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Research Article

Some Inequalities for Modified Bessel Functions

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We denote by I_ν and K_ν the Bessel functions of the first and third kind, respectively. Motivated by the relevance of the function $w_\nu(t) = t(I_{\nu-1}(t)/I_\nu(t))$, $t > 0$, in many contexts of applied mathematics and, in particular, in some elasticity problems Simpson and Spector (1984), we establish new inequalities for $I_\nu(t)/I_{\nu-1}(t)$. The results are based on the recurrence relations for I_ν and $I_{\nu-1}$ and the Turán-type inequalities for such functions. Similar investigations are developed to establish new inequalities for $K_\nu(t)/K_{\nu-1}(t)$.

1. Introduction

Inequalities for modified Bessel functions $I_\nu(t)$ and $K_\nu(t)$ have been established by many authors. For example, Bordelon [1] and Ross [2] proved the bounds

$$e^{x-y}\left(\frac{x}{y}\right)^\nu < \frac{I_\nu(x)}{I_\nu(y)} < e^{y-x}\left(\frac{x}{y}\right)^\nu, \quad \nu > 0,\ 0 < x < y. \tag{1.1}$$

The lower bound was also proved by Laforgia [3] for larger domain $\nu > -1/2$. In [3] also the following bounds:

$$\frac{I_\nu(x)}{I_\nu(y)} < e^{x-y}\left(\frac{y}{x}\right)^\nu, \quad \nu \geq \frac{1}{2},\ 0 < x < y, \tag{1.2}$$

$$\frac{K_\nu(x)}{K_\nu(y)} < e^{y-x}\left(\frac{y}{x}\right)^\nu, \quad \nu > \frac{1}{2},\ 0 < x < y, \tag{1.3}$$

$$\frac{K_\nu(x)}{K_\nu(y)} > e^{y-x}\left(\frac{y}{x}\right)^\nu, \quad 0<\nu<\frac{1}{2},\ 0<x<y, \tag{1.4}$$

have been established; see also [4]

In this paper we continue our investigations on new inequalities for $I_\nu(t)$ and $K_\nu(t)$, but now our results refer not only to a function I_ν or K_ν at two different points x and y, as in (1.1)–(1.4), but to two functions $I_\nu(t)$ and $I_{\nu-1}(t)$ ($K_\nu(t)$ and $K_{\nu-1}(t)$) and, more precisely, to the ratio $(I_\nu(t)/I_{\nu-1}(t))(K_\nu(t)/K_{\nu-1}(t))$. This kind of ratios appears often in applied sciences. Recently, for example, the ratio $I_\nu(t)/I_{\nu-1}(t)$ has been used by Baricz to prove an important lemma (see [5, Lemma 1]) which provides new lower and upper bounds for the generalized Marcum Q-function

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}}\int_b^\infty t^\nu e^{-(t^2+a^2)/2} I_{\nu-1}(at)\,dt, \quad b\geq 0,\ a,\nu>0 \tag{1.5}$$

(see also [6]). This generalized function and the classical one, $Q_1(a,b)$, are widely used in the electronic field, in particular in radar communications [7, 8] and in error performance analysis of multichannel dealing with partially coherent, differentially coherent, and noncoherent detections over fading channels [7, 9, 10].

The results obtained in this paper are proved as consequence of the recurrence relations [11, page 376; 9.6.26]

$$I_{\nu+1}(t) = I_{\nu-1}(t) - \frac{2\nu}{t} I_\nu(t), \tag{1.6}$$

$$K_{\nu+1}(t) = K_{\nu-1}(t) + \frac{2\nu}{t} K_\nu(t), \tag{1.7}$$

and the Turán-type inequalities

$$I_{\nu-1}(t) I_{\nu+1}(t) < I_\nu^2(t), \quad t>0,\ \nu \geq -\frac{1}{2}, \tag{1.8}$$

$$K_{\nu-1}(t) K_{\nu+1}(t) > K_\nu^2(t), \quad t>0,\ \forall \nu\in\mathbb{R} \tag{1.9}$$

proved in [12, 13], respectively (see also [14] for (1.9)). Inequalities (1.8)-(1.9) have been used, recently, by Baricz in [15], to prove, in different way, the known inequalities

$$t\frac{I_\nu'(t)}{I_\nu(t)} < \sqrt{t^2+\nu^2}, \quad \nu\geq -\frac{1}{2} \tag{1.10}$$

$$t\frac{K_\nu'(t)}{K_\nu(t)} < -\sqrt{t^2+\nu^2}, \quad \nu\in\mathbb{R}. \tag{1.11}$$

The results are given by the following theorems.

Theorem 1.1. *For real ν let $I_\nu(t)$ be the modified Bessel function of the first kind and order ν. Then*

$$\frac{-\nu + \sqrt{\nu^2 + t^2}}{t} < \frac{I_\nu(t)}{I_{\nu-1}(t)}, \quad \nu \geq 0. \tag{1.12}$$

In particular, for $\nu \geq 1/2$, the inequality $I_\nu(t)/I_{\nu-1}(t) < 1$ holds also true.

Theorem 1.2. *For real ν let $K_\nu(t)$ be the modified Bessel function of the third kind and order ν. Then*

$$\frac{K_\nu(t)}{K_{\nu-1}(t)} < \frac{\nu + \sqrt{\nu^2 + t^2}}{t}, \quad \forall \nu \in \mathbb{R}. \tag{1.13}$$

In particular, for $\nu > 1/2$, the inequality $K_\nu(t)/K_{\nu-1}(t) > 1$ holds also true.

2. The Proofs

Proof of Theorem 1.1. The upper bound for the ratio $I_\nu(t)/I_{\nu-1}(t)$ follows from the inequality

$$I_\nu(t) < I_{\nu-1}(t), \quad \nu \geq \frac{1}{2} \tag{2.1}$$

proved by Soni for $\nu > 1/2$ [16], and extended by Näsell to $\nu = 1/2$ [17].

To prove the lower bound in (1.12), we substitute the function $I_{\nu+1}(t)$ given by (1.6) in the Turán-type inequality (1.8). We get, for $\nu \geq -1/2$,

$$I_{\nu-1}(t)\left[I_{\nu-1}(t) - \frac{2\nu}{t} I_\nu(t)\right] < I_\nu^2(t), \tag{2.2}$$

that is,

$$1 - \frac{2\nu}{t}\frac{I_\nu(t)}{I_{\nu-1}(t)} < \frac{I_\nu^2(t)}{I_{\nu-1}^2(t)}. \tag{2.3}$$

We denote $I_\nu(t)/I_{\nu-1}(t)$ by u and observe that for $\nu \geq 1/2$, by (2.1), $u < 1$. With this notation (2.3) can be written as

$$u^2 + \frac{2\nu}{t} u - 1 > 0, \tag{2.4}$$

which gives, for $\nu \geq 0$,

$$-\frac{\nu}{t} + \sqrt{\frac{\nu^2}{t^2} + 1} < u, \tag{2.5}$$

that is,

$$\left[-\nu + \sqrt{\nu^2 + t^2}\right] I_{\nu-1}(t) < t I_\nu(t) \tag{2.6}$$

which is the desired result. □

Remark 2.1. For $\nu > 0$, Jones [18] proved stronger result than (2.1) that the function $I_\nu(t)$ decreases with respect to ν, when $t > 0$.

Proof of Theorem 1.2. The proof is similar to the one used to prove Theorem 1.1. By

$$K_{\nu+1}(t) > K_\nu(t), \quad \nu > -\frac{1}{2}, \tag{2.7}$$

we get $K_\nu(t)/K_{\nu-1}(t) > 1$, for $\nu > 1/2$.

We substitute the function $K_{\nu+1}(t)$ given by (1.7) in (1.9). We get

$$K_{\nu-1}(t)\left[K_{\nu-1}(t) + \frac{2\nu}{t} K_\nu(t)\right] \geq K_\nu^2(t), \quad \forall \nu \in \mathbb{R} \tag{2.8}$$

or, equivalently

$$1 + \frac{2\nu}{t}\frac{K_\nu(t)}{K_{\nu-1}(t)} - \left(\frac{K_\nu(t)}{K_{\nu-1}(t)}\right)^2 \geq 0, \tag{2.9}$$

that is,

$$u^2 - \frac{2\nu}{t} u - 1 \leq 0, \quad u = \frac{K_\nu(t)}{K_{\nu-1}(t)}. \tag{2.10}$$

Finally, we obtain

$$\frac{K_\nu(t)}{K_{\nu-1}(t)} < \frac{\nu + \sqrt{\nu^2 + t^2}}{t}, \quad \forall \nu \in \mathbb{R} \tag{2.11}$$

which is the desired result (1.13). □

Remark 2.2. By means the integral formula [11, page 181]

$$K_\nu(t) = \int_0^\infty e^{-t\cosh z} \cosh(\nu z) dz, \quad \nu > -1, \tag{2.12}$$

follows immediately the inequality

$$K_{\nu-1}(t) > K_\nu(t), \quad 0 < \nu < \frac{1}{2}, \tag{2.13}$$

and consequently

$$\frac{K_\nu(t)}{K_{\nu-1}(t)} < 1, \quad 0 < \nu < \frac{1}{2}. \tag{2.14}$$

Since $1 < (\nu + \sqrt{\nu^2 + t^2})/t$ when $0 < \nu < 1/2$, only in this case the above upper bound for $K_\nu(t)/K_{\nu-1}(t)$ improves the (1.13) one.

Remark 2.3. We observe that by Theorem 1.1 we obtain an upper bound for the function $w_\nu(t) = t(I_{\nu-1}(t)/I_\nu(t))$, $\nu \geq -1/2$. The investigations of the properties of $w_\nu(t)$ are motivated by some problems of finite elasticity [19, 20]. By (1.12) we find

$$w_\nu(t) < \frac{t^2}{-\nu + \sqrt{t^2 + \nu^2}}, \quad \nu \geq -\frac{1}{2}, \tag{2.15}$$

in particular, for $\nu \geq 1/2$, we also have $t < w_\nu(t)$.

3. Numerical Considerations

Baricz obtained, for each $\nu \geq 1$, the following similar lower bound for the ratio $I_\nu(t)/I_{\nu-1}(t)$ (see [5, formula (5)])

$$\frac{t}{t + 2\nu - 1} \leq \frac{I_\nu(t)}{I_{\nu-1}(t)}, \quad t \geq \rho_\nu, \tag{3.1}$$

where ρ_ν is the unique simple positive root of the equation $(t + 2\nu - 1)I_\nu = tI_{\nu-1}$. Inequality (3.1) is reversed when $0 < t \leq \rho_\nu$. It is possible to prove that, for $\nu > 1$, our lower bound in (1.12) for the ratio $I_\nu(t)/I_{\nu-1}(t)$ provides an improvement of (3.1).

Proposition 3.1. *Let be $\nu > 1$. Putting $f_\nu(t) = (-\nu + \sqrt{\nu^2 + t^2})/t$ and $g_\nu(t) = t/(t + 2\nu - 1)$, one has $f_\nu(t) > g_\nu(t)$, for all $t > \max\{1/(2-\nu)/(1-\nu), \rho_\nu\}$.*

Proof. From the inequality $f_\nu(t) > g_\nu(t)$ we obtain, by simple calculations, the following one $t(1-\nu) + \nu - 1/2 < 0$ which is satisfied for all $t > 1/(2-\nu)/(1-\nu)$ when $\nu > 1$. □

We report here some numerical experiments, computed by using mathematica.

Example 3.2. In the first case we assume $\nu = 8$. In Figure 1 we report the graphics of the functions $I_\nu(t)/I_{\nu-1}(t)$ (solid line) and the respective lower bounds $f_\nu(t)$ (short dashed line) and $g_\nu(t)$ (long dashed line) on the interval [100, 600].

In Table 1 we report also the respective numerical values of the differences $I_\nu(t)/I_{\nu-1}(t) - f_\nu(t)$ and $I_\nu(t)/I_{\nu-1}(t) - g_\nu(t)$ in some points t.

Remark 3.3. By some numerical experiments we can conjecture that the lower bound (3.1) holds true also when $1/2 \leq \nu < 1$ and, in particular, for these values of ν we have $f_\nu(t) < g_\nu(t)$. See, for example, in Figure 2 the graphics of the functions $I_\nu(t)/I_{\nu-1}(t)$ (solid line) and the respective lower bounds $f_\nu(t)$ (short dashed line) and $g_\nu(t)$ (long dashed line) on the interval [100, 600] when $\nu = 0.7$.

Table 1

	$t = 600$	$t = 6000$	$t = 60000$
$I_\nu(t)/I_{\nu-1}(t) - f_\nu(t)$	0.00081226756	0.00008312164	8.3312153×10^{-6}
$I_\nu(t)/I_{\nu-1}(t) - g_\nu(t)$	0.01195806306	0.00124444278	0.00012494429

Table 2

	$t = 600$	$t = 6000$	$t = 60000$
$h_\nu(t) - K_\nu(t)/K_{\nu-1}(t)$	0.000854625	0.0000835453	8.33545×10^{-6}

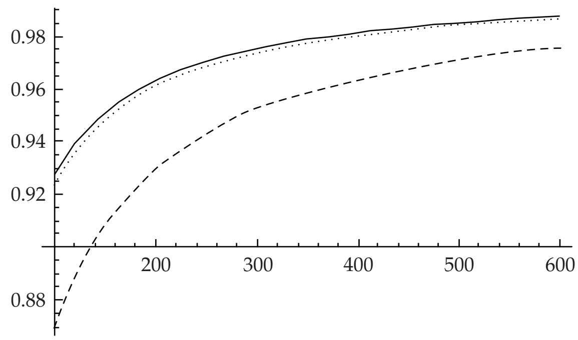

Figure 1

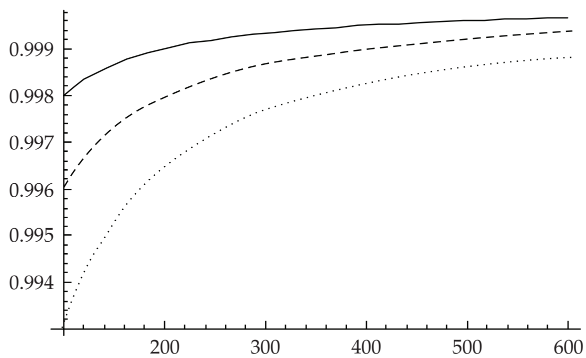

Figure 2

Example 3.4. In this case we assume $\nu = 1/3$, then we report, in Figure 3, the graphics of the functions $I_\nu(t)/I_{\nu-1}(t)$ (solid line) and the respective lower bounds $f_\nu(t)$ (short dashed line) on the interval $[100, 600]$.

In Table 3 we report also the respective numerical values of the differences $I_\nu(t)/I_{\nu-1}(t) - f_\nu(t)$ in some points t.

Example 3.5. Also in this case we assume $\nu = 8$. In Figure 4 we report the graphics of the functions $K_\nu(t)/K_{\nu-1}(t)$ (solid line) and the respective upper bound $h_\nu(t) = (\nu + \sqrt{\nu^2 + t^2})/t$ (short dashed line) on the interval $[100, 600]$.

In Table 2, we report also the respective numerical values of the difference $h_\nu(t) - K_\nu(t)/K_{\nu-1}(t)$ in some points t.

Table 3

	$t = 600$	$t = 6000$	$t = 60000$
$I_\nu(t)/I_{\nu-1}(t) - f_\nu(t)$	0.00083345	0.0000833345	8.33334×10^{-6}

Table 4

	$t = 600$	$t = 6000$	$t = 60000$
$h_\nu(t) - K_\nu(t)/K_{\nu-1}(t)$	0.000821238	0.0000832119	8.33212×10^{-6}

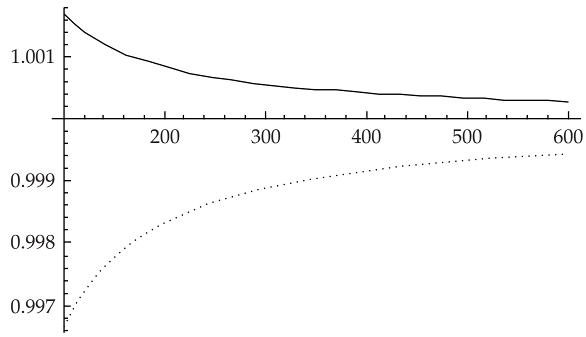

Figure 3

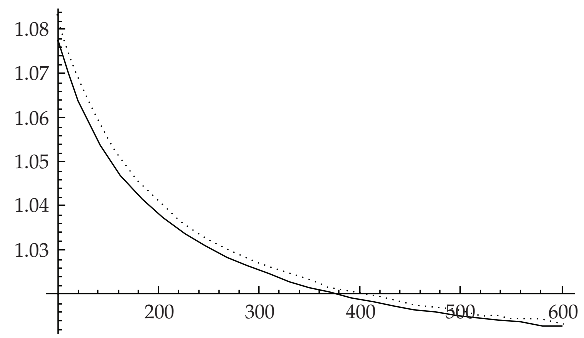

Figure 4

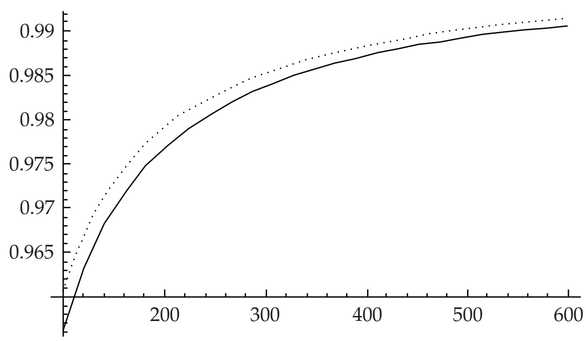

Figure 5

Table 5

	$\nu = 0.3$	$\nu = 8$	$\nu = 30$
$a_\nu(x,y)$	0.109926	0.000528653	1.26041×10^{-10}
$b_\nu(x,y)$	0.133826	0.00272121	8.42928×10^{-10}
$I_\nu(x)/I_\nu(y)$	0.195323	0.00282361	8.45623×10^{-10}

Table 6

	$\nu = 0.2$	$\nu = 0.4$	$\nu = 8$	$\nu = 30$
$c_\nu(x,y)$	8.4878	9.74992	—	—
$d_\nu(x,y)$	7.42604	7.53746	367.483	1.18634×10^9
$K_\nu(x)/K_\nu(y)$	10.2446	10.3615	384.34	1.19039×10^9

Example 3.6. In this last case we assume $\nu = -4$. In Figure 5 we report the graphics of the functions $K_\nu(t)/K_{\nu-1}(t)$ (solid line) and the respective upper bound $h_\nu(t) = (\nu + \sqrt{\nu^2 + t^2})/t$ (short dashed line) on the interval $[100, 600]$.

In Table 4 we report also the respective numerical values of the difference $h_\nu(t) - K_\nu(t)/K_{\nu-1}(t)$ in some points t.

Remark 3.7. We conclude this paper observing that, dividing by t inequalities (1.10)-(1.11) and integrating them from x to y $(0 < x < y)$, we obtain the following new lower bounds for the ratios $I_\nu(x)/I_\nu(y)$ and $K_\nu(x)/K_\nu(y)$:

$$\left(\frac{x}{y}\right)^\nu \left(\frac{\nu + \sqrt{\nu^2 + y^2}}{\nu + \sqrt{\nu^2 + x^2}}\right)^\nu e^{\sqrt{\nu^2+x^2}-\sqrt{\nu^2+y^2}} < \frac{I_\nu(x)}{I_\nu(y)}, \quad \nu \ge -\frac{1}{2}, \tag{3.2}$$

$$\left(\frac{y}{x}\right)^\nu \left(\frac{\nu + \sqrt{\nu^2 + x^2}}{\nu + \sqrt{\nu^2 + y^2}}\right)^\nu e^{\sqrt{\nu^2+y^2}-\sqrt{\nu^2+x^2}} < \frac{K_\nu(x)}{K_\nu(y)}, \quad \forall \nu \in \mathbb{R}. \tag{3.3}$$

For a survey on inequalities of the type (3.2) and (3.3) see [4].

In the following Tables 5 and 6 we confront the lower bounds (1.1)–(3.2) and (1.4)–(3.3), respectively, for different values of ν in the particular cases $x = 2$ and $y = 4$. Let

$$a_\nu(x,y) = \left(\frac{x}{y}\right)^\nu e^{x-y},$$

$$b_\nu(x,y) = \left(\frac{x}{y}\right)^\nu \left(\frac{\nu + \sqrt{\nu^2 + y^2}}{\nu + \sqrt{\nu^2 + x^2}}\right)^\nu e^{\sqrt{\nu^2+x^2}-\sqrt{\nu^2+y^2}},$$

$$c_\nu(x,y) = \left(\frac{y}{x}\right)^\nu e^{y-x},$$

$$d_\nu(x,y) = \left(\frac{y}{x}\right)^\nu \left(\frac{\nu + \sqrt{\nu^2 + x^2}}{\nu + \sqrt{\nu^2 + y^2}}\right)^\nu e^{\sqrt{\nu^2+y^2}-\sqrt{\nu^2+x^2}}, \tag{3.4}$$

then we have Tables 5 and 6.

By the values reported on Table 5 it seems that $b_\nu(x,y)$ is a lower bound much more stringent with respect to $a_\nu(x,y)$ for every $\nu > 0$ (moreover we recall that (3.2) holds true also for $-1/2 < \nu \leq 0$), while by the values reported on Table 6 it seems that $c_\nu(x,y)$ is a lower bound more stringent with respect to $d_\nu(x,y)$ for $0 < \nu < 1/2$ (but we recall that (3.3) holds true also for $\nu \leq 0$ and $\nu \geq 1/2$).

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