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Research Article

Generalized Bi-Quasivariational Inequalities for Quasi-Pseudomonotone Type II Operators on Noncompact Sets

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We prove some existence results of solutions for a new class of generalized bi-quasivariational inequalities (GBQVI) for quasi-pseudomonotone type II and strongly quasi-pseudomonotone type II operators defined on noncompact sets in locally convex Hausdorff topological vector spaces. To obtain these results on GBQVI for quasi-pseudomonotone type II and strongly quasi-pseudomonotone type II operators, we use Chowdhury and Tan's generalized version (1996) of Ky Fan's minimax inequality (1972) as the main tool.

1. Introduction and Preliminaries

In this paper, we obtain some results on generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type II and strongly quasi-pseudo-monotone type II operators defined on noncompact sets in locally convex Hausdorff topological vector spaces. Thus we begin this section by defining the generalized bi-quasi-variational inequalities. For this, we need to introduce some notations which will be used throughout this paper.

Let X be a nonempty set and let 2^X be the family of all nonempty subsets of X. If X and Y are topological spaces and $T : X \to 2^Y$, then the graph of T is the set $G(T) := \{(x, y) \in X \times Y : y \in T(x)\}$. Throughout this paper, Φ denotes either the real field R or the complex field C.

Let E be a topological vector space over Φ, let F be a vector space over Φ and let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional.

For any $x_0 \in E$, any nonempty subset A of E, and any $\epsilon > 0$, let $W(x_0; \epsilon) := \{y \in F : |\langle y, x_0 \rangle| < \epsilon\}$ and $U(A; \epsilon) := \{y \in F : \sup_{x \in A} |\langle y, x \rangle| < \epsilon\}$. Let $\sigma\langle F, E \rangle$ be the (weak) topology on F generated by the family $\{W(x;\epsilon) : x \in E \text{ and } \epsilon > 0\}$ as a subbase for the neighbourhood system at 0 and let $\delta\langle F, E\rangle$ be the (strong) topology on F generated by the family $\{U(A;\epsilon) :$ A is a nonempty bounded subset of E and $\epsilon > 0\}$ as a base for the neighbourhood system at 0. We note then that F, when equipped with the (weak) topology $\sigma\langle F, E\rangle$ or the (strong) topology $\delta\langle F, E\rangle$, becomes a locally convex topological vector space which is not necessarily Hausdorff. But, if the bilinear functional $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ separates points in F, that is, for any $y \in F$ with $y \neq 0$, there exists $x \in E$ such that $\langle y, x\rangle \neq 0$, then F also becomes Hausdorff. Furthermore, for any net $\{y_\alpha\}_{\alpha\in\Gamma}$ in F and $y \in F$,

(1) $y_\alpha \to y$ in $\sigma\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ for any $x \in E$,

(2) $y_\alpha \to y$ in $\delta\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ uniformly for any $x \in A$, where a nonempty bounded subset of E.

The generalized bi-quasi-variational inequality problem was first introduced by Shih and Tan [1] in 1989. Since Shih and Tan, some authors have obtained many results on generalized (quasi)variational inequalities, generalized (quasi)variational-like inequalities and generalized bi-quasi-variational inequalities (see [2–15]).

The following is the definition due to Shih and Tan [1].

Definition 1.1. Let E and F be a vector spaces over Φ, let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional, and let X be a nonempty subset of E. If $S : X \to 2^X$ and $M, T : X \to 2^F$, the *generalized bi-quasi variational inequality problem* (GBQVI) for the triple (S, M, T) is to find $\hat{y} \in X$ satisfying the following properties:

(1) $\hat{y} \in S(\hat{y})$,

(2) $\inf_{w\in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x\rangle \leq 0$ for any $x \in S(\hat{y})$ and $f \in M(\hat{y})$.

The following definition of the generalized bi-quasi-variational inequality problem is a slight modification of Definition 1.1.

Definition 1.2. Let E and F be vector spaces over Φ, let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional, and let X be a nonempty subset of E. If $S : X \to 2^X$ and $M, T : X \to 2^F$, then the *generalized bi-quasivariational inequality* (GBQVI) problem for the triple (S, M, T) is:

(1) to find a point $\hat{y} \in X$ and a point $\hat{w} \in T(\hat{y})$ such that

$$\hat{y} \in S(\hat{y}), \quad \operatorname{Re}\langle f - \hat{w}, \hat{y} - x\rangle \leq 0, \quad \forall x \in S(\hat{y}),\ f \in M(\hat{y}), \tag{1.1}$$

(2) to find a point $\hat{y} \in X$, a point $\hat{w} \in T(\hat{y})$, and a point $\hat{f} \in M(\hat{y})$ such that

$$\hat{y} \in S(\hat{y}), \quad \operatorname{Re}\left\langle \hat{f} - \hat{w}, \hat{y} - x\right\rangle \leq 0, \quad \forall x \in S(\hat{y}). \tag{1.2}$$

Let X be a nonempty subset of E and let $T : X \to 2^{E^*}$ be a set-valued mapping. Then T is said to be *monotone* on X if, for any $x, y \in X$, $u \in T(x)$, and $w \in T(y)$, $\operatorname{Re}\langle w - u, y - x\rangle \geq 0$.

Let X and Y be topological spaces and let $T : X \to 2^Y$ be a set-valued mapping. Then T is said to be:

(1) *upper* (resp., *lower*) *semicontinuous* at $x_0 \in X$ if, for each open set G in Y with $T(x_0) \subset G$ (resp., $T(x_0) \cap G \neq \emptyset$), there exists an open neighbourhood U of x_0 in X such that $T(x) \subset G$ (resp., $T(x) \cap G \neq \emptyset$) for all $x \in U$,

(2) *upper* (resp., *lower*) *semicontinuous* on X if T is upper (resp., lower) semicontinuous at each point of X,

(3) *continuous* on X if T is both lower and upper semi-continuous on X.

Let X be a convex set in a topological vector space E. Then $f : X \to R$ is said to be *lower semi-continuous* if, for all $\lambda \in R$, $\{x \in X : f(x) \leq \lambda\}$ is closed in X.

If X is a convex set in a vector space E, then $f : X \to R$ is said to be *concave* if, for all $x, y \in X$ and $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y). \tag{1.3}$$

Our main results in this paper are to obtain some existence results of solutions of the generalized bi-quasi-variational inequalities using Chowdhury and Tan's following definition of quasi-pseudo-monotone type II and strongly quasi-pseudo-monotone type II operators given in [3].

Definition 1.3. Let E be a topological vector space, let X be a nonempty subset of E, and let F be a topological vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional. Consider a mapping $h : X \to R$ and two set-valued mappings $M : X \to 2^F$ and $T : X \to 2^F$.

(1) T is called an *h-quasi-pseudo-monotone* (resp., *strongly h-quasi-pseudo-monotone*) *type II operator* if, for any $y \in X$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y (resp., weakly to y) with

$$\begin{gathered}
\limsup_{\alpha} \left[\inf_{f \in M(y_\alpha)} \inf_{u \in T(y_\alpha)} \operatorname{Re}\langle f - u, y_\alpha - y \rangle + h(y_\alpha) - h(y) \right] \leq 0, \\
\limsup_{\alpha} \left[\inf_{f \in M(y_\alpha)} \inf_{u \in T(y_\alpha)} \operatorname{Re}\langle f - u, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right] \\
\geq \inf_{f \in M(y)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x), \quad \forall x \in X.
\end{gathered} \tag{1.4}$$

(2) T is said to be a *quasi-pseudo-monotone* (resp., *strongly quasi-pseudo-monotone*) *type II operator* if T is an h-quasi-pseudo-monotone (resp., strongly h-quasi-pseudo-monotone) type II operator with $h \equiv 0$.

The following is an example on quasi-pseudo-monotone type II operators given in [3].

Example 1.4. Consider $X = [-1, 1]$ and $E = R$. Then $E^* = R$. Let $M : X \to 2^R$ be a set-valued mapping defined by

$$M(x) = \begin{cases} [0, 2x] & \text{if } x \geq 0, \\ [2x, 0] & \text{if } x < 0. \end{cases} \tag{1.5}$$

Again, let $T : X \to 2^R$ be a set-valued mapping defined by

$$T(x) = \begin{cases} \{1,3\} & \text{if } x < 1, \\ \{1,2,3\} & \text{if } x = 1. \end{cases} \tag{1.6}$$

Then M is lower semi-continuous and T is upper semi-continuous. It can be shown that T becomes a quasi-pseudo-monotone type II operator on $X = [-1,1]$.

(i) To show that M is lower semi-continuous, consider $x_0 \geq 0$. Then $M(x_0) = [0, 2x_0]$. Let $\epsilon > 0$ be given. Then, if $G = (2x_0 - \epsilon, 2x_0 + \epsilon)$, then $M(x_0) \cap G = [0, 2x_0] \cap (2x_0 - \epsilon, 2x_0 + \epsilon) \neq \emptyset$. Let $\epsilon > 0$ be so chosen that $0 < x_0 - \epsilon/2 < x_0 < x_0 + \epsilon/2 < 1$. Now, if we take $U = (x_0 - \epsilon/2, x_0 + \epsilon/2)$, then, for all $x \in U$, we have $2x_0 - \epsilon < 2x < 2x_0 + \epsilon$. Thus $2x \in M(x) \cap G$. Hence $M(x) \cap G \neq \emptyset$.

If $x_0 = 0$, $M(0) = 0$. Then, for $0 \in G = (\epsilon, \epsilon)$, we can take $U = (-\epsilon/2, \epsilon/2)$. Thus for all $x \in U$, $M(x) = [0, 2x] \cap (\epsilon, \epsilon) \neq \emptyset$ because $-\epsilon < x < \epsilon/2$ implies $2x \in G = (-\epsilon, \epsilon)$.

Finally, if $x_0 < 0$, then $M(x_0) = [2x_0, 0]$. We take $G = (2x_0 - \epsilon, 2x_0 + \epsilon)$ for some $\epsilon > 0$ so that $M(x_0) \cap G \neq \emptyset$ and $x_0 + \epsilon/2 < 0$. Thus, for all $x \in U = (x_0 - \epsilon/2, x_0 + \epsilon/2)$, we have $2x_0 - \epsilon < 2x < 2x_0 + \epsilon$. Hence $2x \in G \cap M(x)$, where $M(x) = [2x, 0]$ for $x < 0$. Consequently, M is lower semi-continuous on $X = [-1, 1]$.

(ii) To show that T is upper semi-continuous, let $x_0 \in [-1, 1]$ be such that $x_0 < 1$. Then $T(x_0) = \{1,3\}$. Let G be an open set in R such that $T(x_0) = \{1,3\} \subset G$. Let $\epsilon > 0$ be such that $-1 < x_0 - \epsilon < x_0 < x_0 + \epsilon < 1$. Consider $U = (x_0 - \epsilon, x_0 + \epsilon)$. Then, for all $x \in U$, $T(x) = \{1,3\} \subset G$ since $x < 1$. Again, if $x = 1$, then $T(1) = \{1,2,3\}$. Let G be an open set in R such that $T(1) = \{1,2,3\} \subset G$. Let $\epsilon > 0$ be such that $-1 < 1 - \epsilon < 1 < 1 + \epsilon$. Let $U = (1 - \epsilon, 1]$ which is an open neighbourhood of 1 in $X = [-1, 1]$. Then for all $x \in U = (1 - \epsilon, 1]$, we have $T(x) = \{1,3\}$ if $1 - \epsilon < x < 1$ and $T(x) = \{1,2,3\}$ if $x = 1$. Now, $T(1) = \{1,2,3\} \subset G$. Also, for all $x \in U$ with $1 - \epsilon < x < 1$, we have $T(x) = \{1,3\} \subset \{1,2,3\} \subset G$. Hence T is upper semi-continuous on $X = [-1, 1]$.

(iii) Finally, we will show that T is also a quasi-pseudo-monotone type II operator. To show this, let us assume first that $\langle y_\alpha \rangle$ is a net in $X = [-1, 1]$ such that $y_\alpha \to y$ in $X = [-1, 1]$. We now show that

$$\limsup_{\alpha} \left[\inf_{f \in M(y_\alpha)} \inf_{u \in T(y_\alpha)} \operatorname{Re}\langle f - u, y_\alpha - y \rangle \right] \leq 0. \tag{1.7}$$

We have

$$\begin{aligned} &\limsup_{\alpha} \left[\inf_{f \in M(y_\alpha)} \inf_{u \in T(y_\alpha)} \operatorname{Re}\langle f - u, y_\alpha - y \rangle \right] \\ &= \limsup_{\alpha} \begin{cases} \left[\inf\limits_{f \in [0, 2y_\alpha]} \inf\limits_{u \in \{1,3\}} (f - u)(y_\alpha - y) \right] \\ = (0 - 3)(y_\alpha - y) = 3(y - y_\alpha), & \text{if } 0 \leq y_\alpha < 1, \\ \left[\inf\limits_{f \in [2y_\alpha, 0]} \inf\limits_{u \in \{1,3\}} (f - u)(y_\alpha - y) \right] \\ = (2y_\alpha - 3)(y_\alpha - y), & \text{if } y_\alpha < 0 \text{ and so } y_\alpha < 1, \\ \left[\inf\limits_{f \in [0, 2y_\alpha]} \inf\limits_{u \in \{1,2,3\}} (f - u)(y_\alpha - y) \right] \\ = (0 - 3)(y_\alpha - y) = 3(y - y_\alpha), & \text{if } y_\alpha = 1, \text{ that is, } y_\alpha \geq 0 \end{cases} \\ &= 0 \end{aligned} \tag{1.8}$$

(considering $y_\alpha - y \geq 0$, the value will be also 0 if we consider $y_\alpha - y \leq 0$). So, it follows that, for all $x \in [-1,1]$,

$$\limsup_{\alpha}\left[\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}(f-u)(y_\alpha - x)\right]$$

$$= \limsup_{\alpha}\begin{cases}\left[\inf_{f\in[0,2y_\alpha]}\inf_{u\in\{1,3\}}(f-u)(y_\alpha - x)\right] & \\ = (0-3)(y_\alpha - x) = 3(x-y_\alpha), & \text{if } 0 \leq y_\alpha < 1, \\ \left[\inf_{f\in[2y_\alpha,0]}\inf_{u\in\{1,3\}}(f-u)(y_\alpha - x)\right] & \\ = (2y_\alpha - 3)(y_\alpha - x), & \text{if } y_\alpha < 0 \text{ and so } y_\alpha < 1, \\ \left[\inf_{f\in[0,2y_\alpha]}\inf_{u\in\{1,2,3\}}(f-u)(y_\alpha - x)\right] & \\ = (0-3)(y_\alpha - x) = 3(x-y_\alpha), & \text{if } y_\alpha = 1, \text{ that is, } y_\alpha \geq 0, \\ (\text{consider } y_\alpha - x \geq 0) & \end{cases} \tag{1.9}$$

$$= \begin{cases} 3(x-y), & \text{if } 0 \leq y_\alpha < 1, \\ (2y-3)(y-x), & \text{if } y_\alpha < 0 \text{ and so } y_\alpha < 1, \\ 3(x-y), & \text{if } y_\alpha = 1, \\ (\text{consider } y - x \geq 0). & \end{cases}$$

The values can be obtained similarly for the cases where $y_\alpha - x \leq 0$ and $y - x \leq 0$. Also, it follows that, for all $x \in X = [-1,1]$,

$$\inf_{f\in M(y)}\inf_{u\in T(y)}(f-u)(y-x)$$

$$= \begin{cases}\left[\inf_{f\in[0,2y]}\inf_{u\in\{1,3\}}(f-u)(y - x)\right] & \\ = (0-3)(y - x) = 3(x-y), & \text{if } 0 \leq y < 1, \\ \left[\inf_{f\in[2y,0]}\inf_{u\in\{1,3\}}(f-u)(y - x)\right] & \\ = (2y - 3)(y - x), & \text{if } y < 0 \text{ and so } y < 1, \\ \left[\inf_{f\in[0,2y]}\inf_{u\in\{1,2,3\}}(f-u)(y - x)\right] & \\ = (0-3)(y - x) = 3(x-y), & \text{if } y = 1, \text{ that is, } y \geq 0, \\ (\text{consider } y - x \geq 0) & \end{cases} \tag{1.10}$$

$$= \begin{cases} 3(x-y), & \text{if } 0 \leq y < 1, \\ (2y-3)(y-x), & \text{if } y < 0 \text{ and so } y < 1, \\ 3(x-y), & \text{if } y = 1, \\ (\text{consider } y - x \geq 0). & \end{cases}$$

The values can be obtained similarly for the cases when $y - x \leq 0$. Therefore, in all the cases, we have shown that

$$\limsup_{\alpha}\left[\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}(f-u)(y_\alpha - x)\right] \geq \inf_{f\in M(y)}\inf_{u\in T(y)}(f-u)(y-x). \tag{1.11}$$

Hence T is a quasi-pseudo-monotone type II operator.

The above example is a particular case of a more general result on quasi-pseudo-monotone type II operators. We will establish this result in the following proposition.

Proposition 1.5. *Let X be a nonempty compact subset of a topological vector space E. Suppose that $M : X \to 2^{E^*}$ and $T : X \to 2^{E^*}$ are two set-valued mappings such that M is lower semi-continuous and T is upper semi-continuous. Suppose further that, for any $x \in X$, $M(x)$ and $T(x)$ are weak*-compact sets in E^*. Then T is both a quasi-pseudo-monotone type II and a strongly quasi-pseudo-monotone type II operator.*

Proof. Suppose that $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X and $y \in X$ with $y_\alpha \to y$ (resp., $y_\alpha \to y$ weakly) and $\limsup_\alpha[\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y_\alpha - y\rangle] \leq 0$. Then it follows that, for any $x \in X$,

$$\begin{aligned}
&\limsup_{\alpha\in\Gamma}\left[\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y_\alpha - x\rangle\right]\\
&\quad\geq \liminf_{\alpha\in\Gamma}\left[\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y_\alpha - x\rangle\right]\\
&\quad\geq \liminf_{\alpha\in\Gamma}\left[\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y_\alpha - y\rangle\right]\\
&\qquad+ \liminf_{\alpha\in\Gamma}\left[\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y - x\rangle\right]\\
&\quad\geq 0 + \liminf_{\alpha\in\Gamma}\left[\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y - x\rangle\right]\\
&\quad= \inf_{f\in M(y)}\inf_{u\in T(y)}\operatorname{Re}\langle f-u, y-x\rangle.
\end{aligned} \tag{1.12}$$

To obtain the above inequalities, we use the following facts. For any $\alpha \in \Gamma$, $u_\alpha \in T(y_\alpha)$ and $f_\alpha \in M(y_\alpha)$. Since X is compact and $T(x)$ and $M(x)$ are weak*-compact valued for any $x \in X$, using the lower semicontinuity of M and the upper semicontinuity of T it can be shown that (details can be verified by the reader easily) $u_\alpha \to u \in T(y)$ and $f_\alpha \to f \in M(y)$. Thus we obtain

$$\liminf_{\alpha\in\Gamma}\left[\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y-x\rangle\right] = \inf_{f\in M(y)}\inf_{u\in T(y)}\operatorname{Re}\langle f-u, y-x\rangle \tag{1.13}$$

in the last inequality above. Consequently, T is both a quasi-pseudo-monotone type II and a strongly quasi-pseudo-monotone type II operator. □

In Section 3 of this paper, we obtain some general theorems on solutions for a new class of generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type II and strongly quasi-pseudo-monotone type II operators defined on noncompact sets in topological vector spaces. To obtain these results, we mainly use the following generalized version of Ky Fan's minimax inequality [16] due to Chowdhury and Tan [17].

Theorem 1.6. *Let E be a topological vector space, let X be a nonempty convex subset of E, and let $f : X \times X \to R \cup \{-\infty, +\infty\}$ be such that*

(a) *for any $A \in F(X)$ and fixed $x \in \text{co}(A)$, $y \mapsto f(x, y)$ is lower semi-continuous on $\text{co}(A)$,*

(b) *for any $A \in F(X)$ and $y \in \text{co}(A)$, $\min_{x\in A} f(x, y) \le 0$,*

(c) *for any $A \in F(X)$ and $x, y \in \text{co}(A)$, every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y with $f(tx + (1-t)y, y_\alpha) \le 0$ for all $\alpha \in \Gamma$ and $t \in [0,1]$, one has $f(x, y) \le 0$,*

(d) *there exist a nonempty closed and compact subset K of X and $x_0 \in K$ such that $f(x_0, y) > 0$ for all $y \in X \setminus K$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \le 0$ for all $x \in X$.

Now, we use the following lemmas for our main results in this paper.

Lemma 1.7 (see [18]). *Let X be a nonempty subset of a Hausdorff topological vector space E and let $S : X \to 2^E$ be an upper semi-continuous mapping such that $S(x)$ is a bounded subset of E for any $x \in X$. Then, for any continuous linear functional p on E, the mapping $f_p : X \to R$ defined by $f_p(y) = \sup_{x\in S(y)} \text{Re}\langle p, x\rangle$ is upper semi-continuous; that is, for any $\lambda \in R$, the set $\{y \in X : f_p(y) = \sup_{x\in S(y)} \text{Re}\langle p, x\rangle < \lambda\}$ is open in X.*

Lemma 1.8 (see [1, 19]). *Let X and Y be topological spaces, let $f : X \to R$ be nonnegative and continuous and let $g : Y \to R$ be lower semi-continuous. Then the mapping $F : X \times Y \to R$ defined by $F(x, y) = f(x)g(y)$ for all $(x, y) \in X \times Y$ is lower semi-continuous.*

Theorem 1.9 (see [20, 21]). *Let X be a nonempty convex subset of a vector space and let Y be a nonempty compact convex subset of a Hausdorff topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that, for each fixed $x \in X$, the mapping $y \mapsto f(x, y)$, that is, $f(x, \cdot)$ is lower semi-continuous and convex on Y and, for each fixed $y \in Y$, the mapping $x \mapsto f(x, y)$, that is, $f(\cdot, y)$ is concave on X. Then*

$$\min_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \min_{y\in Y} f(x, y). \tag{1.14}$$

2. Existence Results

In this section, we will obtain and prove some existence theorems for the solutions to the generalized bi-quasi-variational inequalities of quasi-pseudo-monotone type II and strongly quasi-pseudo-monotone type II operators with noncompact domain in locally convex Hausdorff topological vector spaces. Our results extend and/or generalize the corresponding results in [1].

Before we establish our main results, we state the following result which is Lemma 3.1 in [3].

Lemma 2.1. *Let E be a Hausdorff topological vector space over Φ, let F be a vector space over Φ, and let X be a nonempty compact subset of E. Let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle\cdot,\cdot\rangle$ separates points in F. Suppose that the F equips with the $\sigma\langle F, E\rangle$-topology; for any $w \in F$, $x \mapsto \operatorname{Re}\langle w, x\rangle$ is continuous on E and T, $M : X \to 2^F$ are upper semi-continuous maps such that $T(x)$ and $M(x)$ are compact for any $x \in X$. Let $x_0 \in X$ and $h : X \to R$ be continuous. Define a mapping $g : X \to R$ by*

$$g(y) = \left[\inf_{f\in M(y)} \inf_{w\in T(y)} \operatorname{Re}\langle f - w, y - x_0\rangle\right] + h(y) - h(x_0), \quad \forall y \in X. \tag{2.1}$$

Suppose that $\langle\cdot,\cdot\rangle$ is continuous over the (compact) subset $[\bigcup_{y\in X} M(y) - \bigcup_{y\in X} T(y)] \times X$ of $F \times E$. Then g is lower semi-continuous on X.

Now, we establish our first main result as follows.

Theorem 2.2. *Let E be a locally convex Hausdorff topological vector space over Φ, let X be a nonempty paracompact convex and bounded subset of E, and let F be a Hausdorff topological vector space over Φ. Let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional which is continuous over compact subsets of $F \times X$. Suppose that*

(a) *$S : X \to 2^X$ is upper semi-continuous such that each $S(x)$ is compact and convex,*

(b) *$h : E \to R$ is convex and $h(X)$ is bounded,*

(c) *$T : X \to 2^F$ is an h-quasi-pseudo-monotone type II (resp., strongly h-quasi-pseudo-monotone type II) operator and is upper semi-continuous such that each $T(x)$ is compact (resp., weakly compact) and convex and $T(X)$ is strongly bounded,*

(d) *$M : X \to 2^F$ is an upper semi-continuous mapping such that each $M(x)$ is weakly compact and convex,*

(e) *the set $\Sigma = \{y \in X : \sup_{x\in S(y)}(\inf_{f\in M(y)}\inf_{u\in T(y)} \operatorname{Re}\langle f - u, y - x\rangle + h(y) - h(x)) > 0\}$ is open in X.*

Suppose further that there exist a nonempty closed and compact (resp., weakly closed and weakly compact) subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and

$$\inf_{w\in T(y)} \inf_{f\in M(y)} \operatorname{Re}\langle f - w, y - x_0\rangle + h(y) - h(x_0) > 0, \quad \forall y \in X \setminus K. \tag{2.2}$$

Then there exists a point $\hat{y} \in X$ such that

(1) *$\hat{y} \in S(\hat{y})$,*

(2) *there exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that*

$$\operatorname{Re}\left\langle \hat{f} - \hat{w}, \hat{y} - x\right\rangle \le h(x) - h(\hat{y}), \quad \forall x \in S(\hat{y}). \tag{2.3}$$

Moreover, if $S(x) = X$ for all $x \in X$, then E is not required to be locally convex, and if $T \equiv 0$, then the continuity assumption on $\langle\cdot,\cdot\rangle$ can be weakened to the assumption that, for any $f \in F$, the mapping $x \mapsto \langle f, x\rangle$ is continuous (resp., weakly continuous) on X.

Proof. We need to show that there exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x\in S(\hat{y})}\left[\inf_{f\in M(\hat{y})}\inf_{u\in T(\hat{y})}\operatorname{Re}\langle f-u,\hat{y}-x\rangle+h(\hat{y})-h(x)\right]\leq 0. \tag{2.4}$$

Suppose the contrary. Then, for any $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that

$$\inf_{f\in M(y)}\inf_{u\in T(y)}\operatorname{Re}\langle f-u,y-x\rangle+h(y)-h(x)>0, \tag{2.5}$$

that is, for any $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then, by a separation theorem for convex sets in locally convex Hausdorff topological vector spaces, there exists $p \in E^*$ such that

$$\operatorname{Re}\langle p,y\rangle-\sup_{x\in S(y)}\operatorname{Re}\langle p,x\rangle>0. \tag{2.6}$$

Let

$$\begin{gathered}\gamma(y)=\sup_{x\in S(y)}\inf_{f\in M(y)}\inf_{u\in T(y)}\operatorname{Re}\langle f-u,y-x\rangle+h(y)-h(x),\\ V_0:=\{y\in X:\gamma(y)>0\}=\Sigma,\end{gathered} \tag{2.7}$$

and, for any $p \in E^*$, set

$$V_p:=\left\{y\in X:\operatorname{Re}\langle p,y\rangle-\sup_{x\in S(y)}\operatorname{Re}\langle p,x\rangle>0\right\}. \tag{2.8}$$

Then $X = V_0 \cup \bigcup_{p\in E^*} V_p$. Since each V_p is open in X by Lemma 1.7 and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X. Since X is paracompact, there exists a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the covering $\{V_0, V_p : p \in E^*\}$ (see Dugundji [22, Theorem VIII, 4.2]); that is, for any $p \in E^*$, $\beta_p : X \to [0,1]$ and $\beta_0 : X \to [0,1]$ are continuous functions such that, for any $p \in E^*$, $\beta_p(y) = 0$ for all $y \in X \setminus V_p$ and $\beta_0(y) = 0$ for all $y \in X \setminus V_0$ and $\{\text{support } \beta_0, \text{support } \beta_p : p \in E^*\}$ is locally finite and $\beta_0(y) + \sum_{p\in E^*} \beta_p(y) = 1$ for any $y \in X$. Note that, for any $A \in F(X)$, h is continuous on $\operatorname{co}(A)$ (see [23, Corollary 10.1.1]). Define a mapping $\phi : X \times X \to R$ by

$$\begin{aligned}\phi(x,y)=\beta_0(y)&\left[\inf_{f\in M(y)}\inf_{u\in T(y)}\operatorname{Re}\langle f-u,y-x\rangle+h(y)-h(x)\right]\\&+\sum_{p\in E^*}\beta_p(y)\operatorname{Re}\langle p,y-x\rangle,\quad\forall x,y\in X.\end{aligned} \tag{2.9}$$

Then we have the following.

(i) Since E is Hausdorff, for any $A \in F(X)$ and fixed $x \in \mathrm{co}(A)$, the mapping

$$y \longmapsto \inf_{f \in M(y)} \inf_{u \in T(y)} \mathrm{Re}\langle f - u, y - x\rangle + h(y) - h(x) \tag{2.10}$$

is lower semi-continuous (resp., weakly lower semi-continuous) on $\mathrm{co}(A)$ by Lemma 2.1 and so the mapping

$$y \longmapsto \beta_0(y)\left[\inf_{f \in M(y)} \inf_{u \in T(y)} \mathrm{Re}\langle f - u, y - x\rangle + h(y) - h(x)\right] \tag{2.11}$$

is lower semi-continuous (resp., weakly lower semi-continuous) on $\mathrm{co}(A)$ by Lemma 1.8. Also, for any fixed $x \in X$,

$$y \longmapsto \sum_{p \in E^*} \beta_p(y)\,\mathrm{Re}\langle p, y - x\rangle \tag{2.12}$$

is continuous on X. Hence, for any $A \in F(X)$ and fixed $x \in \mathrm{co}(A)$, the mapping $y \mapsto \phi(x, y)$ is lower semi-continuous (resp., weakly lower semi-continuous) on $\mathrm{co}(A)$.

(ii) For any $A \in F(X)$ and $y \in \mathrm{co}(A)$, $\min_{x \in A}\phi(x, y) \le 0$. Indeed, if this is false, then, for some $A = \{x_1, x_2, \ldots, x_n\} \in F(X)$ and $y \in \mathrm{co}(A)$ (say $y = \sum_{i=1}^{n} \lambda_i x_i$ where $\lambda_1, \lambda_2, \ldots, \lambda_n \ge 0$ with $\sum_{i=1}^{n} \lambda_i = 1$), we have $\min_{1 \le i \le n}\phi(x_i, y) > 0$. Then, for any $i = 1, 2, \ldots, n$,

$$\beta_0(y)\left[\inf_{f \in M(y)} \inf_{u \in T(y)} \mathrm{Re}\langle f - u, y - x_i\rangle + h(y) - h(x_i)\right] + \sum_{p \in E^*} \beta_j(y)\,\mathrm{Re}\langle p, y - x_i\rangle > 0, \tag{2.13}$$

which implies that

$$\begin{aligned}
0 &= \phi(y, y) \\
&= \beta_0(y)\left[\inf_{f \in M(y)} \inf_{u \in T(y)} \mathrm{Re}\left\langle f - u, y - \sum_{i=1}^{n}\lambda_i x_i\right\rangle + h(y) - h\left(\sum_{i=1}^{n}\lambda_i x_i\right)\right] \\
&\quad + \sum_{p \in E^*} \beta_j(y)\,\mathrm{Re}\left\langle p, y - \sum_{i=1}^{n}\lambda_i x_i\right\rangle \\
&\ge \sum_{i=1}^{n}\lambda_i\left\{\left(\beta_0(y)\left[\inf_{f \in M(y)} \inf_{u \in T(y)} \mathrm{Re}\langle f - u, y - x_i\rangle + h(y) - h(x_i)\right]\right.\right. \\
&\qquad\qquad \left.\left. + \sum_{p \in E^*} \beta_p(y)\,\mathrm{Re}\langle p, y - x_i\rangle\right)\right\} \\
&> 0,
\end{aligned} \tag{2.14}$$

which is a contradiction.

(iii) Suppose that $A \in A(X)$, $x, y \in \mathrm{co}(A)$, and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X converging to y (resp., weakly to y) with $\phi(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and $t \in [0,1]$.

Case 1 ($\beta_0(y) = 0$). Note that $\beta_0(y_\alpha) \geq 0$ for any $\alpha \in \Gamma$ and $\beta_0(y_\alpha) \to 0$. Since $T(X)$ is strongly bounded and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a bounded net, it follows that

$$\limsup_{\alpha}\left[\beta_0(y_\alpha)\left(\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\mathrm{Re}\langle f-u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right)\right] = 0. \tag{2.15}$$

Also, we have

$$\beta_0(y)\left[\min_{f\in M(y)}\min_{u\in T(y)}\mathrm{Re}\langle f-u, y-x\rangle + h(y) - h(x)\right] = 0. \tag{2.16}$$

Thus, from (2.15), it follows that

$$\begin{aligned}
\limsup_{\alpha}&\left[\beta_0(y_\alpha)\left(\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\mathrm{Re}\langle f-u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right)\right]\\
&\quad + \sum_{p\in E^*}\beta_p(y)\,\mathrm{Re}\langle p, y-x\rangle = \sum_{p\in E^*}\beta_p(y)\,\mathrm{Re}\langle p, y-x\rangle\\
&= \beta_0(y)\left[\min_{f\in M(y)}\min_{u\in T(y)}\mathrm{Re}\langle f-u, y-x\rangle + h(y) - h(x)\right]\\
&\quad + \sum_{p\in E^*}\beta_p(y)\,\mathrm{Re}\langle p, y-x\rangle.
\end{aligned} \tag{2.17}$$

When $t = 1$, we have $\phi(x, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, that is,

$$\beta_0(y_\alpha)\left[\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\mathrm{Re}\langle f-u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] + \sum_{p\in E^*}\beta_p(y_\alpha)\,\mathrm{Re}\langle p, y_\alpha - x\rangle \leq 0. \tag{2.18}$$

Therefore, by (2.18), we have

$$\begin{aligned}
\limsup_{\alpha}&\left[\beta_0(y_\alpha)\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\mathrm{Re}\langle f-u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right]\\
&\quad + \liminf_{\alpha}\left[\sum_{p\in E^*}\beta_p(y_\alpha)\,\mathrm{Re}\langle p, y_\alpha - x\rangle\right]\\
&\leq \limsup_{\alpha}\left[\beta_0(y_\alpha)\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\mathrm{Re}\langle f-u, y_\alpha - x\rangle + h(y_\alpha) - h(x) + \sum_{p\in E^*}\beta_p(y_\alpha)\,\mathrm{Re}\langle p, y_\alpha - x\rangle\right]\\
&\leq 0,
\end{aligned} \tag{2.19}$$

which implies that

$$\limsup_{\alpha}\left[\beta_0(y_\alpha)\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u,y_\alpha-x\rangle+h(y_\alpha)-h(x)\right]+\sum_{p\in E^*}\beta_p(y)\operatorname{Re}\langle p,y-x\rangle\le 0. \tag{2.20}$$

Hence, by (2.17) and (2.20), we have $\phi(x,y)\le 0$.

Case 2 ($\beta_0(y)>0$). Since $\beta_0(y_\alpha)\to\beta_0(y)$, there exists $\lambda\in\Gamma$ such that $\beta_0(y_\alpha)>0$ for any $\alpha\ge\lambda$. When $t=0$, we have $\phi(y,y_\alpha)\le 0$ for all $\alpha\in\Gamma$, that is,

$$\beta_0(y_\alpha)\left[\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u,y_\alpha-y\rangle+h(y_\alpha)-h(y)\right]+\sum_{p\in E^*}\beta_p(y_\alpha)\operatorname{Re}\langle p,y_\alpha-y\rangle\le 0. \tag{2.21}$$

Thus it follows that

$$\limsup_{\alpha}\left[\beta_0(y_\alpha)\left(\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}\Re\langle f-u,y_\alpha-y\rangle+h(y_\alpha)-h(y)\right)+\sum_{p\in E^*}\beta_p(y_\alpha)\operatorname{Re}\langle p,y_\alpha-y\rangle\right]\le 0. \tag{2.22}$$

Hence, by (2.22), we have

$$\begin{aligned}
&\limsup_{\alpha}\left[\beta_0(y_\alpha)\left(\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u,y_\alpha-y\rangle+h(y_\alpha)-h(y)\right)\right]\\
&\quad+\liminf_{\alpha}\left[\sum_{p\in E^*}\beta_p(y_\alpha)\operatorname{Re}\langle p,y_\alpha-y\rangle\right]\\
&\le\limsup_{\alpha}\left[\beta_0(y_\alpha)\left(\inf_{f\in M(y_\alpha)}\inf_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u,y_\alpha-y\rangle+h(y_\alpha)-h(y)\right)\right.\\
&\qquad\left.+\sum_{p\in E^*}\beta_p(y_\alpha)\operatorname{Re}\langle p,y_\alpha-y\rangle\right]\\
&\le 0.
\end{aligned} \tag{2.23}$$

Since $\liminf_\alpha[\sum_{p\in E^*}\beta_p(y_\alpha)\operatorname{Re}\langle p,y_\alpha-y\rangle]=0$, we have

$$\limsup_{\alpha}\left[\beta_0(y_\alpha)\left(\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u,y_\alpha-y\rangle+h(y_\alpha)-h(y)\right)\right]\le 0. \tag{2.24}$$

Since $\beta_0(y_\alpha) > 0$ for all $\alpha \geq \lambda$, it follows that

$$\beta_0(y)\limsup_{\alpha}\left[\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right]$$
$$= \limsup_{\alpha}\left[\beta_0(y_\alpha)\left(\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right)\right]. \tag{2.25}$$

Since $\beta_0(y) > 0$, by (2.24) and (2.25), we have

$$\limsup_{\alpha}\left[\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right] \leq 0. \tag{2.26}$$

Since T is an h-quasi-pseudo-monotone type II (resp., strongly h-quasi-pseudo-monotone type (II) operator, we have

$$\limsup_{\alpha}\left[\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right]$$
$$\geq \min_{f\in M(y)}\min_{w\in T(y)}\operatorname{Re}\langle f-w, y - x\rangle + h(y) - h(x), \quad \forall x \in X. \tag{2.27}$$

Since $\beta_0(y) > 0$, we have

$$\beta_0(y)\left[\limsup_{\alpha}\left(\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right)\right]$$
$$\geq \beta_0(y)\left[\min_{f\in M(y)}\min_{w\in T(y)}\operatorname{Re}\langle f-w, y - x\rangle + h(y) - h(x)\right] \tag{2.28}$$

and so

$$\beta_0(y)\left[\limsup_{\alpha}\left(\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right)\right]$$
$$+ \sum_{p\in E^*}\beta_p(y)\operatorname{Re}\langle p, y - x\rangle$$
$$\geq \beta_0(y)\left[\min_{f\in M(y)}\min_{w\in T(y)}\operatorname{Re}\langle f-w, y - x\rangle + h(y) - h(x)\right]$$
$$+ \sum_{p\in E^*}\beta_p(y)\operatorname{Re}\langle p, y - x\rangle. \tag{2.29}$$

When $t = 1$, we have $\phi(x, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, that is,

$$\beta_0(y_\alpha)\left[\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] + \sum_{p\in E^*}\beta_p(y_\alpha)\operatorname{Re}\langle p, y_\alpha - x\rangle \leq 0 \tag{2.30}$$

and so, by (2.29),

$$\begin{aligned}0 &\geq \limsup_{\alpha}\left[\beta_0(y_\alpha)\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u,y_\alpha-x\rangle+h(y_\alpha)-h(x)+\sum_{p\in E^*}\beta_p(y_\alpha)\operatorname{Re}\langle p,y_\alpha-x\rangle\right]\\ &\geq \limsup_{\alpha}\left[\beta_0(y_\alpha)\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u,y_\alpha-x\rangle+h(y_\alpha)-h(x)\right]\\ &\quad+\liminf_{\alpha}\left[\sum_{p\in E^*}\beta_p(y_\alpha)\operatorname{Re}\langle p,y_\alpha-x\rangle\right]\\ &=\beta_0(y)\left[\limsup_{\alpha}\left\{\min_{f\in M(y_\alpha)}\min_{u\in T(y_\alpha)}\operatorname{Re}\langle f-u,y_\alpha-x\rangle+h(y_\alpha)-h(x)\right\}\right]+\sum_{p\in E^*}\beta_p(y)\operatorname{Re}\langle p,y-x\rangle\\ &\geq \beta_0(y)\left[\min_{f\in M(y)}\min_{w\in T(y)}\operatorname{Re}\langle f-w,y-x\rangle+h(y)-h(x)\right]+\sum_{p\in E^*}\beta_p(y)\operatorname{Re}\langle p,y-x\rangle.\end{aligned} \tag{2.31}$$

Hence we have $\phi(x,y)\leq 0$.

(iv) By the hypothesis, there exists a nonempty compact and so a closed (resp., weakly closed and weakly compact) subset K of X and a point $x_0\in X$ such that $x_0\in K\cap S(y)$ and

$$\inf_{f\in M(y)}\inf_{w\in T(y)}\operatorname{Re}\langle f-w,y-x_0\rangle+h(y)-h(x_0)>0,\quad \forall y\in X\setminus K. \tag{2.32}$$

Thus it follows that

$$\beta_0(y)\left[\inf_{w\in T(y)}\inf_{f\in M(y)}\operatorname{Re}\langle f-w,y-x_0\rangle+h(y)-h(x_0)\right]>0,\quad \forall y\in X\setminus K, \tag{2.33}$$

whenever $\beta_0(y)>0$ and $\operatorname{Re}\langle p,y-x_0\rangle>0$ whenever $\beta_p(y)>0$ for all $p\in E^*$. Consequently, we have

$$\begin{aligned}\phi(x_0,y)=\beta_0(y)&\left[\inf_{f\in M(y)}\inf_{w\in T(y)}\operatorname{Re}\langle f-w,y-x_0\rangle+h(y)-h(x_0)\right]\\ &+\sum_{p\in E^*}\beta_p(y)\operatorname{Re}\langle p,y-x_0\rangle>0,\quad \forall y\in X\setminus K.\end{aligned} \tag{2.34}$$

(If T is a strongly h-quasi-pseudo-monotone type II operator, then we equip E with the weak topology.) Thus ϕ satisfies all the hypotheses of Theorem 1.6 and so, by Theorem 1.6, there exists a point $\hat{y}\in K$ such that $\phi(x,\hat{y})\leq 0$ for all $x\in X$, that is,

$$\beta_0(\hat{y})\left[\inf_{f\in M(\hat{y})}\inf_{u\in T(\hat{y})}\operatorname{Re}\langle f-u,\hat{y}-x\rangle+h(\hat{y})-h(x)\right]+\sum_{p\in E^*}\beta_p(\hat{y})\operatorname{Re}\langle p,\hat{y}-x\rangle\leq 0,\quad \forall x\in X. \tag{2.35}$$

Now, the rest of the proof is similar to the proof in Step 1 of Theorem 1 in [24]. Hence we have shown that

$$\sup_{x\in S(\hat{y})}\left[\inf_{f\in M(\hat{y})}\inf_{u\in T(\hat{y})}\operatorname{Re}\langle f-u,\hat{y}-x\rangle+h(\hat{y})-h(x)\right]\leq 0. \tag{2.36}$$

Then, by applying Theorem 1.9 as we proved in Step 3 of Theorem 1 in [24], we can show that there exist a point $\hat{f}\in M(\hat{y})$ and a point $\hat{w}\in T(\hat{y})$ such that

$$\operatorname{Re}\left\langle \hat{f}-\hat{w},\hat{y}-x\right\rangle \leq h(x)-h(\hat{y}),\quad \forall x\in S(\hat{y}). \tag{2.37}$$

We observe from the above proof that the requirement that E is locally convex is needed if and only if the separation theorem is applied to the case $y\notin S(y)$. Thus, if $S:X\to 2^X$ is the constant mapping $S(x)=X$ for all $x\in X$, the E is not required to be locally convex.

Finally, if $T\equiv 0$, in order to show that, for any $x\in X$, $y\mapsto\phi(x,y)$ is lower semi-continuous (resp., weakly lower semi-continuous), Lemma 2.1 is no longer needed and the weaker continuity assumption on $\langle\cdot,\cdot\rangle$ that, for any $f\in F$, the mapping $x\mapsto\langle f,x\rangle$ is continuous (resp., weakly continuous) on X is sufficient. This completes the proof. □

We will now establish our last result of this section.

Theorem 2.3. *Let E be a locally convex Hausdorff topological vector space over Φ, let X be a nonempty paracompact convex and bounded subset of E, and let F be a vector space over Φ. Let $\langle\cdot,\cdot\rangle:F\times E\to\Phi$ be a bilinear functional such that $\langle\cdot,\cdot\rangle$ separates points in F, $\langle\cdot,\cdot\rangle$ is continuous over compact subsets of $F\times X$, and, for any $f\in F$, the mapping $x\mapsto\langle f,x\rangle$ is continuous on X. Suppose that F equips with the strong topology $\delta\langle F,E\rangle$ and*

(a) *$S:X\to 2^X$ is a continuous mapping such that each $S(x)$ is compact and convex,*

(b) *$h:X\to R$ is convex and $h(X)$ is bounded,*

(c) *$T:X\to 2^F$ is an h-quasi-pseudo-monotone type II (resp., strongly h-quasi-pseudo-monotone type II) operator and is an upper semi-continuous mapping such that each $T(x)$ is strongly, that is, $\delta\langle F,E\rangle$-compact and convex (resp., weakly, i.e., $\sigma\langle F,E\rangle$-compact and convex),*

(d) *$M:X\to 2^F$ is an upper semi-continuous mapping such that each $M(x)$ is $\delta\langle F,E\rangle$-compact convex and, for any $y\in\Sigma$, M is upper semi-continuous at some point x in $S(y)$ with $\inf_{f\in M(y)}\inf_{u\in T(y)}\operatorname{Re}\langle f-u,y-x\rangle+h(y)-h(x)>0$, where*

$$\Sigma=\left\{y\in X:\sup_{x\in S(y)}\left[\inf_{f\in M(y)}\inf_{u\in T(y)}\operatorname{Re}\langle f-u,y-x\rangle+h(y)-h(x)\right]>0\right\}. \tag{2.38}$$

Suppose further that there exist a nonempty closed and compact (resp., weakly closed and weakly compact) subset K of X and a point $x_0\in X$ such that $x_0\in K\cap S(y)$ and

$$\inf_{f\in M(y)}\inf_{w\in T(y)}\operatorname{Re}\langle f-w,y-x_0\rangle+h(y)-h(x_0)>0,\quad \forall y\in X\setminus K. \tag{2.39}$$

Then there exists a point $\widehat{y} \in X$ *such that*

(1) $\widehat{y} \in S(\widehat{y})$,

(2) *there exist a point* $\widehat{f} \in M(\widehat{y})$ *and a point* $\widehat{w} \in T(\widehat{y})$ *with*

$$\operatorname{Re}\left\langle \widehat{f} - \widehat{w}, \widehat{y} - x \right\rangle \leq h(x) - h(\widehat{y}), \quad \forall x \in S(\widehat{y}). \tag{2.40}$$

Moreover, if $S(x) = X$ *for all* $x \in X$*, then* E *is not required to be locally convex.*

Proof. The proof is similar to the proof of Theorem 2 in [24] and so the proof is omitted here. □

Remark 2.4. (1) Theorems 2.2 and 2.3 of this paper are generalizations of Theorems 3.2 and 3.3 in [3], respectively, on noncompact sets. In Theorems 2.2 and 2.3, X is considered to be a paracompact convex and bounded subset of locally convex Hausdorff topological vector space E whereas, in [3], X is just a compact and convex subset of E. Hence our results generalize the corresponding results in [3].

(2) The first paper on generalized bi-quasi-variational inequalities was written by Shih and Tan in 1989 in [1] and the results were obtained on compact sets where the set-valued mappings were either lower semi-continuous or upper semi-continuous. Our present paper is another extension of the original work in [1] using quasi-pseudo-monotone type II operators on noncompact sets.

(3) The results in [4] were obtained on compact sets where one of the set-valued mappings is a quasi-pseudo-monotone type I operators which were defined first in [4] and extends the results in [1]. The quasi-pseudo-monotone type I operators are generalizations of pseudo-monotone type I operators introduced first in [17]. In all our results on generalized bi-quasi-variational inequalities, if the operators $M \equiv 0$ and the operators T are replaced by $-T$, then we obtain results on generalized quasi-variational inequalities which generalize the corresponding results in the literature (see [18]).

(4) The results on generalized bi-quasi-variational inequalities given in [5] were obtained for set-valued quasi-semi-monotone and bi-quasi-semi-monotone operators and the corresponding results in [2] were obtained for set-valued upper-hemi-continuous operators introduced in [6]. Our results in this paper are also further extensions of the corresponding results in [2, 5] using set-valued quasi-pseudo-monotone type II operators on noncompact sets.

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