\\&\sum_{j=1}^{m}\xi_j G_{g_j}\left(g_j(\overline{x})\right)=0,\\&(\lambda_1,\dots,\lambda_k,\xi_1,\dots,\xi_m)\neq 0.\end{aligned}\tag{2.7}$$

Since $\partial b_i(\overline{x}) = \{w_i \in C_i \mid \langle w_i, \overline{x}\rangle = s(\overline{x} \mid C_i)\}$, we obtain the desired result. □

Theorem 2.2 (G-Karush-Kuhn-Tucker Necessary Optimality Conditions). *Suppose that $G_{F_i}, i \in I$, are differentiable real-valued strictly increasing functions defined on $I_{F_i}(D)$, G_{g_j}, $j \in J$, are differentiable real-valued strictly increasing functions defined on $I_{g_j}(D)$, and G_{h_t}, $t \in T$, are differentiable real-valued strictly increasing functions defined on $I_{h_t}(D)$, and G_{h_t}, $t \in T$, are linearly independent, and let $F_i = f_i(\cdot) + (\cdot)^T w_i$. Moreover, we assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle G'_{g_j}(g_j(\overline{x}))\nabla g_j(\overline{x}), z^*\rangle < 0$, $j \in J(\overline{x})$, and $\langle G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x}), z^*\rangle = 0$, $t = 1,\dots,p$. If $\overline{x} \in D$*

is a weak Pareto optimal point in problem (NMP), then there exist $\lambda \in \mathbb{R}^k_+, \xi \in \mathbb{R}^m_+$, $\mu \in \mathbb{R}^p$, *and* $w_i \in C_i$, $i = 1, \dots, k$ *such that*

$$\begin{aligned}
&\sum_{i=1}^{k} \lambda_i G'_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right)(\nabla f_i(\overline{x}) + w_i) + \sum_{j=1}^{m} \xi_j G'_{g_j}(g_j(\overline{x}))\nabla g_j(\overline{x}) \\
&\quad + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x}) = 0, \\
&\xi_j G_{g_j}(g_j(\overline{x})) = 0, \quad j \in J, \\
&\langle w_i, \overline{x}\rangle = s(\overline{x} \mid C_i), \quad i = 1, \dots, k, \\
&\lambda \geq 0, \quad \sum_{i=1}^{k} \lambda_i = 1, \quad \xi \geqq 0.
\end{aligned} \tag{2.8}$$

Proof. Since $\overline{x}$ is a weak Pareto optimal point of (NMP), by Theorem 2.1, there exist $\widehat{\lambda} \in \mathbb{R}^k_+, \widehat{\xi} \in \mathbb{R}^m_+$, $\widehat{\mu} \in \mathbb{R}^p$, and $w_i \in C_i, i = 1, \dots, k$ such that

$$\begin{aligned}
&\sum_{i=1}^{k} \widehat{\lambda}_i G'_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right)(\nabla f_i(\overline{x}) + w_i) + \sum_{j=1}^{m} \widehat{\xi}_j G'_{g_j}(g_j(\overline{x}))\nabla g_j(\overline{x}) \\
&\quad + \sum_{t=1}^{p} \widehat{\mu}_t G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x}) = 0, \\
&\widehat{\xi}_j G_{g_j}(g_j(\overline{x})) = 0, \quad j \in J, \\
&\langle w_i, \overline{x}\rangle = s(\overline{x} \mid C_i), \quad i = 1, \dots, k, \\
&\widehat{\lambda} \geqq 0, \ \widehat{\xi} \geqq 0, \quad \left(\widehat{\lambda}_1, \dots, \widehat{\lambda}_k, \widehat{\xi}_1, \dots, \widehat{\xi}_m, \widehat{\mu}_1, \dots, \widehat{\mu}_p\right) \neq 0.
\end{aligned} \tag{2.9}$$

Assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle G'_{g_j}(g_j(\overline{x}))\nabla g_j(\overline{x}), z^*\rangle < 0$, $j \in J(\overline{x})$, and $\langle G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x}), z^*\rangle = 0$, $t = 1, \dots, p$. Then $(\widehat{\lambda}_1, \dots, \widehat{\lambda}_k) \neq (0, \dots, 0)$. Assume to the contrary that $(\widehat{\lambda}_1, \dots, \widehat{\lambda}_k) = (0, \dots, 0)$. Then $(\widehat{\xi}_1, \dots, \widehat{\xi}_m, \widehat{\mu}_1, \dots, \widehat{\mu}_p) \neq (0, \dots, 0)$. If $\widehat{\xi} = 0$, then $\widehat{\mu} \neq 0$. Since G_{h_t}, $t \in T$, are linearly independent, $\widehat{\mu}_1 G_{h_1}(h_1(\overline{x})) + \cdots + \widehat{\mu}_p G_{h_p}(h_p(\overline{x})) = 0$ has a trivial solution $\widehat{\mu} = 0$, this contradicts to the fact that $\widehat{\mu} \neq 0$. So $\widehat{\xi} \geq 0$. Define $\widehat{\xi}_{j \in J(\overline{x})} > 0$, $\widehat{\xi}_{j \notin J(\overline{x})} = 0$. Since $\langle G'_{g_j}(g_j(\overline{x}))\nabla g_j(\overline{x}), z^*\rangle < 0$, $j \in J(\overline{x})$, we have $\sum_{j=1}^{m} \langle G'_{g_j}(g_j(\overline{x}))\nabla g_j(\overline{x}), z^*\rangle < 0$ and so $\sum_{j=1}^{m} \langle G'_{g_j}(g_j(\overline{x}))\nabla g_j(\overline{x}), z^*\rangle + \sum_{t=1}^{p} \langle G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x}), z^*\rangle < 0$. This is a contradiction. Hence $(\widehat{\lambda}_1, \dots, \widehat{\lambda}_k) \neq (0, \dots, 0)$. Indeed, it is sufficient only to show that there exist $\lambda \in \mathbb{R}^k_+$, $\xi \in \mathbb{R}^m_+$, and $\mu \in \mathbb{R}^p$ such that $\sum_{i=1}^{k} \lambda_i = 1$. We set

$$\begin{aligned}
\lambda_q &= \frac{1}{1 + \sum_{i=1, i \neq j}^{k} \widehat{\lambda}_i}, \quad \text{for some } q \in I(\overline{x}), \\
\lambda_i &= \frac{\widehat{\lambda}_i}{1 + \sum_{i=1, i \neq j}^{k} \widehat{\lambda}_i}, \quad \text{for } i \in I, \ i \notin q,
\end{aligned}$$

$$\xi_j = \frac{\widehat{\xi}_j}{1 + \sum_{i=1, i\neq j}^{k} \widehat{\lambda}_i}, \quad \text{for } j \in J,$$
$$\mu_t = \frac{\widehat{\mu}_t}{1 + \sum_{i=1, i\neq j}^{k} \widehat{\lambda}_i}, \quad \text{for } t \in T. \tag{2.10}$$

It is not difficult to see that the G-Karush-Kuhn-Tucker necessary optimality conditions are satisfied with Lagrange multipliers, there exist $\lambda \in \mathbb{R}_+^k$, $\xi \in \mathbb{R}_+^m$; and $\mu \in \mathbb{R}^p$ given by (2.10). □

We denote by $T^+(\overline{x})$ and $T^-(\overline{x})$ the sets of equality constraints indices for which a corresponding Lagrange multiplier is positive and negative, respectively, that is, $T^+(\overline{x}) = \{t \in T : \mu_t > 0\}$ and $T^-(\overline{x}) = \{t \in T : \mu_t < 0\}$.

Theorem 2.3 (G-Fritz John Sufficient Optimality Conditions). *Let $(\overline{x}, \lambda, \xi, \mu, w)$ satisfy the G-Fritz John optimality conditions as follow:*

$$\sum_{i=1}^{k} \lambda_i G'_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right)\left(\nabla f_i(\overline{x}) + w_i\right) + \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(\overline{x})\right)\nabla g_j(\overline{x}) + \sum_{t=1}^{p} \mu_t G'_{h_t}\left(h_t(\overline{x})\right)\nabla h_t(\overline{x}) = 0, \tag{2.11}$$

$$\xi_j G_{g_j}\left(g_j(\overline{x})\right) = 0, \quad j \in J,\ \forall \overline{x} \in D, \tag{2.12}$$

$$\langle w_i, \overline{x}\rangle = s(\overline{x} \mid C_i), \quad i = 1, \ldots, k, \tag{2.13}$$

$$\lambda \geqq 0,\ \xi \geqq 0, \quad (\lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_m) \neq 0. \tag{2.14}$$

Further, assume that $F(= f(\cdot) + (\cdot)^T w)$ is vector G_F-invex with respect to η at $\overline{x}$ on D, g is strictly G_g-invex with respect to η at $\overline{x}$ on D, $h_t, t \in T^+(\overline{x})$, is G_{h_t}-invex with respect to η at $\overline{x}$ on D, and $h_t, t \in T^-(\overline{x})$, is G_{h_t}-incave with respect to η at $\overline{x}$ on D. Moreover, suppose that $G_{g_j}(0) = 0$ for $j \in J$ and $G_{h_t}(0) = 0$ for $t \in T^+(\overline{x}) \cup T^-(\overline{x})$. Then $\overline{x}$ is a weak Pareto optimal point in problem (NMP).

Proof. Suppose that $\overline{x}$ is not a weak Pareto optimal point in problem (NMP). Then there exists $x^* \in D$ such that $G_{F_i}(f_i(x^*) + s(x^* \mid C_i)) < G_{F_i}(f_i(\overline{x}) + s(\overline{x} \mid C_i))$, $i = 1, \ldots, k$. Since $\langle w_i, \overline{x}\rangle = s(\overline{x} \mid C_i)$, $i = 1, \ldots, k$,

$$\begin{aligned} G_{F_i}\left(f_i(x^*) + x^{*T} w_i\right) &< G_{F_i}\left(f_i(x^*) + s(x^* \mid C_i)\right) \\ &< G_{F_i}\left(f_i(\overline{x}) + s(\overline{x} \mid C_i)\right) \\ &= G_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right). \end{aligned} \tag{2.15}$$

Thus we get

$$G_{F_i}\left(f_i(x^*) + x^{*T} w_i\right) < G_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right), \quad i \in I. \tag{2.16}$$

By assumption, $F(= f(\cdot) + (\cdot)^T w)$ is G_F-invex with respect to η at $\overline{x}$ on D. Then by Definition 1.3, for any $i \in I$,

$$\begin{aligned}&\left[G_{F_i}\left(f_i(x^*) + x^{*T} w_i\right)\right] - \left[G_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right)\right]\\&\geqq \left[G'_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right)\left(\nabla f_i(\overline{x}) + w_i\right)\right]\eta(x^*, \overline{x}).\end{aligned} \tag{2.17}$$

Hence by (2.16) and (2.17), we obtain

$$\left[G'_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right)\left(\nabla f_i(\overline{x}) + w_i\right)\right]\eta(x^*, \overline{x}) < 0, \quad i \in I. \tag{2.18}$$

Since $(\overline{x}, \lambda, \xi, \mu, w)$ satisfy the G-Fritz John conditions, by $\lambda \geqq 0$,

$$\left[\sum_{i=1}^{k} \lambda_i G'_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right)\left(\nabla f_i(\overline{x}) + w_i\right)\right]\eta(x^*, \overline{x}) \leqq 0, \quad i \in I. \tag{2.19}$$

Since g is strictly G_g-invex with respect to η at $\overline{x}$ on D,

$$G_{g_j}\left(g_j(x^*)\right) - G_{g_j}\left(g_j(\overline{x})\right) > G'_{g_j}\left(g_j(\overline{x})\right)\nabla g_j(\overline{x})\eta(x^*, \overline{x}). \tag{2.20}$$

Thus, by $\xi \geqq 0$,

$$\xi_j G_{g_j}\left(g_j(x^*)\right) - \xi_j G_{g_j}\left(g_j(\overline{x})\right) \geqq \xi_j G'_{g_j}\left(g_j(\overline{x})\right)\nabla g_j(\overline{x})\eta(x^*, \overline{x}). \tag{2.21}$$

Then, (2.12) implies

$$\sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(\overline{x})\right)\nabla g_j(\overline{x})\eta(x^*, \overline{x}) \leqq 0. \tag{2.22}$$

By assumption, h_t, $t \in T^+(\overline{x})$, is G_{h_t}-invex with respect to η at $\overline{x}$ on D, and $h_t, t \in T^-(\overline{x})$, is G_{h_t}-incave with respect to η at $\overline{x}$ on D. Then, by Definition 1.3, we have,

$$\begin{aligned}&G_{h_t}(h_t(x^*)) - G_{h_t}(h_t(\overline{x})) \geqq G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x})\eta(x^*, \overline{x}), \quad t \in T^+(\overline{x}),\\&G_{h_t}(h_t(x^*)) - G_{h_t}(h_t(\overline{x})) \leqq G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x})\eta(x^*, \overline{x}), \quad t \in T^-(\overline{x}).\end{aligned} \tag{2.23}$$

Thus, for any $t \in T^+$,

$$\mu_t G_{h_t}(h_t(x^*)) - \mu_t G_{h_t}(h_t(\overline{x})) \geqq \mu_t G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x})\eta(x^*, \overline{x}). \tag{2.24}$$

Since $x^* \in D$ and $\overline{x} \in D$, then the inequality above implies

$$\sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(\overline{x})) \nabla h_t(\overline{x}) \eta(x^*, \overline{x}) \leqq 0. \tag{2.25}$$

Adding both sides of inequalities (2.19), (2.22), (2.25), and by (2.14),

$$\begin{aligned}&\left[\sum_{i=1}^{k} \lambda_i G'_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right)(\nabla f_i(\overline{x}) + w_i)\right.\\ &\left.\quad + \sum_{j=1}^{m} \xi_j G'_{g_j}(g_j(\overline{x})) \nabla g_j(\overline{x}) + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(\overline{x})) \nabla h_t(\overline{x})\right] \eta(x^*, \overline{x}) < 0,\end{aligned} \tag{2.26}$$

which contradicts (2.11). Hence, $\overline{x}$ is a weak Pareto optimal for (NMP). □

Theorem 2.4 (G-Karush-Kuhn-Tucker Sufficient Optimality Conditions). *Let* $(\overline{x}, \lambda, \xi, \mu, w)$ *satisfy the G-Karush-Kuhn-Tucker conditions as follow:*

$$\begin{aligned}&\sum_{i=1}^{k} \lambda_i G'_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right)(\nabla f_i(\overline{x}) + w_i) + \sum_{j=1}^{m} \xi_j G'_{g_j}(g_j(\overline{x})) \nabla g_j(\overline{x})\\ &\quad + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(\overline{x})) \nabla h_t(\overline{x}) = 0,\end{aligned} \tag{2.27}$$

$$\xi_j G_{g_j}(g_j(\overline{x})) = 0, \quad j \in J, \ \forall \overline{x} \in D, \tag{2.28}$$

$$\langle w_i, \overline{x} \rangle = s(\overline{x} \mid C_i), \quad i = 1, \ldots, k, \tag{2.29}$$

$$\lambda \geq 0, \quad \sum_{i=1}^{k} \lambda_i = 1, \quad \xi \geqq 0. \tag{2.30}$$

Further, assume that $F(= f(\cdot) + (\cdot)^T w)$ *is vector* G_F*-invex with respect to* η *at* $\overline{x}$ *on* D*,* g *is strictly* G_g*-invex with respect to* η *at* $\overline{x}$ *on* D*,* $h_t, t \in T^+(\overline{x})$*, is* G_{h_t}*-invex with respect to* η *at* $\overline{x}$ *on* D*, and* $h_t, t \in T^-(\overline{x})$*, is* G_{h_t}*-incave with respect to* η *at* $\overline{x}$ *on* D*. Moreover, suppose that* $G_{g_j}(0) = 0$ *for* $j \in J$ *and* $G_{h_t}(0) = 0$ *for* $t \in T^+(\overline{x}) \cup T^-(\overline{x})$*. Then* $\overline{x}$ *is a weak Pareto optimal point in problem (NMP).*

Proof. Suppose that $\overline{x}$ is not a weak Pareto optimal point in problem (NMP). Then there exists $x^* \in D$ such that $G_{F_i}(f_i(x^*) + s(x^* \mid C_i)) < G_{F_i}(f_i(\overline{x}) + s(\overline{x} \mid C_i))$, $i = 1, \ldots, k$. Since $\langle w_i, \overline{x} \rangle = s(\overline{x} \mid C_i)$, $i = 1, \ldots, k$,

$$\begin{aligned}G_{F_i}\left(f_i(x^*) + x^{*T} w_i\right) &< G_{F_i}(f_i(x^*) + s(x^* \mid C_i))\\ &< G_{F_i}(f_i(\overline{x}) + s(\overline{x} \mid C_i))\\ &= G_{F_i}\left(f_i(\overline{x}) + \overline{x}^T w_i\right).\end{aligned} \tag{2.31}$$

Thus we get

$$G_{F_i}\Big(f_i(x^*) + x^{*T}w_i\Big) < G_{F_i}\Big(f_i(\overline{x}) + \overline{x}^T w_i\Big), \quad i \in I. \tag{2.32}$$

By assumption, $F(= f(\cdot) + (\cdot)^T w)$ is G_F-invex with respect to η at $\overline{x}$ on D. Then by Definition 1.3, for any $i \in I$,

$$\begin{aligned}&\Big[G_{F_i}\Big(f_i(x^*) + x^{*T}w_i\Big)\Big] - \Big[G_{F_i}\Big(f_i(\overline{x}) + \overline{x}^T w_i\Big)\Big]\\ &\geqq \Big[G'_{F_i}\Big(f_i(\overline{x}) + \overline{x}^T w_i\Big)(\nabla f_i(\overline{x}) + w_i)\Big]\eta(x^*, \overline{x}).\end{aligned} \tag{2.33}$$

Hence by (2.32) and (2.33), we obtain

$$\Big[G'_{F_i}\Big(f_i(\overline{x}) + \overline{x}^T w_i\Big)(\nabla f_i(\overline{x}) + w_i)\Big]\eta(x^*, \overline{x}) < 0, \quad i \in I. \tag{2.34}$$

Since $(\overline{x}, \lambda, \xi, \mu, w)$ satisfy the G-Karush-Kuhn-Tucker conditions, by $\lambda \geq 0$,

$$\sum_{i=1}^{k}\lambda_i\Big[G'_{F_i}\Big(f_i(\overline{x}) + \overline{x}^T w_i\Big)(\nabla f_i(\overline{x}) + w_i)\Big]\eta(x^*, \overline{x}) < 0, \quad i \in I. \tag{2.35}$$

Since g is strictly G_g-invex with respect to η at $\overline{x}$ on D,

$$G_{g_j}(g_j(x^*)) - G_{g_j}(g_j(\overline{x})) > G'_{g_j}(g_j(\overline{x}))\nabla g_j(\overline{x})\eta(x^*, \overline{x}). \tag{2.36}$$

Thus, by $\xi \geqq 0$,

$$\xi_j G_{g_j}(g_j(x^*)) - \xi_j G_{g_j}(g_j(\overline{x})) \geqq \xi_j G'_{g_j}(g_j(\overline{x}))\nabla g_j(\overline{x})\eta(x^*, \overline{x}). \tag{2.37}$$

Then, (2.28),(2.30) imply

$$\sum_{j=1}^{m}\xi_j G'_{g_j}(g_j(\overline{x}))\nabla g_j(\overline{x})\eta(x^*, \overline{x}) \leqq 0. \tag{2.38}$$

By assumption, $h_t,\ t \in T^+(\overline{x})$, is G_{h_t}-invex with respect to η at $\overline{x}$ on D, and $h_t,\ t \in T^-(\overline{x})$, is G_{h_t}-incave with respect to η at $\overline{x}$ on D. Then, by Definition 1.3, we have,

$$\begin{aligned}G_{h_t}(h_t(x^*)) - G_{h_t}(h_t(\overline{x})) &\geqq G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x})\eta(x^*, \overline{x}), \quad t \in T^+(\overline{x}),\\ G_{h_t}(h_t(x^*)) - G_{h_t}(h_t(\overline{x})) &\leqq G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x})\eta(x^*, \overline{x}), \quad t \in T^-(\overline{x}).\end{aligned} \tag{2.39}$$

Thus, for any $t \in T^+$,

$$\mu_t G_{h_t}(h_t(x^*)) - \mu_t G_{h_t}(h_t(\overline{x})) \geqq \mu_t G'_{h_t}(h_t(\overline{x}))\nabla h_t(\overline{x})\eta(x^*, \overline{x}). \tag{2.40}$$

Since $x^* \in D$ and $\overline{x} \in D$, then the inequality above implies

$$\sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(\overline{x})) \nabla h_t(\overline{x}) \eta(x^*, \overline{x}) \leqq 0. \tag{2.41}$$

Adding both sides of inequalities (2.35), (2.38) and (2.41),

$$\begin{aligned} &\left[\sum_{i=1}^{k} \lambda_i G'_{F_i}\left(f_i(\overline{x}) + x^T w_i \right) \left(\nabla f_i(\overline{x}) + w_i \right) \right. \\ &\quad \left. + \sum_{j=1}^{m} \xi_j G'_{g_j}\left(g_j(\overline{x})\right) \nabla g_j(\overline{x}) + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(\overline{x})) \nabla h_t(\overline{x}) \right] \eta(x^*, \overline{x}) < 0, \end{aligned} \tag{2.42}$$

which contradicts (2.27). Hence, $\overline{x}$ is a weak Pareto optimal for (NMP). □

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