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Research Article

Regularity for Solutions of Second-Order Nonlinear Integrodifferential Functional Equations

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We deal with the well-posedness for solutions of nonlinear integrodifferential equations of second-order in Hilbert spaces by converting the problem into the contraction mapping principle with more general conditions on the principal operators and the nonlinear terms and obtain a variation of constant formula of solutions of the given nonlinear equations.

1. Introduction

Let H and V be two complex Hilbert spaces. Assume that V is a dense subspace in H and the injection of V into H is continuous. If H is identified with its dual space then we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V, H, and V^* will be denoted by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$, respectively.

The subset of this paper is to consider the initial value problem of the following second-order nonlinear integrodifferential control system on Hilbert spaces:

$$\begin{gathered} x''(t) + Ax(t) = f(t, x(t)) + h(t), \quad 0 < t \leq T, \\ x(0) = x_0, \qquad x'(0) = x_1, \end{gathered} \tag{1.1}$$

where the nonlinear term is given by

$$f(t, x) = \int_0^t k(t-s) g(s, x(s)) ds. \tag{1.2}$$

Let A be a continuous linear operator from V into V^* which is assumed to satisfy Gårding's inequality. Here V^* stands for the dual space of V. The nonlinear mapping g is Lipschitz continuous from $\mathbb{R} \times V$ into H.

The regularity results of the evolution equations of second-order in case $f(\cdot, x(\cdot)) + h \in L^2(0, T; V^*)$ are given by [1–3]. We also call attention to the method of Baoicchi [4] who has obtained the results directly, without duality, under slightly different hypotheses, which is valid for first- and second-order equations. The well-posedness of solutions for delay evolution equations of second-order in time was referred to in [5]. As for the equation of first order, it is well known as the quasi-autonomous differential equation (see Theorem 2.6 in [1, 11] and [6, 7]).

The paper is a generalization of a particular case of the equation by Brézis [8, 9] based on the Lipschitz continuity of nonlinear term f where the nonlinear differential equation of hyperbolic type

$$\begin{gathered} x''(t) + Ax(t) + f(x(t)) = h(t), \quad 0 < t \leq T, \\ x(0) = x_0, \qquad x'(0) = x_1 \end{gathered} \tag{1.3}$$

has a unique solution $u(t)$ which satisfies what follows: if $x_0 \in V$, $x_1 \in H$, and $h \in W^{1,2}(0, T; H)$, then

$$\begin{gathered} u \in C([0, T]; H) \cap C^1((0, T]; V^*), \\ u \in L^\infty(0, T; V), \qquad u' \in L^\infty(0, T; H), \qquad u'' \in L^\infty(0, T; V^*). \end{gathered} \tag{1.4}$$

By different methods, results similar to those in result mentioned above were established by Browder [10], Barbu [11], Lions and Strauss [12], and many others. Tanabe [13] proved the existence of local solution of (1.3) when the nonlinear mapping f is locally Lipschitz continuous. Even under the weakest assumption on nonlinear term, the existence of weak solution can be shown though the uniqueness is not vertain. For example, it is referred to in a work by Strauss [14]. [15] dealt with an L_2-approach to second-order nonlinear functional evolutions involving m-accretive operators in Banach spaces.

In this paper, under the assumption that $h \in L^2(0, T; V^*)$ and the local Lipschitz continuity of the nonlinear mapping from V into H (not from H into itself), we deal with regularity for the solution of the given equation (1.1) which will enable us to obtain a global existence theorem for the strict solution x belonging to $C^1((0, T]; V) \cap C([0, T]; H) \cap W^{2,2}(0, T; V^*)$; namely, assuming the Lipschitz continuity of nonlinear terms, we show that the well-posedness and stability properties for a class with nonlinear perturbation of second-order are similar to those of its corresponding linear system.

we begin to study well-posedness and stability properties for a class with nonlinear perturbation of second-order.

We will develop and apply the existence theory for first-order differential equations (see [16, 17]) to study certain second-order differential equations associated with nonlinear maximal monotone operators in Hilbert spaces.

2. Linear Hyperbolic Equations

The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$. By considering $H = H^*$, we may write $V \subset H \subset V^*$ where H^* and V^* denote the dual spaces of H and V, respectively. For $l \in V^*$ we denoted (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$\|l\|_* = \sup_{v \in V} \frac{|(l, v)|}{\|v\|}. \tag{2.1}$$

Therefore, we assume that V has a stronger topology than that of H and, for the brevity, we may regard that

$$\|u\|_* \leq |u| \leq \|u\|, \quad \forall u \in V. \tag{2.2}$$

Definition 2.1. Let X and Y be complex Banach spaces. An operator S from X to Y is called antilinear if $S(u+v) = S(u) + S(v)$ and $S(\lambda u) = \overline{\lambda} S(u)$ for $u, v \in X$ and for $\lambda \in \mathbb{C}$.

Let $a(u, v)$ be a quadratic form defined on $V \times V$ which is linear in u and antilinear in v.

We make the following assumptions:

(i) $a(u, v)$ is bounded, that is,

$$|a(u, v)| \leq c_0 \|u\| \cdot \|v\|, \tag{2.3}$$

(ii) $a(u, v)$ is symmetric, that is,

$$a(u, v) = \overline{a(v, u)}, \tag{2.4}$$

(iii) $a(u, v)$ satisfies the Gårding's inequality, that is,

$$\operatorname{Re} a(u, u) \geq \delta \|u\|^2 - \kappa |u|^2, \quad \delta > 0, \ \kappa \in \mathbb{R}. \tag{2.5}$$

Let A be the operator such that $(Au, v) = a(u, v) u, v \in V$. Then, as seen in [13, Theorem 2.2.3], the operator A is positive, definite, and self-adjoint, $D(A^{1/2}) = V$, and

$$a(u, v) = \left(A^{1/2} u, A^{1/2} v\right), \quad u, v \in V. \tag{2.6}$$

It is also known that the operator A is a bounded linear from V to V^*. The realization of A in H which is the restriction of A to $D(A) = \{v \in V : Av \in H\}$ is also denoted by A, which is structured as a Hilbert space with the norm $\|v\|_{D(A)} = |Av|$. Then the operator A generates an analytic semigroup in both of H and V^*. Thus we have the following sequence:

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.7}$$

where each space is dense in the next one which is continuous injection.

If X is a Banach space and $1 < p < \infty$, then $L^p(0,T;X)$ is the collection of all strongly measurable functions from $(0,T)$ into X the pth powers whose norms are integrable and $W^{m,p}(0,T;X)$ is the set of all functions f whose derivatives $D^\alpha f$ up to degree m in the distribution sense belong to $L^p(0,T;X)$, and $C^m([0,T];X)$ is the set of all m-times continuously differentiable functions from $[0,T]$ into X. Let X and Y be complex Banach spaces. Denote by $B(X,Y)$ (resp., $\overline{B}(X,Y)$) the set of all bounded linear (resp., antilinear) operators from X and Y. Let $B(X) = B(X,X)$.

First, consider the following linear hyperbolic equation:

$$\begin{gathered} x''(t) + Ax(t) = h(t), \quad 0 < t \le T, \\ x(0) = x_0, \qquad x'(0) = x_1. \end{gathered} \tag{2.8}$$

By virtue of Theorem 8.2 of [18], we have the following result on the corresponding linear equation of (2.8) in case $f \equiv 0$.

Proposition 2.2. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold: For $(x_0, x_1) \in V \times H$ and $h \in L^2(0,T;H)$, $T > 0$, there exists a unique solution x of* (2.8) *belonging to*

$$C([0,T];V) \cap C^1((0,T];H) \cap W^{2,2}(0,T;V^*) \tag{2.9}$$

and satisfying

$$\|x\|_{C([0,T];V)\cap C^1((0,T];H)\cap W^{2,2}(0,T;V^*)} \le C_1\Big(\|x_0\| + |x_1| + \|h\|_{L^2(0,T;H)}\Big), \tag{2.10}$$

where C_1 is a constant depending on T. Moreover, the mapping

$$\{x_0, x_1, h) \longmapsto \{x, x'\} \tag{2.11}$$

is a linear continuous map of $V \times H \times L^2(0,T;H) \to C([0,T];V) \cap C^1((0,T];H)$.

As for the solution which belongs to L^2 spaces, we refer to Chapter IV of [17]. The applications of these problems are mixed problems in the sense of Ladyzenskaya [16].

3. Hyperbolic Equations with Nonlinear Perturbations

Let $g : [0,T] \times V \to H$ be a nonlinear mapping such that $t \mapsto g(t,x)$ is measurable and

$$|g(t,x) - g(t,y)| \le L\|x - y\|, \tag{F}$$

for a positive constant L. We assume that $g(t,0) = 0$ for the sake of simplicity.

For $x \in L^2(0,T;V)$ we set

$$f(t,x) = \int_0^t k(t-s)g(s,x(s))ds, \tag{3.1}$$

where k belongs to $L^2(0,T)$.

Lemma 3.1. *Let $x \in L^2(0,T;V)$, $T > 0$. Then $f(\cdot,x) \in L^2(0,T;H)$,*

$$\begin{aligned} |f(t,x_1(t)) - f(t,x_2(t))| &\le L\sqrt{t}\|k\|_{L^2(0,t)}\|x_1 - x_2\|_{C([0,t];V)}, \\ \|f(\cdot,x)\|_{L^2(0,T;H)} &\le L\|k\|_{L^2(0,T)}\sqrt{T}\|x\|_{L^2(0,T;V)}. \end{aligned} \tag{3.2}$$

Moreover if $x_1, x_2 \in L^2(0,T;V)$, then

$$\|f(\cdot,x_1) - f(\cdot,x_2)\|_{L^2(0,T;H)} \le L\|k\|_{L^2(0,T)}\sqrt{T}\|x_1 - x_2\|_{L^2(0,T;V)}. \tag{3.3}$$

Proof. By Hölder inequality we obtain

$$\begin{aligned} |f(t,x_1(t)) - f(t,x_2(t))| &\le \int_0^t |k(t-s)\{g(s,x_1(s)) - g(s,x_2(s))\}|ds \\ &\le \left\{\int_0^t |k(t-s)|^2 ds\right\}^{1/2} \left\{\int_0^t L^2\|x_1(s) - x_2(s)\|^2 ds\right\}^{1/2} \\ &\le L\sqrt{t}\|k\|_{L^2(0,t)}\|x_1 - x_2\|_{C([0,t];V)}. \end{aligned} \tag{3.4}$$

From (F) and using the Hölder inequality, it is easily seen that

$$\begin{aligned} \|f(\cdot,x)\|^2_{L^2(0,T;H)} &\le \int_0^T \left|\int_0^t k(t-s)g(s,x(s))ds\right|^2 dt \\ &\le \|k\|^2_{L^2(0,T)} \int_0^T\int_0^t L^2\|x(s)\|^2 dsdt \\ &\le L^2\|k\|^2_{L^2(0,T)} T\|x\|^2_{L^2(0,T;V)}. \end{aligned} \tag{3.5}$$

The proof of the second paragraph is similar. □

The following lemma is from Brézis ([9], Lemma A.5)

Lemma 3.2. *Let $m \in L^1(0,T;\mathbb{R})$ satisfying $m(t) \geq 0$ for all $t \in (0,T)$ and let $a \geq 0$ be a constant. Let b be a continuous function on $[0,T] \subset \mathbb{R}$ satisfying the following inequalities:*

$$\frac{1}{2}b^2(t) \leq \frac{1}{2}a^2 + \int_0^t m(s)b(s)ds, \quad t \in [0,T]. \tag{3.6}$$

Then,

$$|b(t)| \leq a + \int_0^t m(s)ds, \quad t \in [0,T]. \tag{3.7}$$

Let $x \in C([0,T];V) \cap C^1((0,T];H)$. Then invoking Proposition 2.2, we obtain that the problem

$$\begin{gathered} y''(t) + Ay(t) = f(t,x(t)) + h(t), \quad 0 < t \leq T, \\ y(0) = x_0, \qquad y'(0) = x_1 \end{gathered} \tag{3.8}$$

has a unique solution $y \in C([0,T];V) \cap C^1((0,T];H) \cap W^{2,2}(0,T;V^*)$.

The following Lemma is one of the useful integral inequalities.

Lemma 3.3. *Let $b, a, m \in C(\mathbb{R}^+, \mathbb{R}^+)$ and consider the following inequality:*

$$b(t) \leq a(t) + \int_{t_0}^t m(s)b(s)ds, \quad t \geq t_0. \tag{3.9}$$

Then,

$$b(t) \leq a(t) + \int_{t_0}^t [a(s)m(s)] \exp\left\{\int_s^t m(\tau)d\tau\right\}ds, \quad t \geq t_0. \tag{3.10}$$

Lemma 3.4. *Let y_1, y_2 be the solutions of (3.8) with x replaced of by $x_1, x_2 \in C([0,T];V) \cap C^1((0,T];H)$, respectively. Then the following inequality holds:*

$$\left|y_1'(t) - y_2'(t)\right|^2 + \delta\left\|y_1(s) - y_2(s)\right\|^2 \leq \alpha(t)\int_0^t e^{2\kappa(t-s)}H(s)\left|y_1'(s) - y_2'(s)\right|ds, \tag{3.11}$$

$$\left|y_1'(t) - y_2'(t)\right| \leq \frac{1}{2}\alpha(t)\int_0^t e^{\kappa(t-s)}H(s)ds, \tag{3.12}$$

where $\alpha(t) = 1 + 2te^{2t}$ and

$$H(t) = 2L\sqrt{t}\|k\|_{L^2(0,t)}\|x_1 - x_2\|_{C([0,t];V)}. \tag{3.13}$$

Proof. For $i = 1,\ 2$, we consider the following equation:

$$\begin{gathered} y_i''(t) + Ay_i(t) = f(t, x_i(t)) + h(t), \quad 0 < t \le T, \\ y(0) = x_0, \qquad y'(0) = x_1. \end{gathered} \tag{3.14}$$

Then, we have that

$$\left(y_1(t) - y_2(t)\right)'' + A\left(y_1(t) - y_2(t)\right) = f(t, x_1(t)) - f(t, x_2(t)) \tag{3.15}$$

for $t > 0$. Acting on both sides of (3.15) by $y_1'(t) - y_2'(t)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left|y_1'(t) - y_2'(t)\right|^2 + \left(A\left(y_1(t) - y_2(t)\right), y_1'(t) - y_2'(t)\right) \\ &\quad = \left(f(t, x_1(t)) - f(t, x_2(t)), y_1'(t) - y_2'(t)\right). \end{aligned} \tag{3.16}$$

By integration of parts, it holds that

$$\int_0^t \left(Ay(s), y'(s)\right)ds = \left(Ay(t), y(t)\right) - \left(Ay(0), y(0)\right) - \int_0^t \left(Ay'(s), y(s)\right)ds. \tag{3.17}$$

Consequently, since A is self-adjoint, we have

$$2\ \operatorname{Re}\int_0^t \left(Ay(s), y'(s)\right)ds = \operatorname{Re}\left(Ay(t), y(t)\right) - \operatorname{Re}\left(Ay(0), y(0)\right), \tag{3.18}$$

and by Lemma 3.1,

$$\left|f(t, x_1(t)) - f(t, x_2(t))\right| \le \frac{1}{2}H(t), \tag{3.19}$$

where

$$H(t) = 2L\sqrt{t}\ \|k\|_{L^2(0,t)}\|x_1 - x_2\|_{C([0,t];V)}. \tag{3.20}$$

By integrating over $(0, t)$, (3.16) implies that

$$\left|y_1'(t) - y_2'(t)\right|^2 + \operatorname{Re}\left(A\left(y_1(t) - y_2(t)\right), y_1(t) - y_2(t)\right) \le \int_0^t H(s) \cdot \left|y_1'(s) - y_2'(s)\right|ds, \tag{3.21}$$

which yields that

$$\left|y_1'(t) - y_2'(t)\right|^2 + \delta\left\|y_1(t) - y_2(t)\right\|^2 \le \kappa\left|y_1(t) - y_2(t)\right|^2 + \int_0^t H(s) \cdot \left|y_1'(s) - y_2'(s)\right|ds. \tag{3.22}$$

From (3.22) and Schwarz's inequality it follows that

$$\begin{aligned}
&\frac{d}{dt}\left\{e^{-2\kappa t}\left|y_1(t)-y_2(t)\right|^2\right\}\\
&\quad= 2e^{-2\kappa t}\left\{\frac{1}{2}\frac{d}{dt}\left|y_1(t)-y_2(t)\right|^2-\kappa\left|y_1(t)-y_2(t)\right|^2\right\}\\
&\quad= 2e^{-2\kappa t}\left\{\operatorname{Re}\left(y_1'(t)-y_2'(t),y_1(t)-y_2(t)\right)-\kappa\left|y_1(t)-y_2(t)\right|^2\right\}\\
&\quad\le 2e^{-2\kappa t}\left(\left|y_1'(t)-y_2'(t)\right|^2+\left|y_1(t)-y_2(t)\right|^2-\kappa\left|y_1(t)-y_2(t)\right|^2\right)\\
&\quad\le 2e^{-2\kappa t}\left\{\left|y_1(t)-y_2(t)\right|^2+\int_0^t H(s)\cdot\left|y_1'(s)-y_2'(s)\right|ds\right\}.
\end{aligned}\tag{3.23}$$

Integrating (3.23) over $(0,t)$ we have

$$\begin{aligned}
&e^{-2\kappa t}\left|y_1(t)-y_2(t)\right|^2\\
&\quad\le 2\int_0^t e^{-2\kappa s}\left|y_1(s)-y_2(s)\right|^2 ds+2\int_0^t e^{-2\kappa\tau}\int_0^\tau H(s)\cdot\left|y_1'(s)-y_2'(s)\right|ds\,d\tau\\
&\quad= 2\int_0^t e^{-2\kappa s}\left|y_1(s)-y_2(s)\right|^2 ds+\frac{1}{\kappa}\int_0^t\left(e^{-2\kappa s}-e^{-2\kappa t}\right)H(s)\cdot\left|y_1'(s)-y_2'(s)\right|ds.
\end{aligned}\tag{3.24}$$

By Lemma 3.2, we get

$$e^{-2\kappa t}\left|y_1(t)-y_2(t)\right|^2\le\frac{\alpha(t)}{\kappa}\int_0^t\left(e^{-2\kappa s}-e^{-2\kappa t}\right)H(s)\cdot\left|y_1'(s)-y_2'(s)\right|ds\tag{3.25}$$

where $\alpha(t)=1+2te^{2t}$, that is,

$$\kappa\left|y_1(t)-y_2(t)\right|^2\le\alpha(t)\int_0^t\left(e^{2\kappa(t-s)}-1\right)H(s)\cdot\left|y_1'(s)-y_2'(s)\right|ds.\tag{3.26}$$

Hence, inequality (3.11) is obtained from (3.22) and (3.26), which implies that

$$\begin{aligned}
&\frac{1}{2}\left(e^{-\kappa t}\left|y_1'(t)-y_2'(t)\right|\right)^2+\frac{1}{2}\delta e^{-2\kappa t}\left\|y_1(t)-y_2(t)\right\|^2\\
&\quad\le\frac{1}{2}\alpha(t)\int_0^t e^{-\kappa s}H(s)\cdot e^{-\kappa s}\left|y_1'(s)-y_2'(s)\right|ds.
\end{aligned}\tag{3.27}$$

So, by using Lemma 3.1, we obtain (3.12). □

Theorem 3.5. *Let the assumption* (F) *be satisfied. Assume that* $h\in L^2(0,T;H)$ *and* $(x_0,x_1)\in V\times H$. *Then there exists a time* $T_0>0$ *such that the functional differential equation* (1.1) *admits a unique solution* x *in* $C([0,T_0];V)\cap C^1((0,T_0];H)\cap W^{2,2}(0,T_0;V^*)$.

Proof. Let us fix $T_0 > 0$ such that

$$\frac{\left\{L\|k\|_{L^2(0,T)}T_0\alpha(T_0)\right\}^2(e^{2\kappa T_0}-1)}{(2\kappa)\min\{1,\delta\}} < 1. \tag{3.28}$$

We are going to show that $x \mapsto y$ is strictly contractive from $C([0,T];V)\cap C^1((0,T];H)$ to itself if condition (3.28) is satisfied. The norm in $C([0,T];V)\cap C^1((0,T];H)$ is given by

$$\|\cdot\|_{C([0,T];V)\cap C^1((0,T];H)} = \max\left\{\|\cdot\|_{C([0,T_0];V)}, \|\cdot\|_{C^1((0,T_0];H)}\right\}. \tag{3.29}$$

From (3.11) and (3.12) it follows that

$$\begin{aligned}
&\left|y_1'(t)-y_2'(t)\right|^2+\delta\left\|y_1(t)-y_2(t)\right\|^2\\
&\quad\le \frac{1}{2}\alpha(t)^2\int_0^t e^{2\kappa(t-s)}H(s)\int_0^s e^{\kappa(s-\tau)}H(\tau)d\tau\, ds\\
&\quad= \frac{1}{2}\alpha(t)^2e^{2\kappa t}\int_0^t e^{-\kappa s}H(s)\int_0^s e^{-\kappa\tau}H(\tau)d\tau\, ds\\
&\quad= \frac{1}{2}\alpha(t)^2e^{2\kappa t}\int_0^t \frac{1}{2}\frac{d}{ds}\left\{\int_0^s e^{-\kappa\tau}H(\tau)d\tau\right\}^2 ds = \frac{1}{4}\alpha(t)^2e^{2\kappa t}\left\{\int_0^t e^{-\kappa\tau}H(\tau)d\tau\right\}^2\\
&\quad\le \frac{1}{4}\alpha(t)^2e^{2\kappa t}\int_0^t e^{-2\kappa\tau}d\tau\int_0^t H(\tau)^2d\tau = \frac{1}{4}\alpha(t)^2e^{2\kappa t}\frac{1-e^{-2\kappa t}}{2\kappa}\int_0^t H(\tau)^2d\tau\\
&\quad= \frac{\alpha(t)^2}{8\kappa}\left(e^{2\kappa t}-1\right)\int_0^t H(s)^2ds.
\end{aligned} \tag{3.30}$$

Starting from initial value $x_0(t) = x_0$, consider a sequence $\{x_n(\cdot)\}$ satisfying

$$\begin{gathered}
x_{n+1}''(t) + Ax_{n+1}(t) = f(t, x_n(t)) + h(t), \quad 0 < t \le T,\\
x_{n+1}(0) = x_0, \qquad x_{n+1}'(0) = x_1.
\end{gathered} \tag{3.31}$$

Then from (3.30) it follows that

$$\begin{aligned}
&\left|x_{n+1}'(t)-x_n'(t)\right|^2+\delta\|x_{n+1}(t)-x_n(t)\|^2\\
&\quad\le \left\{L\|k\|_{L^2(0,T)}T_0\alpha(T_0)\right\}^2\left(e^{2\kappa T_0}-1\right)(2\kappa)^{-1}\|x_n-x_{n-1}\|^2_{C(0,t;V)}.
\end{aligned} \tag{3.32}$$

Hence, we obtain that

$$\begin{aligned}&\|x_{n+1}-x_n\|^2_{C([0,T];V)\cap C^1((0,T];H)}\\&\quad\le\frac{\left\{L\|k\|_{L^2(0,T)}T_0\alpha(T_0)\right\}^2(e^{2\kappa T_0}-1)}{(2\kappa)\min\{1,\delta\}}\|x_n-x_{n-1}\|^2_{C([0,T];V))\cap C^1([0,T];H)}.\end{aligned}\tag{3.33}$$

So by virtue of condition (3.28) the contraction principle gives that there exists $x(\cdot)\in C([0,T];V)\cap C^1((0,T];H)$ such that

$$x_n(\cdot)\longrightarrow x(\cdot)\quad\text{in } C([0,T];V)\cap C^1((0,T];H).\tag{3.34}$$

Since $h\in L^2(0,T;H)$ and the operator A is bounded linear from V to V^*, it follows from (1.1) that the solution x belongs to $W^{2,2}(0,T;V^*)$. Thus, it completes the proof of theorem. □

From now on, we give a norm estimation of the solution of (1.1) and establish the global existence of solutions with the aid of norm estimations.

Theorem 3.6. *Let the assumption* (F) *be satisfied. Assume that $h\in L^2(0,T;H)$ and $(x_0,x_1)\in V\times H$. Then the solution x of* (1.1) *exists and is unique in $\mathcal{X}\equiv C([0,T];V)\cap C^1((0,T];H)\cap W^{2,2}(0,T;V^*)$, and there exists a constant C_2 depending on T such that*

$$\|x\|_{\mathcal{X}}\le C_2\Big(1+\|x_0\|+|x_1|+\|h\|_{L^2(0,T;H)}\Big).\tag{3.35}$$

Proof. We establish the estimates of solution. Let y be the solution of

$$\begin{aligned}y''(t)+Ay(t)&=h(t),\quad 0<t\le T_0,\\ y(0)=x_0,&\qquad y'(0)=x_1.\end{aligned}\tag{3.36}$$

Then if x is a solution of (1.1), since

$$\big(x(t)-y(t)\big)''+A\big(x(t)-y(t)\big)=f(t,x(t)),\tag{3.37}$$

by multiplying $x(t)-y(t)$ and using the monotonicity of A, then we obtain

$$\left|x'(t)-y'(t)\right|^2+\delta\left\|x(t)-y(t)\right\|^2\le\kappa\left|x(t)-y(t)\right|^2+\int_0^t\left|f(t,x(t))\right|\cdot\left|x(t)-y(t)\right|dt.\tag{3.38}$$

By the procedure similar to (3.33) we have

$$\|x-y\|^2_{C([0,T];V)\cap C^1((0,T];H)}\le\frac{\left\{L\|k\|_{L^2(0,T)}T_0\alpha(T_0)\right\}^2(e^{2\kappa T_0}-1)}{(2\kappa)\min\{1,\delta\}}\|x\|^2_{C([0,T];V)\cap C^1((0,T];H)}.\tag{3.39}$$

Put

$$N^2 = \frac{\left\{L\|k\|_{L^2(0,T)} T_0 \alpha(T_0)\right\}^2 (e^{2\kappa T_0} - 1)}{(2\kappa)\min\{1, \delta\}}. \tag{3.40}$$

Then from Proposition 2.2, we have that

$$\begin{aligned}\|x\|_{C([0,T];V)\cap C^1((0,T];H))} &\le \frac{1}{1-N}\|y\|_{L^2(0,T_0;V)} \\ &\le \frac{C_1}{1-N}\left(\|x_0\| + |x_1| + \|h\|_{L^2(0,T_0;H)}\right) \\ &\le C_2\left(\|x_0\| + |x_1| + \|h\|_{L^2(0,T_0;H)}\right),\end{aligned} \tag{3.41}$$

for some positive constant C_2. Noting that by Lemma 3.1

$$\begin{aligned}\|f(\cdot, x(\cdot))\|_{L^2(0,T_0;H)} &\le \|f(\cdot, x(\cdot)) - f(\cdot, 0)\|_{L^2(0,T_0;H)} + \|f(\cdot, 0)\|_{L^2(0,T_0;H)} \\ &\le \text{const.}\left(1 + \|x\|_{L^2(0,T_0;V)}\right)\end{aligned} \tag{3.42}$$

and by Proposition 2.2

$$\|x\|_{W^{2,2}(0,T_0;V^*)} \le \left\{1 + \|x_0\| + |x_1| + \|f(\cdot, x(\cdot)) + h\|_{L^2(0,T_0;H)}\right\}, \tag{3.43}$$

it is easy to obtain the norm estimate of x in $W^{2,2}(0, T_0; V^*)$ satisfying (3.35).

Now from (3.41) it follows that

$$|x(T_0)| \le \|x\|_{C([0,T_0],H)} \le C_2\left(1 + \|x_0\| + |x_1| + \|h\|_{L^2(0,T_0;H)}\right). \tag{3.44}$$

So, we can solve the equation in $[T_0, 2T_0]$ and obtain an analogous estimate to (3.41). Since condition (3.28) is independent of initial values, the solution of (1.1) can be extended to the internal $[0, nT_0]$ for natural number n, that is, for the initial $x(nT_0)$ in the interval $[nT_0, (n+1)T_0]$, as analogous estimate (3.41) holds for the solution in $[0, (n+1)T_0]$. Furthermore, the estimate (3.35) is easily obtained from (3.41) and (3.43). □

Example 3.7. Let

$$\begin{gathered}H = L^2(0,\pi), \qquad V = H_0^1(0,\pi), \qquad V^* = H^{-1}(0,\pi), \\ a(u,v) = \int_0^\pi \frac{du(x)}{dx}\frac{\overline{dv(x)}}{dx}dx, \\ A = \frac{\partial^2}{\partial x^2} \quad \text{with } D(A) = \left\{y \in H^2(0,\pi) : y(0) = y(\pi) = 0\right\}.\end{gathered} \tag{3.45}$$

Let U be another Hilbert space and let B be a bounded linear operator from U to H. We consider the following semilinear control equation of second-order:

$$y_{tt}(t,x) = Ay(t,x) + \int_0^t k(t-s)g(s,x(s))ds + Bu(t), \quad (t,x) \in [0,T] \times (0,\pi),$$
$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad x \in (0,\pi), \tag{3.46}$$

where k belongs to $L^2(0,T)$. Let the assumption (F) be satisfied. Assume that $(y_0, y_1) \in H_0^1(0,\pi) \times L^2(0,\pi)$. Then the solution y of (3.46) exists and is unique in $C([0,T]; H_0^1(0,\pi)) \cap C^1((0,T]; L^2(0,\pi)) \cap W^{2,2}(0,T; H^{-1}(0,\pi))$, and the variation of constant formula (3.35) is established.

Remark 3.8. Since the operator $A : D(A) \subset H \to H$ is an unbounded operator, we will make use of the hypothesis (F). If A is a bounded operator from H into itself, then we may assume that $g : [0,T] \times H \to H$ is a nonlinear mapping satisfying

$$|g(t,x) - g(t,y)| \le L|x-y|, \tag{3.47}$$

for a positive constant L; therefore, our results can be obtained directly.

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