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*Research Article*

# Further Results on the Reverse Order Law for $\{1,3\}$-Inverse and $\{1,4\}$-Inverse of a Matrix Product

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Both Djordjević (2007) and Takane et al. (2007) have studied the equivalent conditions for $B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\}$ and $B\{1,4\}A\{1,4\} \subseteq (AB)\{1,4\}$. In this note, we derive the necessary and sufficient conditions for $B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\}$, $B\{1,4\}A\{1,4\} \supseteq (AB)\{1,4\}$, $B\{1,3\}A\{1,3\} = (AB)\{1,3\}$ and $B\{1,4\}A\{1,4\} = (AB)\{1,4\}$.

## 1. Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of all $m\times n$ matrices over the complex field $\mathbb{C}$. For $A \in \mathbb{C}^{m\times n}$, its range space, null space, rank, and conjugate transpose will be denoted by $\mathcal{R}(A)$, $\mathcal{N}(A)$, $r(A)$, and $A^*$, respectively. The symbol $\dim \mathcal{R}(A)$ denotes the dimension of $\mathcal{R}(A)$. The $n\times n$ identity matrix is denoted by $I_n$, and if the size is obvious from the context, then the subscript on $I_n$ can be neglected.

For a matrix $A \in \mathbb{C}^{m\times n}$, a generalized inverse $X$ of $A$ is a matrix which satisfies some of the following four Penrose equations:

$$(1)\ AXA = A, \qquad (2)\ XAX = X, \qquad (3)\ (AX)^* = AX, \qquad (4)\ (XA)^* = XA. \tag{1.1}$$

Let $\emptyset \neq \eta \subseteq \{1,2,3,4\}$. Then $A\eta$ denotes the set of all matrices $X$ which satisfy $(i)$ for all $i \in \eta$. Any matrix $X \in A\eta$ is called an $\eta$-inverse of $A$. One usually denotes any $\{1\}$-inverse of $A$ by $A^{(1)}$ or $A^-$, and any $\{1,3\}$-inverse of $A$ by $A^{(1,3)}$ which is also called a least squares g-inverses of $A$. Any $\{1,4\}$-inverse of $A$ is denoted by $A^{(1,4)}$ which is also called a minimum norm g-inverses of $A$. The unique $\{1,2,3,4\}$-inverse of $A$ is denoted by $A^\dagger$, which is called the Moore-Penrose generalized inverse of $A$. General properties of the above generalized inverses can be found in [1–3]. The research in this area is active, especially about the $\{2\}$-inverse and the reverse order law for generalized inverse; see [4–7].

There are very good results for the reverse order law for $\{1\}$-inverse and $\{1,2\}$-inverse of two-matrix or multi-matrix products, and Liu and Yang [8] studied equivalent conditions for $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3,4\}$, $B\{1,3,4\}A\{1,3,4\} \supseteq (AB)\{1,3,4\}$, and $B\{1,3,4\}A\{1,3,4\} = (AB)\{1,3,4\}$. Moreover, Wei and Guo [9] derived the reverse order law for $\{1,3\}$-inverse and $\{1,4\}$-inverse of two-matrix products by using the product singular value decomposition (P-SVD). However, there is a fly in the ointment in Wei and Guo's results. That is, those results contain the information of subblock produced by P-SVD. In other words, they are related to P-SVD. In order to overcome this shortcoming, two methods are employed. One is operator theory; the other is maximal and minimal rank of matrix expressions. Using these two different methods, both [6, 10] obtain

$$B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\} \Longleftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B), \tag{1.2}$$

$$B\{1,4\}A\{1,4\} \subseteq (AB)\{1,4\} \Longleftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*). \tag{1.3}$$

These results are our hope because there is no information of the P-SVD in them. Note that $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$ are equivalent to $r(B, A^*AB) = r(B)$ and $r(A^*, BB^*A^*) = r(A)$, respectively. Therefore, these results are only related to the range space (or the rank) of $A$, $A^*$, $B$, $B^*$ or their expressions. However, there are no analogs for $B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\}$ and $B\{1,4\}A\{1,4\} \supseteq (AB)\{1,4\}$. In this note, we derive the necessary and sufficient conditions for them. And after this we present a new equivalent conditions for $B\{1,3\}A\{1,3\} = (AB)\{1,3\}$ and $B\{1,4\}A\{1,4\} = (AB)\{1,4\}$, and this results are not related to P-SVD. To our knowledge, there is no article discussing these in the literature.

In this note we will need the following two lemmas.

**Lemma 1.1** (see [11, 12]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $X \in \mathbb{C}^{k\times l}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$. Then*

$$(1)\ r(A,B) = r(A) + r(B) - \dim \mathcal{R}(A) \cap \mathcal{R}(B); \tag{1.4}$$

$$(2)\ r(BX) = r(X) - \dim \mathcal{N}(B) \cap \mathcal{R}(X); \tag{1.5}$$

$$(3)\ r\begin{pmatrix} C \\ A \end{pmatrix} = r(A) + r\left[C\left(I - A^\dagger A\right)\right]; \tag{1.6}$$

$$(4)\ \max_X r(A - BXC) = \min\left\{ r[A,\ B],\ r\begin{pmatrix} A \\ C \end{pmatrix} \right\}; \tag{1.7}$$

$$(5)\ \max_{A^{(1,3)}} r\left(D - CA^{(1,3)}B\right) = \min\left\{ r\begin{pmatrix} A^*A & A^*B \\ C & D \end{pmatrix} - r(A), r\begin{pmatrix} B \\ D \end{pmatrix} \right\}; \tag{1.8}$$

$$(6)\ \min_{A^{(1,3)}} r\left(D - CA^{(1,3)}B\right) = r\begin{pmatrix} A^*A & A^*B \\ C & D \end{pmatrix} + r\begin{pmatrix} B \\ D \end{pmatrix} - r\begin{pmatrix} A & 0 \\ 0 & B \\ C & D \end{pmatrix}. \tag{1.9}$$

**Lemma 1.2** (see [13]). *Let $A_{i,j} \in \mathbb{C}^{m_i \times n_j}$ $(1 \le i,\, j \le 3)$ be given; $X \in \mathbb{C}^{m_1 \times n_3}$ and $Y \in \mathbb{C}^{m_3 \times n_1}$ are two arbitrary matrices. Then*

$$\min_{X,Y} r\begin{pmatrix} A_{11} & A_{12} & X \\ A_{21} & A_{22} & A_{23} \\ Y & A_{32} & A_{33} \end{pmatrix} = r(A_{21}, A_{22}, A_{23}) + r\begin{pmatrix} A_{12} \\ A_{22} \\ A_{32} \end{pmatrix}$$
$$+ \max\left\{ r\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} - r\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}\right)$$
$$-r(A_{21}, A_{22}),\ r\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} - \begin{pmatrix} A_{22} \\ A_{32} \end{pmatrix} - r(A_{22}, A_{23}) \right\}. \tag{1.10}$$

## 2. Main Results

In this section, we first give the minimal rank of $D - B^{(1,3)}A^{(1,3)}$ with respect to any $B^{(1,3)}$ and $A^{(1,3)}$. Secondly, the necessary and sufficient conditions for the inclusion $B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\}$ are obtained by using our previous result. Finally, we also give the necessary and sufficient conditions for $B\{1,3\}A\{1,3\} = (AB)\{1,3\}$, $B\{1,4\}A\{1,4\} \supseteq (AB)\{1,4\}$, and $B\{1,4\}A\{1,4\} = (AB)\{1,4\}$.

**Lemma 2.1.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$ and $D \in \mathbb{C}^{k\times m}$. Then*

$$\min_{B^{(1,3)},A^{(1,3)}} r\left(D - B^{(1,3)}A^{(1,3)}\right) = r\begin{pmatrix} B^*BD & B^* \\ A^* & A^*A \end{pmatrix} - \min\left\{ r\begin{pmatrix} B^* \\ A \end{pmatrix}, r\begin{pmatrix} BD \\ A^* \end{pmatrix} - r\begin{pmatrix} D \\ A^* \end{pmatrix} + n \right\}. \tag{2.1}$$

*Proof.* The expression of $\{1,3\}$-inverses of $A$ can be written as $A^{(1,3)} = A^\dagger + F_A V$, where $F_A = I - A^\dagger A$ and the matrix $V$ is arbitrary; see [1]. By combining this fact with elementary block matrix operations, it follows that

$$\begin{aligned} r\left(D - B^{(1,3)}A^{(1,3)}\right) &= r\left[\left(B^\dagger + F_B\widetilde{V}\right)\left(A^\dagger + F_A V\right) - D\right] \\ &= r\left(B^\dagger A^\dagger + B^\dagger F_A V + F_B\widetilde{V}A^\dagger + F_B\widetilde{V}F_A V - D\right) \\ &= r\begin{pmatrix} 0 & 0 & 0 & 0 & I_n & V \\ 0 & 0 & -I_m & 0 & 0 & I_m \\ 0 & 0 & 0 & I_n & F_A & 0 \\ -B^\dagger & F_B & -D & 0 & 0 & 0 \\ I_n & 0 & A^\dagger & I_n & 0 & 0 \\ \widetilde{V} & I_k & 0 & 0 & 0 & 0 \end{pmatrix} - k - m - 3n. \end{aligned} \tag{2.2}$$

Applying (1.10) to (2.2) gives

$$\min_{B^{(1,3)},A^{(1,3)}} r\left(D - B^{(1,3)}A^{(1,3)}\right) = r\left(F_B, B^{\dagger}A^{\dagger} - D, -B^{\dagger}F_A\right) + \max\left\{-r\left(F_B, B^{\dagger}F_A\right), r\begin{pmatrix} -D & 0 \\ A^{\dagger} & F_A \end{pmatrix} - r(F_A) - r\begin{pmatrix} F_B & -D & 0 \\ 0 & A^{\dagger} & -F_A \end{pmatrix}\right\}. \tag{2.3}$$

By using the elementary block matrix operations, the rank of the first partitioned matrix in the right-hand side of (2.3) is simplified as follows:

$$\begin{aligned} r&\left(F_B, B^{\dagger}A^{\dagger} - D, -B^{\dagger}F_A\right) \\ &= r\begin{pmatrix} -B^{\dagger} & F_B & -D & 0 \\ I_n & 0 & A^{\dagger} & -F_A \end{pmatrix} - n \\ &= r\begin{pmatrix} B^{\dagger} & 0 & 0 & 0 & 0 & 0 \\ B^{\dagger} & -B^{\dagger} & I_k - B^{\dagger}B & -D & 0 & 0 \\ 0 & I_n & 0 & A^{\dagger} & -I_n + A^{\dagger}A & A^{\dagger} \\ 0 & 0 & 0 & 0 & 0 & A^{\dagger} \end{pmatrix} - n - r\left(A^{\dagger}\right) - r\left(B^{\dagger}\right) \\ &= r\begin{pmatrix} B^{\dagger} & B^{\dagger} & B^{\dagger}B & 0 & 0 & 0 \\ B^{\dagger} & 0 & I_k & -D & 0 & 0 \\ 0 & I_n & 0 & 0 & -I_n & A^{\dagger} \\ 0 & 0 & 0 & -A^{\dagger} & -A^{\dagger}A & A^{\dagger} \end{pmatrix} - n - r(A) \ - \mathrm{r}(B) \\ &= r\begin{pmatrix} B^{\dagger}BD & B^{\dagger} \\ A^{\dagger} & A^{\dagger}A \end{pmatrix} + k - r(A) - r(B). \end{aligned} \tag{2.4}$$

Using the formula $r(AB) \leq \min\{r(A), r(B)\}$ together with the fact that

$$\begin{gathered} \begin{pmatrix} B^{*}B & 0 \\ 0 & A^{*}A \end{pmatrix}\begin{pmatrix} B^{\dagger}BD & B^{\dagger} \\ A^{\dagger} & A^{\dagger}A \end{pmatrix} = \begin{pmatrix} B^{*}BD & B^{*} \\ A^{*} & A^{*}A \end{pmatrix}, \\ \begin{pmatrix} B^{\dagger}\left(B^{\dagger}\right)^{*} & 0 \\ 0 & A^{\dagger}\left(A^{\dagger}\right)^{*} \end{pmatrix}\begin{pmatrix} B^{*}BD & B^{*} \\ A^{*} & A^{*}A \end{pmatrix} = \begin{pmatrix} B^{\dagger}BD & B^{\dagger} \\ A^{\dagger} & A^{\dagger}A \end{pmatrix} \end{gathered} \tag{2.5}$$

means that

$$r\begin{pmatrix} B^{\dagger}BD & B^{\dagger} \\ A^{\dagger} & A^{\dagger}A \end{pmatrix} = r\begin{pmatrix} B^{*}BD & B^{*} \\ A^{*} & A^{*}A \end{pmatrix}. \tag{2.6}$$

Substituting (2.6) into (2.4) yields

$$r\left(F_B, B^\dagger A^\dagger - D, -B^\dagger F_A\right) = r\begin{pmatrix} B^*BD & B^* \\ A^* & A^*A \end{pmatrix} + k - r(A) - r(\mathrm{B}). \tag{2.7}$$

Similarly, we obtain

$$\begin{gathered} r\left(F_B, B^\dagger F_A\right) = r\begin{pmatrix} B^* \\ A \end{pmatrix} + k - r(A) - r(B), \\ r\begin{pmatrix} -D & 0 \\ A^\dagger & -F_A \end{pmatrix} = r\begin{pmatrix} A^* \\ D \end{pmatrix} + n - r(A)\ , \\ r\begin{pmatrix} F_B & -D & 0 \\ 0 & A^\dagger & -F_A \end{pmatrix} = r\begin{pmatrix} BD \\ A^* \end{pmatrix} + n + k - r(A) - r(B). \end{gathered} \tag{2.8}$$

It is always ture that $\mathcal{R}(I - A^\dagger A) = \mathcal{N}(A)$. Therefore,

$$r(F_A) = r\left(I - A^\dagger A\right) = n - r(A). \tag{2.9}$$

Substituting (2.7)–(2.9) into (2.3) yields (2.1). □

**Theorem 2.2.** *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$. Then the following statements are equivalent:*

(1) $B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\}$;

(2) $r(A^*AB, B) + r(A) = r(AB) + \min\{r(A^*, B),\ \max\{n + r(A) - m,\ n + r(B) - k\}\}$.

*Proof.* We know that $B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\}$ is equivalent to saying that for an arbitrary $\{1,3\}$-inverse $(AB)^{(1,3)}$, there are $\{1,3\}$-inverses $A^{(1,3)}$ and $B^{(1,3)}$ satisfying $B^{(1,3)}A^{(1,3)} = (AB)^{(1,3)}$. That is,

$$B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\} \Longleftrightarrow \max_{(AB)^{(1,3)}} \min_{A^{(1,3)},B^{(1,3)}} r\left[(AB)^{(1,3)} - B^{(1,3)}A^{(1,3)}\right] = 0. \tag{2.10}$$

By using the formula (2.1), we get

$$\begin{aligned} &\min_{B^{(1,3)},A^{(1,3)}} r\left[(AB)^{(1,3)} - B^{(1,3)}A^{(1,3)}\right] \\ &\quad = r\begin{pmatrix} B^*B(AB)^{(1,3)} & B^* \\ A^* & A^*A \end{pmatrix} - \min\left\{r\begin{pmatrix} B^* \\ A \end{pmatrix},\ r\begin{pmatrix} B(AB)^{(1,3)} \\ A^* \end{pmatrix} - r\begin{pmatrix} (AB)^{(1,3)} \\ A^* \end{pmatrix} + n\right\}. \end{aligned} \tag{2.11}$$

Using the formulas (1.9) and (1.8) together with elementary block matrix operations, the maximal and minimal ranks of first partitioned matrix in the right-hand side of (2.11) are as follows:

$$\begin{aligned}
&\min_{(AB)^{(1,3)}} r\begin{pmatrix} B^*B(AB)^{(1,3)} & B^* \\ A^* & A^*A \end{pmatrix} \\
&= \min_{(AB)^{(1,3)}} \left[ r\left( \begin{pmatrix} 0 & B^* \\ A^* & A^*A \end{pmatrix} - \begin{pmatrix} -B^*B \\ 0 \end{pmatrix} (AB)^{(1,3)} (I, 0) \right) \right] \\
&= r\begin{pmatrix} B^*A^*AB & B^*A^* & 0 \\ -B^*B & 0 & B^* \\ 0 & A^* & A^*A \end{pmatrix} + r\begin{pmatrix} I & 0 \\ 0 & B^* \\ A^* & A^*A \end{pmatrix} - r\begin{pmatrix} AB & 0 & 0 \\ 0 & I & 0 \\ -B^*B & 0 & B^* \\ 0 & A^* & A^*A \end{pmatrix} \\
&= r\begin{pmatrix} B^*A^*A \\ B^* \end{pmatrix} + r(A) - r(AB) = \max_{(AB)^{(1,3)}} r\begin{pmatrix} B^*B(AB)^{(1,3)} & B^* \\ A^* & A^*A \end{pmatrix}.
\end{aligned} \tag{2.12}$$

Therefore, for an arbitrary $\{1,3\}$-inverse $(AB)^{(1,3)}$,

$$r\begin{pmatrix} B^*B(AB)^{(1,3)} & B^* \\ A^* & A^*A \end{pmatrix} = r\begin{pmatrix} B^*A^*A \\ B^* \end{pmatrix} + r(A) - r(AB). \tag{2.13}$$

Using formulas (1.6) and (1.5), we get

$$\begin{aligned}
r\begin{pmatrix} B(AB)^{(1,3)} \\ A^* \end{pmatrix} - r\begin{pmatrix} (AB)^{(1,3)} \\ A^* \end{pmatrix} &= r\left[ B(AB)^{(1,3)}\left(I - AA^{\dagger}\right)\right] - r\left[(AB)^{(1,3)}\left(I - AA^{\dagger}\right)\right] \\
&= -\dim \mathcal{N}(B) \cap \mathcal{R}\left[(AB)^{(1,3)}\left(I - AA^{\dagger}\right)\right].
\end{aligned} \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.11) produces

$$\begin{aligned}
\min_{B^{(1,3)}, A^{(1,3)}} r\left[(AB)^{(1,3)} - B^{(1,3)}A^{(1,3)}\right] &= r\begin{pmatrix} B^*A^*A \\ B^* \end{pmatrix} + r(A) - r(AB) \\
&\quad - \min\left\{ r\begin{pmatrix} B^* \\ A \end{pmatrix}, n - \dim \mathcal{N}(B) \cap \mathcal{R}\left[(AB)^{(1,3)}\left(I - AA^{\dagger}\right)\right] \right\}.
\end{aligned} \tag{2.15}$$

Furthermore, we have

$$\max_{(AB)^{(1,3)}} \min_{B^{(1,3)},A^{(1,3)}} r\left[(AB)^{(1,3)} - B^{(1,3)}A^{(1,3)}\right] = r\begin{pmatrix} B^*A^*A \\ B^* \end{pmatrix} + r(A) - r(AB) - \min\left\{ r\begin{pmatrix} B^* \\ A \end{pmatrix}, n - a \right\}, \tag{2.16}$$

where $a = \max_{(AB)^{(1,3)}} \dim \mathcal{N}(B) \cap \mathcal{R}[(AB)^{(1,3)}(I - AA^{\dagger})]$.

Next, we want to prove that $a$ is equal to $\min\{k - r(B),\ m - r(A)\}$. First observe that $a \le \min\{k - r(B),\ m - r(A)\}$ since $a \le \dim \mathcal{N}(B) = k - r(B)$ and $a \le \max_{(AB)^{(1,3)}} r[(AB)^{(1,3)}(I - AA^{\dagger})] \le r(I - AA^{\dagger}) = \dim \mathcal{N}(A^*) = m - r(A)$. Therefore, $a = \min\{k - r(B),\ m - r(A)\}$ holds if and only if there is a $\{1,3\}$-inverse $(AB)^{(1,3)}$ such that

$$\dim \mathcal{N}(B) \cap \mathcal{R}\left[(AB)^{(1,3)}\left(I - AA^{\dagger}\right)\right] = \min\{k - r(B), m - r(A)\}. \tag{2.17}$$

Suppose that $m - r(A) \le k - r(B)$. Also note that $r[(AB)^{(1,3)}(I - AA^{\dagger})] \le m - r(A)$ for arbitrary $\{1,3\}$-inverses $(AB)^{(1,3)}$. Therefore, for some $(AB)^{(1,3)}$, (2.17) holds if and only if there is a $\{1,3\}$-inverse $(AB)^{(1,3)}$ such that $\mathcal{R}[(AB)^{(1,3)}(I - AA^{\dagger})] \subseteq \mathcal{N}(B)$ and $r[(AB)^{(1,3)}(I - AA^{\dagger})] = m - r(A)$ hold—that is,

$$\min_{(AB)^{(1,3)}} r\left[\begin{pmatrix} B \\ I \end{pmatrix}(AB)^{(1,3)}\left(I - AA^{\dagger}\right) - \begin{pmatrix} 0 \\ C \end{pmatrix}\right] = 0, \tag{2.18}$$

where $C$ is any $k \times m$ matrix and $r(C) = m - r(A)$. It follows from the formula (1.7) that $\max_X r(I - B^{\dagger}B)X(I - AA^{\dagger}) = \min\{r(I - B^{\dagger}B),\ r(I - AA^{\dagger})\} = m - r(A)$. Therefore, there is a matrix $X_0$ satisfying $r(I - B^{\dagger}B)X_0(I - AA^{\dagger}) = m - r(A)$. Let $C = (I - B^{\dagger}B)X_0(I - AA^{\dagger})$. It is always true that $r(C) = m - r(A)$, $BC = 0$, and $B^*A^*(I - AA^{\dagger}) = 0$. Use these equations together with the formula (1.9) to conclude that (2.18) holds. Therefore, if $m - r(A) \le k - r(B)$, then there is a $\{1,3\}$-inverse $(AB)^{(1,3)}$ such that (2.17) holds.

On the other hand, suppose that $m - r(A) > k - r(B)$. Also note that $\dim \mathcal{N}(B) = k - r(B)$. Therefore, for some $(AB)^{(1,3)}$ (2.17) holds if and only if there is a $\{1,3\}$-inverse $(AB)^{(1,3)}$ such that $\mathcal{N}(B) = \mathcal{R}(I - B^{\dagger}B) \subseteq \mathcal{R}[(AB)^{(1,3)}(I - AA^{\dagger})]$ holds, that is,

$$\min_{(AB)^{(1,3)}} r\left[I - B^{\dagger}B - (AB)^{(1,3)}\left(I - AA^{\dagger}\right)X\right] = 0, \tag{2.19}$$

where $X$ is some $m \times k$ matrix. Use the formula (1.9) to find that

$$\min_{(AB)^{(1,3)}} r\left[I - B^{\dagger}B - (AB)^{(1,3)}\left(I - AA^{\dagger}\right)X\right]$$
$$= r\begin{pmatrix} B^*A^*AB & B^*A^*(I - AA^{\dagger})X \\ I & I - B^{\dagger}B \end{pmatrix} + r\begin{pmatrix} (I - AA^{\dagger})X \\ I - B^{\dagger}B \end{pmatrix} - r\begin{pmatrix} AB & 0 \\ 0 & (I - AA^{\dagger})X \\ I & I - B^{\dagger}B \end{pmatrix}$$
$$= r\begin{pmatrix} (I - AA^{\dagger})X \\ I - B^{\dagger}B \end{pmatrix} - r\left[\left(I - AA^{\dagger}\right)X\right]$$
$$= r\left(X^*\left(I - AA^{\dagger}\right), I - B^{\dagger}B\right) - r\left[X^*\left(I - AA^{\dagger}\right)\right]. \tag{2.20}$$

We know from (2.20) that (2.19) holds if and only if there is an $m \times k$ matrix $X$ such that $\mathcal{R}(I - B^{\dagger}B) \subseteq \mathcal{R}[X^*(I - AA^{\dagger})]$. In fact, note that $r(I - B^{\dagger}B) = \dim \mathcal{N}(B) = k - r(B)$ and $r(I - A^{\dagger}A) = \dim \mathcal{N}(A^*) = m - r(A)$, and let $P_1$, $P_2$, $Q_1$, and $Q_2$ be nonsingular matrices such that $I - B^{\dagger}B = P_1\begin{pmatrix} I_{k-r(B)} & 0 \\ 0 & 0 \end{pmatrix}Q_1$ and $I - A^{\dagger}A = P_2\begin{pmatrix} I_{m-r(A)} & 0 \\ 0 & 0 \end{pmatrix}Q_2$. Using this together with $m - r(A) > k - r(B)$ means that if $X^* = P_1P_2^{-1}$, then $\mathcal{R}(I - B^{\dagger}B) \subseteq \mathcal{R}[X^*(I - AA^{\dagger})]$. Therefore, if $m - r(A) > k - r(B)$, then there is a $\{1,3\}$-inverse $(AB)^{(1,3)}$ such that (2.17) holds.

In summary, there is a $\{1,3\}$-inverse $(AB)^{(1,3)}$ such that (2.17) holds. That is, $a = \min\{k - r(B),\ m - r(A)\}$. Apply this to (2.16) to obtain that

$$\max_{(AB)^{(1,3)}} \min_{B^{(1,3)}, A^{(1,3)}} r\left[(AB)^{(1,3)} - B^{(1,3)}A^{(1,3)}\right] = r(A^*AB, B) + r(A) - r(AB)$$
$$- \min\{r(A^*, B),\ \max\{n + r(B) - k,\ n + r(A) - m\}\}. \tag{2.21}$$

Noting that (2.10) and letting the right-hand side in (2.21) be equal to zero, then the equivalence between (1) and (2) follows immediately. □

It is obvious that $B\{1,3\}A\{1,3\} = (AB)\{1,3\}$ if and only if $B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\}$ and $B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\}$. Also note Theorem 2.2 and formula (1.2). It is easy to obtain the following theorem.

**Theorem 2.3.** *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$. Then the following statements are equivalent:*

(1) $B\{1,3\}A\{1,3\} = (AB)\{1,3\}$;

(2) $r(B, A^*AB) = r(B)$ *and* $r(A) + r(B) = r(AB) + \min\{r(A^*, B),\ \max\{n + r(B) - k,\ n + r(A) - m\}\}$.

*The following theorems can be obtained by applying Theorem 2.2 or Theorem 2.3 to the product $B^*A^*$ and using the fact that $X \in D\{1,3\}$ if and only if $X^* \in D^*\{1,4\}$.*

**Theorem 2.4.** *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$. Then the following statements are equivalent:*

(1) $B\{1,4\}A\{1,4\} \supseteq (AB)\{1,4\}$;

(2) $r(BB^*A^*, A^*) + r(B) = r(AB) + \min\{r(A^*, B),\ \max\{n + r(A) - m,\ n + r(B) - k\}\}$.

**Theorem 2.5.** *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$. Then the following statements are equivalent:*

(1) $B\{1,4\}A\{1,4\} = (AB)\{1,4\}$;

(2) $r(BB^*A^*,\ A^*) = r(A)$ *and* $r(A) + r(B) = r(AB) + \min\{r(A^*, B),\ \max\{n + r(A) - m,\ n + r(B) - k\}\}$.

## 3. Examples

In this section, we give two examples. The first example comes from [14], and they verify that $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$. However, this example does not only satisfy this result. In Example 3.1, we know that this example satisfies Theorems 2.3 and 2.5, and so we have $B\{1,3\}A\{1,3\} = (AB)\{1,3\}$ and $B\{1,4\}A\{1,4\} = (AB)\{1,4\}$. In this example, we will verify these results. Secondly, we give another example which only satisfies $B\{1,3\}A\{1,3\} \supset (AB)\{1,3\}$ and $B\{1,4\}A\{1,4\} \supset (AB)\{1,4\}$.

*Example 3.1.* Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}. \tag{3.1}$$

It is easy to obtain that

$$r(B, \mathrm{A}^*AB) = r(A^*, BB^*A^*) = r(B) = r(A) = r(B, A^*) = 2. \tag{3.2}$$

From Theorems 2.3 and 2.5, we can conclude that

$$B\{1,3\}A\{1,3\} = (AB)\{1,3\}, \qquad B\{1,4\}A\{1,4\} = (AB)\{1,4\}. \tag{3.3}$$

Now we verify this statement. Since

$$A\{1,3\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a_1 & a_2 & a_3 \\ -a_1 & -a_2 + \frac{1}{2} & -a_3 + \frac{1}{2} \end{pmatrix} \mid a_1, a_2, a_3 \in \mathbb{C} \right\},$$

$$B\{1,3\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ a_4 & a_5 & a_6 \end{pmatrix} \mid a_4, a_5, a_6 \in \mathbb{C} \right\},$$

$$(AB)\{1,3\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ -1 & \dfrac{1}{4} & \dfrac{1}{4} \\ a_7 & a_8 & a_9 \end{pmatrix} \mid a_7, a_8, a_9 \in \mathbb{C} \right\}, \tag{3.4}$$

we easily find that

$$B\{1,3\}A\{1,3\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ -1 & \dfrac{1}{4} & \dfrac{1}{4} \\ a & b & c \end{pmatrix} \mid a_i \in \mathbb{C},\ i = 1,2,\ldots,6 \right\}, \tag{3.5}$$

where $a = a_4 + a_1a_5 - a_1a_6$, $b = a_2a_5 - a_2a_6 + (1/2)a_6$, and $c = a_3a_5 - a_3a_6 + (1/2)a_6$. It is obvious that $B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\}$. If $a_1 = a_2 = 0$, $a_3 = 1$, $a_4 = a_7$, $a_5 = a_8 + a_9$, and $a_6 = 2a_8$, then we have $a = a_7$, $b = a_8$, and $c = a_9$, that is, $B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\}$. Therefore, $B\{1,3\}A\{1,3\} = (AB)\{1,3\}$.

On the other hand, since

$$A\{1,4\} = \left\{ \begin{pmatrix} 1 & a_1 & -a_1 \\ 0 & a_2 & -a_2 + \dfrac{1}{2} \\ 0 & -a_3 + \dfrac{1}{2} & a_3 \end{pmatrix} \mid a_1, a_2, a_3 \in \mathbb{C} \right\},$$

$$B\{1,4\} = \left\{ \begin{pmatrix} 1 & a_4 & -a_4 \\ -1 & a_5 & 1 - a_5 \\ 0 & a_6 & -a_6 \end{pmatrix} \mid a_4, a_5, a_6 \in \mathbb{C} \right\}, \tag{3.6}$$

$$(AB)\{1,4\} = \left\{ \begin{pmatrix} 1 & a_7 & -a_7 \\ -1 & a_8 & -a_8 + \dfrac{1}{2} \\ 0 & a_9 & -a_9 \end{pmatrix} \mid a_7, a_8, a_9 \in \mathbb{C} \right\},$$

we easily see that

$$B\{1,4\}A\{1,4\} = \left\{ \begin{pmatrix} 1 & d & -d \\ -1 & e & -e + \dfrac{1}{2} \\ 0 & f & -f \end{pmatrix} \mid a_i \in \mathbb{C},\ i = 1,2,\ldots,6 \right\}, \tag{3.7}$$

where $d = a_1 - (1/2)a_4 + a_2a_4 + a_3a_4$, $e = (1/2) - a_1 - a_3 - (1/2)a_5 + a_2a_5 + a_3a_5$, and $f = a_2a_6 + a_3a_6 - (1/2)a_6$. It is obvious that $B\{1,4\}A\{1,4\} \subseteq (AB)\{1,4\}$. If $a_1 = a_7$, $a_2 = a_7 + a_8 + a_9$, $a_3 = 1/2 - a_7 - a_8$, $a_4 = a_5 = 0$ and $a_6 = 1$, then we have $d = a_7$, $e = a_8$, and $f = a_9$, that is, $B\{1,4\}A\{1,4\} \supseteq (AB)\{1,4\}$. Therefore, $B\{1,4\}A\{1,4\} = (AB)\{1,4\}$.

*Example 3.2.* Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.8}$$

It is easy to obtain that

$$r(A) = r(B) = r(AB) = 2, \qquad r(B, A^*AB) = r(A^*, BB^*A^*) = r(B, A^*) = 3. \tag{3.9}$$

From Theorems 2.2 and 2.4, we can find that

$$B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\}, \qquad B\{1,4\}A\{1,4\} \supseteq (AB)\{1,4\}. \tag{3.10}$$

Furthermore, note that $r(B, A^*AB) = r(A^*, BB^*A^*) = 3 \neq r(B) = r(A) = 2$. Using Theorems 2.3 and 2.5, we can conclude that

$$B\{1,3\}A\{1,3\} \supset (AB)\{1,3\}, \qquad B\{1,4\}A\{1,4\} \supset (AB)\{1,4\}. \tag{3.11}$$

Now we verify this statement. Since

$$\begin{gathered}
A\{1,3\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a_1 & a_2 & a_3 \\ -a_1 & -a_2 + \frac{1}{2} & -a_3 + \frac{1}{2} \\ a_4 & a_5 & a_6 \end{pmatrix} \mid a_1, a_2, \ldots, a_6 \in \mathbb{C} \right\}, \\
B\{1,3\} = \left\{ \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \mid a_7, a_8, a_9, a_{10} \in \mathbb{C} \right\}, \\
(AB)\{1,3\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ a_{11} & a_{12} & a_{13} \end{pmatrix} \mid a_{11}, a_{12}, a_{13} \in \mathbb{C} \right\},
\end{gathered} \tag{3.12}$$

we easily get that

$$B\{1,3\}A\{1,3\} = \left\{ \begin{pmatrix} \frac{2}{3}+\frac{2}{3}a_1 & -\frac{1}{6}+\frac{2}{3}a_2 & -\frac{1}{6}+\frac{2}{3}a_3 \\ -\frac{1}{3}-\frac{1}{3}a_1 & \frac{1}{3}-\frac{1}{3}a_2 & \frac{1}{3}-\frac{1}{3}a_3 \\ a & b & c \end{pmatrix} \mid a_1, a_2, \ldots, a_{10} \in \mathbb{C} \right\}, \tag{3.13}$$

where $a = a_7 + a_1a_8 - a_1a_9 + a_4a_{10}$, $b = (1/2)a_9 + a_2a_8 - a_2a_9 + a_5a_{10}$, and $c = (1/2)a_9 + a_3a_8 - a_3a_9 + a_6a_{10}$. It is obvious that if $a_1 = 1/2)$, $a_2 = 1/4$, $a_3 = 1/4$, $a_4 = a_6 = a_8 = 0$, $a_5 = a_{12} - a_{13}$, $a_7 = 2a_{13} + a_{11}$, $a_9 = 4a_{13}$, and $a_{10} = 1$, then

$$\begin{pmatrix} \frac{2}{3}+\frac{2}{3}a_1 & -\frac{1}{6}+\frac{2}{3}a_2 & -\frac{1}{6}+\frac{2}{3}a_3 \\ -\frac{1}{3}-\frac{1}{3}a_1 & \frac{1}{3}-\frac{1}{3}a_2 & \frac{1}{3}-\frac{1}{3}a_3 \\ a & b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ a_{11} & a_{12} & a_{13} \end{pmatrix}. \tag{3.14}$$

That is, $B\{1,3\}A\{1,3\} \supseteq (AB)\{1,3\}$. Furthermore, note that if $a_1 \neq 1/2$, then there are some $B^{(1,3)}A^{(1,3)}$ which do not belong to $(AB)\{1,3\}$. Therefore, $B\{1,3\}A\{1,3\} \supset (AB)\{1,3\}$.

On the other hand, because

$$\begin{gathered} A\{1,4\} = \left\{ \begin{pmatrix} 1 & a_1 & -a_1 \\ 0 & a_2 & -a_2+\frac{1}{2} \\ 0 & a_3 & -a_3+\frac{1}{2} \\ 0 & a_4 & -a_4 \end{pmatrix} \mid a_1, a_2, a_3, a_4 \in \mathbb{C} \right\}, \\ B\{1,4\} = \left\{ \begin{pmatrix} a_5 & -a_5+1 & a_5-1 & a_6 \\ a_7 & -a_7 & a_7+1 & a_8 \\ a_9 & -a_9 & a_9 & a_{10} \end{pmatrix} \mid a_5, a_6, \ldots, a_{10} \in \mathbb{C} \right\}, \\ (AB)\{1,4\} = \left\{ \begin{pmatrix} 1 & a_{11} & -a_{11} \\ -\frac{1}{2} & a_{12} & -a_{12}+\frac{1}{2} \\ 0 & a_{13} & -a_{13} \end{pmatrix} \mid a_{11}, a_{12}, a_{13} \in \mathbb{C} \right\}, \end{gathered} \tag{3.15}$$

we easily obtain that

$$B\{1,4\}A\{1,4\} = \left\{ \begin{pmatrix} a_5 & d & -d \\ a_7 & e & -e+\frac{1}{2} \\ a_9 & f & -f \end{pmatrix} \mid a_1, a_2, \ldots, a_{10} \in \mathbb{C} \right\}, \tag{3.16}$$

where $d = a_2 - a_3 + a_1a_5 - a_2a_5 + a_3a_5 + a_4a_6$, $e = a_3 + a_1a_7 - a_2a_7 + a_3a_7 + a_4a_8$, and $f = a_1a_9 - a_2a_9 + a_3a_9 + a_4a_{10}$. It is obvious that if $a_1 = a_{11}$, $a_2 = a_6 = a_8 = a_9 = 0$, $a_3 = a_{11} + 2a_{12}$, $a_4 = a_{13}$, $a_5 = a_{10} = 1$ and $a_7 = -1/2$, then

$$\begin{pmatrix} a_5 & d & -d \\ a_7 & e & -e + \dfrac{1}{2} \\ a_9 & f & -f \end{pmatrix} = \begin{pmatrix} 1 & a_{11} & -a_{11} \\ -\dfrac{1}{2} & a_{12} & -a_{12} + \dfrac{1}{2} \\ 0 & -a_{13} & -a_{13} \end{pmatrix}. \tag{3.17}$$

That is, $B\{1,4\}A\{1,4\} \supseteq (AB)\{1,4\}$. Furthermore, note that if $a_5 \neq 1$, then there are some $B^{(1,4)}A^{(1,4)}$ which do not belong to $(AB)\{1,4\}$. Therefore, $B\{1,4\}A\{1,4\} \supset (AB)\{1,4\}$.

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