

Research Article

Solutions and Stability of Generalized Mixed Type QC Functional Equations in Random Normed Spaces

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We achieve the general solution and the generalized stability result for the following functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary t -norms: $f(x + ky) + f(x - ky) = k^2[f(x + y) + f(x - y)] + (2(k^2 - 1)/k^2(k - 2))f(kx) - ((k^3 - k^2 - k + 1)/2(k - 2))f(2x) + \tilde{f}(2y) - 8\tilde{f}(y)$, where $\tilde{f}(y) := f(y) - f(-y)$ for any fixed integer k with $k \neq 0, \pm 1, 2$.

1. Introduction

In 1940, the stability problem of functional equations originated from a question given by Ulam [1], that is, the stability of group homomorphisms.

Let (G_1, \cdot) be a group and $(G_2, *, d)$ be a metric group with the metric d . For any $\epsilon > 0$, does there exist $\delta > 0$, such that, if a mapping $h : G_1 \rightarrow G_2$ satisfies inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$.

For all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In other words, under what condition, does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation.

In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces.

Let f be a mapping between Banach spaces E and E' such that, for some $\delta > 0$,

$$\|f(x+y) - f(x) - f(y)\| \leq \delta, \quad \forall x, y \in E. \quad (1.1)$$

Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta, \quad \forall x, y \in E. \quad (1.2)$$

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for any fixed $x \in E$, then T is linear.

In 1978, Rassias [3] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded. In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as the *Hyers-Ulam-Rassias stability of functional equations* (see [5–17]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad (1.3)$$

is related to symmetric biadditive mapping. It is natural that this equation is called a *quadratic functional equation*. In particular, every solution of the quadratic equation (1.3) is said to be a *quadratic mapping*. It is well known that a mapping f between real vector spaces E and E' is quadratic if and only if there exists a unique symmetric biadditive mapping B such that $f(x) = B(x, x)$ for all $x \in E$ (see [5, 18]). The biadditive mapping B is given by

$$B(x, y) = \frac{1}{4}(f(x+y) - f(x-y)), \quad \forall x, y \in E. \quad (1.4)$$

Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.3) was proved by Skof [19] for a mapping $f : E \rightarrow E'$, where E is a normed space and E' is a Banach space. Cholewa [20] noticed that the theorem of Skof is still true if the relevant domain A is replaced by an abelian group. In [21], Czerwik proved the Hyers-Ulam-Rassias stability of (1.3) and Grabiec [22] generalized these results mentioned above.

Recently, Jun and Kim [23] introduced the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \quad \forall x, y \in X, \quad (1.5)$$

where f is a mapping from a real vector space X into a real vector space Y and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.5). The function $f(x) = x^3$ satisfies the functional equation (1.5), which is thus called a *cubic functional equation*. Every solution of the cubic functional equation is said to be a *cubic function*. Also, Jun and Kim proved that a function f between real vector spaces X and Y is a solution of (1.5) if and only if there exists a unique function $C : X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$ and C is symmetric for each fixed one variable and is additive for fixed two variables.

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces as in [24–28]. Throughout this paper, Δ^+ is the space of distribution

functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \rightarrow [0, 1]$ such that F is left-continuous and nondecreasing on \mathbb{R} , $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^-F(+\infty) = 1$, where $l^-f(x)$ denotes the left limit of the function f at the point x , that is, $l^-f(x) = \lim_{t \rightarrow x^-} f(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases} \quad (1.6)$$

Definition 1.1 (see [27]). A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a *continuous triangular norm* (briefly, a *continuous t -norm*) if T satisfies the following conditions:

- (a) T is commutative and associative;
- (b) T is continuous;
- (c) $T(a, 1) = a$ for all $a \in [0, 1]$;
- (d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t -norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz t -norm).

Recall (see [29, 30]) that, if T is a t -norm and $\{x_n\}$ is a given sequence of numbers in $[0, 1]$, then $T_{i=1}^n x_i$ is defined recurrently by

$$T_{i=1}^1 x_i = x_1, \quad T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n), \quad \forall n \geq 2, \quad (1.7)$$

and $T_{i=1}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$.

It is known [30] that, for the Lukasiewicz t -norm, the following implication holds:

$$\lim_{n \rightarrow \infty} (T_L)_{i=1}^\infty x_{n+i} = 1 \iff \sum_{n=1}^{\infty} (1 - x_n) < \infty. \quad (1.8)$$

Definition 1.2 (see [28]). A *random normed space* (briefly, *RN-space*) is a triple (X, μ, T) , where X is a vector space, T is a continuous t -norm and μ is a mapping from X into D^+ satisfying the following conditions:

- (RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
- (RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$;
- (RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Every normed spaces $(X, \|\cdot\|)$ defines a random normed space (X, μ, T_M) , where

$$\mu_x(t) = \frac{t}{t + \|x\|}, \quad \forall t > 0, \quad (1.9)$$

and T_M is the minimum t -norm. This space is called the *induced random normed space*.

Definition 1.3. Let (X, μ, T) be a RN-space.

- (1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n-x}(\epsilon) > 1 - \lambda$ for all $n \geq N$.
- (2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n-x_m}(\epsilon) > 1 - \lambda$ for all $n \geq m \geq N$.
- (3) A RN-space (X, μ, T) is said to be *complete* if every Cauchy sequence in X is convergent to a point in X .

Theorem 1.4 (see [27]). *If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$, then $\lim_{n \rightarrow \infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

The stability of different functional equations in fuzzy normed spaces and random normed spaces has been studied in [13, 31–42].

In this paper, we deal with the following functional equation for fixed integers k with $k \neq 0, \pm 1, 2$:

$$f(x + ky) + f(x - ky) = k^2[f(x + y) + f(x - y)] + \frac{2(k^2 - 1)}{k^2(k - 2)}f(kx) - \frac{k^3 - k^2 - k + 1}{2(k - 2)}f(2x) + \tilde{f}(2y) - 8\tilde{f}(y), \quad (1.10)$$

where $\tilde{f}(y) := f(y) - f(-y)$ on random normed spaces. It is easy to see that the mapping $f(x) = ax^2 + bx^3 + c$ is a solution of the functional equation (1.10). In Section 2, we investigate the general solution of functional equation (1.10) when f is a mapping between vector spaces and, in Section 3, we establish the stability of the functional equation (1.10) in RN-spaces.

2. General Solutions

Before proceeding to the proof of Theorem 2.3 which is the main result in this section, we need the following two lemmas.

Lemma 2.1. *If an even function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies (1.10), then f is quadratic.*

Proof. Setting $x = 0$ in (1.10), by the evenness of f , we obtain

$$f(ky) = k^2 f(y), \quad \forall y \in X. \quad (2.1)$$

Interchanging x with y in (2.1), we have

$$f(kx) = k^2 f(x), \quad \forall x \in X. \quad (2.2)$$

Letting $y = 0$ in (1.10), we have

$$f(x) = \frac{-1}{k^2(k - 2)}f(kx) + \frac{k - 1}{4(k - 2)}f(2x), \quad \forall x \in X. \quad (2.3)$$

It follows from (2.2) and (2.3) that

$$f(2x) = 4f(x), \quad \forall x \in X. \quad (2.4)$$

According to (2.2), (2.4) and using the evenness of f , (1.10) can be written as

$$f(x + ky) + f(x - ky) = k^2 f(x + y) + k^2 f(x - y) + 2(1 - k^2)f(x), \quad \forall x, y \in X. \quad (2.5)$$

Replacing x by kx in (2.5) and then using (2.2), we obtain

$$f(kx + y) + f(kx - y) = f(x + y) + f(x - y) + 2(k^2 - 1)f(x), \quad \forall x, y \in X. \quad (2.6)$$

Interchanging x with y in (2.5), by the evenness of f , we have

$$f(kx + y) + f(kx - y) = k^2 f(x + y) + k^2 f(x - y) + 2(1 - k^2)f(y), \quad \forall x, y \in X. \quad (2.7)$$

But, since $k \neq 0, \pm 1, 2$, it follows from (2.6) and (2.7) that

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad \forall x, y \in X, \quad (2.8)$$

which shows that f is quadratic. This completes the proof. \square

Lemma 2.2. *If an odd function $f : X \rightarrow Y$ satisfies (1.10), then f is cubic.*

Proof. Letting $x = 0$ in (1.10), by the oddness of f , we have

$$f(2y) = 8f(y), \quad \forall y \in X. \quad (2.9)$$

Setting $y = 0$ in (1.10) and then using (2.9), we obtain

$$f(kx) = k^3 f(x), \quad \forall x \in X. \quad (2.10)$$

According to (2.9), (2.10) and using the oddness of f , (1.10) can be written as

$$f(x + ky) + f(x - ky) = k^2 f(x + y) + k^2 f(x - y) + 2(1 - k^2)f(x), \quad \forall x, y \in X. \quad (2.11)$$

Letting $y = x$ in (2.11) and using (2.9), by the oddness of f , it follows that

$$f((k + 1)x) = f((k - 1)x) + 2(1 + 3k^2)f(x), \quad \forall x \in X. \quad (2.12)$$

Replacing x by $(k-1)x$ in (2.11), we have

$$\begin{aligned} & f((k-1)x+ky) + f((k-1)x-ky) \\ &= k^2 f((k-1)x+y) + k^2 f((k-1)x-y) \\ & \quad + 2(1-k^2)f((k-1)x), \quad \forall x, y \in X. \end{aligned} \tag{2.13}$$

Now, replacing x by $(k+1)x$ in (2.11) and using (2.12), we have

$$\begin{aligned} & f((k+1)x+ky) + f((k+1)x-ky) \\ &= k^2 f((k+1)x+y) + k^2 f((k+1)x-y) \\ & \quad + 2(1-k^2)f((k-1)x) + 4(1-k^2)(3k^2+1)f(x), \quad \forall x, y \in X. \end{aligned} \tag{2.14}$$

Substituting $x+y$ with x in (2.11) and then $x-y$ with x in (2.11), we obtain

$$\begin{aligned} & f(x+(k+1)y) + f(x-(k-1)y) \\ &= k^2 f(x+2y) + 2(1-k^2)f(x+y) + k^2 f(x), \quad \forall x, y \in X, \end{aligned} \tag{2.15}$$

$$\begin{aligned} & f(x-(k+1)y) + f(x+(k-1)y) \\ &= k^2 f(x-2y) + 2(1-k^2)f(x-y) + k^2 f(x), \quad \forall x, y \in X. \end{aligned} \tag{2.16}$$

If we subtract (2.16) from (2.15), we have

$$\begin{aligned} & f(x+(k+1)y) - f(x-(k+1)y) \\ &= k^2 f(x+2y) - k^2 f(x-2y) + f(x+(k-1)y) - f(x-(k-1)y) \\ & \quad + 2(1-k^2)f(x+y) - 2(1-k^2)f(x-y), \quad \forall x, y \in X. \end{aligned} \tag{2.17}$$

Interchanging x with y in (2.17) and using the oddness of f , we get

$$\begin{aligned} & f((k+1)x+y) + f((k+1)x-y) \\ &= k^2 f(2x+y) + k^2 f(2x-y) + f((k-1)x+y) + f((k-1)x-y) \\ & \quad + 2(1-k^2)f(x+y) + 2(1-k^2)f(x-y), \quad \forall x, y \in X. \end{aligned} \tag{2.18}$$

Thus it follows from (2.14) and (2.18) that

$$\begin{aligned}
 & f((k+1)x+ky) + f((k+1)x-ky) \\
 &= k^2 f((k-1)x+y) + k^2 f((k-1)x-y) + k^4 f(2x+y) + k^4 f(2x-y) \\
 & \quad + 2k^2(1-k^2)f(x+y) + 2k^2(1-k^2)f(x-y) \\
 & \quad + 2(1-k^2)f((k-1)x) + 4(1-k^2)(3k^2+1)f(x), \quad \forall x, y \in X.
 \end{aligned} \tag{2.19}$$

Again, substituting $x+y$ with y in (2.11) and then $x-y$ with y in (2.11), we get, by the oddness of f ,

$$\begin{aligned}
 & f((k+1)x+ky) - f((k-1)x+ky) \\
 &= k^2 f(2x+y) + k^2 f(-y) + 2(1-k^2)f(x), \quad \forall x, y \in X,
 \end{aligned} \tag{2.20}$$

$$\begin{aligned}
 & f((k+1)x-ky) - f((k-1)x-ky) \\
 &= k^2 f(2x-y) + k^2 f(y) + 2(1-k^2)f(x), \quad \forall x, y \in X.
 \end{aligned} \tag{2.21}$$

Then, by adding (2.20) to (2.21) and then using (2.13), we have

$$\begin{aligned}
 & f((k+1)x+ky) + f((k+1)x-ky) \\
 &= k^2 f((k-1)x+y) + k^2 f((k-1)x-y) + 2(1-k^2)f((k-1)x) \\
 & \quad + k^2 f(2x+y) + k^2 f(2x-y) + 4(1-k^2)f(x), \quad \forall x, y \in X.
 \end{aligned} \tag{2.22}$$

Finally, if we compare (2.19) with (2.22), then we conclude that

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \quad \forall x, y \in X. \tag{2.23}$$

Therefore, f is a cubic function. This completes the proof. \square

Theorem 2.3. *A function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies (1.10) for all $x, y \in X$ if and only if there exist functions $Q : X \times X \rightarrow Y$ and $C : X \times X \times X \rightarrow Y$ such that $f(x) = Q(x, x) + C(x, x, x)$ for all $x \in X$, where the function C is symmetric for each fixed one variable and is additive for fixed two variables and Q is symmetric biadditive.*

Proof. Let f be a mapping with $f(0) = 0$ satisfies (1.10). We decompose f into the even part and odd part by putting

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)), \quad \forall x \in X. \tag{2.24}$$

It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in X$ and it is easy to show that the functions f_e and f_o satisfy (1.10). Hence, by Lemmas 2.1 and 2.2, we know that the functions f_e and f_o are quadratic and cubic, respectively. Thus there exist a symmetric biadditive function $Q : X \times X \rightarrow Y$ such that $f_e(x) = Q(x, x)$ for all $x \in X$ and the function $C : X \times X \times X \rightarrow Y$ such that $f_o(x) = C(x, x, x)$ for all $x \in X$, where the function C is symmetric for each fixed one variable and is additive for fixed two variables. Therefore, we get $f(x) = C(x, x, x) + Q(x, x)$ for all $x \in X$.

Conversely, let $f(x) = C(x, x, x) + Q(x, x)$ for all $x \in X$, where the function C is symmetric for each fixed one variable and is additive for fixed two variables and Q is biadditive. By a simple computation, one can show that the functions $x \mapsto C(x, x, x)$ and $x \mapsto Q(x, x)$ satisfy the functional equation (1.10). Thus the function f satisfies (1.10). This completes the proof. \square

3. Stability Problems

From now on, we suppose that X is a real linear space, (Y, μ, T) is a complete RN-space and $f : X \rightarrow Y$ is a function with $f(0) = 0$ for which there exists a mapping $\rho : X \times X \rightarrow D^+$ ($\rho(x, y)$ is denoted by $\rho_{x,y}$) with the property:

$$\begin{aligned} & \mu_{f(x+ky)+f(x-ky)-k^2[f(x+y)+f(x-y)]-(2(k^2-1)/k^2(k-2))f(kx)+((k^3-k^2-k+1)/2(k-2))f(2x)-\tilde{f}(2y)+8\tilde{f}(y)}(t) \\ & \geq \rho_{x,y}(t), \quad \forall x, y \in X, t > 0, \end{aligned} \quad (3.1)$$

where $\tilde{f}(y) := f(y) - f(-y)$ for all $y \in X$.

Theorem 3.1. *Let*

$$\begin{aligned} & \lim_{m \rightarrow \infty} T_{j=1}^{\infty} \left(\rho_{0, k^{m+j-1}x} \left(k^{2m+j}t \right) \right) = 1 \\ & = \lim_{m \rightarrow \infty} \rho_{k^m x, k^m y} \left(k^{2m}t \right), \quad \forall x, y \in X, t > 0. \end{aligned} \quad (3.2)$$

Suppose that an even function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies inequality

$$\begin{aligned} & \mu_{f(x+ky)+f(x-ky)-k^2[f(x+y)+f(x-y)]-(2(k^2-1)/k^2(k-2))f(kx)+((k^3-k^2-k+1)/2(k-2))f(2x)-\tilde{f}(2y)+8\tilde{f}(y)}(t) \\ & \geq \rho_{x,y}(t), \quad \forall x, y \in X, t > 0, \end{aligned} \quad (3.3)$$

where $\tilde{f}(y) := f(y) - f(-y)$ for all $y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\mu_{Q(x)-f(x)}(t) \geq T_{j=1}^{\infty} \left(\rho_{0, k^{j-1}x} \left(k^j t \right) \right), \quad \forall x \in X, t > 0. \quad (3.4)$$

Proof. It follows from (3.3) and the evenness of f that

$$\begin{aligned} & \mu_{f(x+ky)+f(x-ky)-k^2[f(x+y)+f(x-y)]-(2(k^2-1)/k^2(k-2))f(kx)+((k^3-k^2-k+1)/2(k-2))f(2x)}(t) \\ & \geq \rho_{x,y}(t), \quad \forall x, y \in X, t > 0. \end{aligned} \quad (3.5)$$

Setting $x = 0$ in (3.5), we get

$$\mu_{2f(ky)-2k^2f(y)}(t) \geq \rho_{0,y}(t), \quad \forall y \in X, t > 0. \quad (3.6)$$

If we replace y by x in (3.6) and divide both sides of (3.6) by 2, we get

$$\mu_{f(kx)-k^2f(x)}(t) \geq \rho_{0,x}(2t) \geq \rho_{0,x}(t), \quad \forall x \in X, t > 0. \quad (3.7)$$

In other words, we have

$$\mu_{f(kx)/k^2-f(x)}(t) \geq \rho_{0,x}(k^2t), \quad \forall x \in X, t > 0. \quad (3.8)$$

Therefore, it follows that

$$\mu_{f(k^{n+1}x)/k^{2(n+1)}-f(k^n x)/k^{2n}}\left(\frac{t}{k^{2n}}\right) \geq \rho_{0,k^n x}(k^2t), \quad \forall x \in X, t > 0, n \in \mathbb{N}. \quad (3.9)$$

Hence we have

$$\mu_{f(k^{n+1}x)/k^{2(n+1)}-f(k^n x)/k^{2n}}(t) \geq \rho_{0,k^n x}(k^{2(n+1)}t), \quad \forall x \in X, t > 0, n \in \mathbb{N}. \quad (3.10)$$

This means that

$$\mu_{f(k^{n+1}x)/k^{2(n+1)}-f(k^n x)/k^{2n}}\left(\frac{t}{k^{n+1}}\right) \geq \rho_{0,k^n x}(k^{(n+1)}t), \quad \forall x \in X, t > 0, n \in \mathbb{N}. \quad (3.11)$$

Since $1 > 1/2 + 1/2^2 + \dots + 1/2^m$, by the triangle inequality, it follows that

$$\begin{aligned} \mu_{f(k^m x)/k^{2m}-f(x)}(t) & \geq T_{n=0}^{m-1} \left(\mu_{f(k^{n+1}x)/k^{2(n+1)}-f(k^n x)/k^{2n}} \left(\sum_{n=0}^{m-1} \frac{t}{k^{n+1}} \right) \right) \\ & \geq T_{n=0}^{m-1} \left(\rho_{0,k^n x}(k^{(n+1)}t) \right) \\ & = T_{j=1}^m \left(\rho_{0,k^{j-1}x}(k^j t) \right), \quad \forall x \in X, t > 0. \end{aligned} \quad (3.12)$$

In order to prove the convergence of the sequence $\{f(k^m x)/k^{2m}\}$, we replace x with $k^{m'}x$ in (3.12) to find that

$$\mu_{f(k^{m+m'}x)/k^{2(m+m')} - f(k^{m'}x)/k^{2m'}}(t) \geq T_{j=1}^m \left(\rho_{0,k^{j+m'-1}x}(k^{j+2m'}t) \right), \quad \forall x \in X, t > 0. \quad (3.13)$$

Since the right hand side of inequality (3.13) tends to 1 as $m', m \rightarrow \infty$, the sequence $\{f(k^m x)/k^{2m}\}$ is a Cauchy sequence. Therefore, we may define

$$Q(x) = \lim_{m \rightarrow \infty} \frac{f(k^m x)}{k^{2m}}, \quad \forall x \in X. \quad (3.14)$$

Now, we show that Q is a quadratic mapping. In fact, replacing x, y with $k^m x$ and $k^m y$, respectively, in (3.3) and then taking the limit as $m \rightarrow \infty$, we find that Q satisfies (1.10) for all $x, y \in X$. Therefore, the mapping $Q : X \rightarrow Y$ is quadratic.

To prove (3.4), taking the limit as $m \rightarrow \infty$ in (3.12), we get (3.4).

Finally, to prove the uniqueness of the quadratic function Q subject to (3.4), assume that there exists a quadratic function Q' which satisfies (3.4). Since $Q(k^m x) = k^{2m}Q(x)$ and $Q'(k^m x) = k^{2m}Q'(x)$ for all $x \in X$ and $m \in \mathbb{N}$, it follows from (3.4) that

$$\begin{aligned} \mu_{Q(x)-Q'(x)}(t) &= \mu_{Q(k^m x)-Q'(k^m x)}(k^{2m}t) \\ &\geq T\left(\mu_{Q(k^m x)-f(k^m x)}(k^{2m-1}t), \mu_{f(k^m x)-Q'(k^m x)}(k^{2m-1}t)\right) \\ &\geq T\left(T_{j=1}^{\infty}\left(\rho_{0, k^{j+m-1}x}(k^{2m+j}t)\right), T_{j=1}^{\infty}\left(\rho_{0, k^{j+m-1}x}(k^{2m+j}t)\right)\right) \end{aligned} \quad (3.15)$$

for all $x \in X$ and $t > 0$. Therefore, by letting $m \rightarrow \infty$ in inequality (3.15), we find that $Q = Q'$. This completes the proof. \square

Theorem 3.2. Let $f : X \rightarrow Y$ be an odd mapping with $f(0) = 0$ satisfies inequality

$$\begin{aligned} \mu_{f(x+ky)+f(x-ky)-k^2[f(x+y)+f(x-y)]-(2(k^2-1)/k^2(k-2))f(kx)+((k^3-k^2-k+1)/2(k-2))f(2x)-\tilde{f}(2y)+8\tilde{f}(y)}(t) \\ \geq \rho_{x,y}(t), \quad \forall x, y \in X, t > 0, \end{aligned} \quad (3.16)$$

where $\tilde{f}(y) := f(y) - f(-y)$ for all $y \in X$. If

$$\lim_{m \rightarrow \infty} T_{j=1}^{\infty} \left(T \left(\rho_{0, k^{m+j-1}x} \left(\frac{2k^{2j+3m}}{k^2(k-1)} t \right), \rho_{k^{m+j-1}x, 0} \left(\frac{k^{(3m+2j)}(1-k^2)}{k^2(k-2)} t \right) \right) \right) = 1, \quad (3.17)$$

$$\lim_{m \rightarrow \infty} \rho_{k^m x, k^m y} (k^{3m} t) = 1, \quad \forall x, y \in X, t > 0, \quad (3.18)$$

then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\mu_{C(x)-f(x)}(t) \geq T_{j=1}^{\infty} \left(T \left(\rho_{0, k^{j-1}x} \left(\frac{2k^{2j}}{k^2(k-1)} t \right), \rho_{k^{j-1}x, 0} \left(\frac{k^{2j}(1-k^2)}{k^2(k-2)} t \right) \right) \right) \quad (3.19)$$

for all $x \in X$ and $t > 0$.

Proof. It follows from (3.15) and the oddness of f that

$$\begin{aligned} & \mu_{f(x+ky)+f(x-ky)-k^2[f(x+y)+f(x-y)]-(2(k^2-1)/k^2(k-2))f(kx)+((k^3-k^2-k+1)/2(k-2))f(2x)-2f(2y)+16f(y)}(t) \\ & \geq \rho_{x,y}(t), \quad \forall x, y \in X, t > 0. \end{aligned} \quad (3.20)$$

By putting $x = 0$ in (3.20) and using (RN2), we obtain

$$\mu_{2f(2y)-16f(y)}(t) \geq \rho_{0,y}(t), \quad \forall y \in X, t > 0. \quad (3.21)$$

If we replace y by x in (3.21) and divide both sides of (3.21) by 2, we get

$$\mu_{f(2x)-8f(x)}(t) \geq \rho_{0,x}(2t) \geq \rho_{0,x}(t), \quad \forall x \in X, t > 0. \quad (3.22)$$

Letting $y = 0$ in (3.20), we get

$$\begin{aligned} & \mu_{2(1-k^2)f(x)-(2(k^2-1)/k^2(k-2))f(kx)+((k^3-k^2-k+1)/2(k-2))f(2x)}(t) \\ & \geq \rho_{x,0}(t), \quad \forall x \in X, t > 0. \end{aligned} \quad (3.23)$$

Therefore, we have

$$\mu_{k^2(k-2)f(x)+f(kx)-(k^2(k-1)/4)f(2x)}(t) \geq \rho_{x,0}\left(\frac{2(1-k^2)}{k^2(k-2)}t\right), \quad \forall x \in X, t > 0. \quad (3.24)$$

It follows from (3.22) and (3.24) that

$$\begin{aligned} \mu_{f(kx)-k^3f(x)}(t) & \geq T\left(\rho_{0,x}\left(\frac{2}{k^2(k-1)}t\right), \rho_{x,0}\left(\frac{(1-k^2)}{k^2(k-2)}t\right)\right) \\ & \quad \forall x \in X, t > 0, k \in \mathbb{N}. \end{aligned} \quad (3.25)$$

Let

$$\psi_{(x,x)}(t) = T\left(\rho_{0,x}\left(\frac{2}{k^2(k-1)}t\right), \rho_{x,0}\left(\frac{(1-k^2)}{k^2(k-2)}t\right)\right), \quad \forall x \in X, t > 0. \quad (3.26)$$

Then we get

$$\mu_{f(kx)-k^3f(x)}(t) \geq \psi_{(x,x)}(t), \quad \forall x \in X, t > 0. \quad (3.27)$$

It follows that

$$\mu_{f(kx)/k^3-f(x)}(t) \geq \psi_{(x,x)}(k^3t), \quad \forall x \in X, t > 0. \quad (3.28)$$

Hence we have

$$\mu_{f(k^{n+1}x)/k^{3(n+1)}-f(k^n x)/k^{3n}}\left(\frac{t}{k^{3n}}\right) \geq \psi_{(k^n x, k^n x)}(k^3 t), \quad \forall x \in X, t > 0, \quad (3.29)$$

which implies that

$$\mu_{f(k^{n+1}x)/k^{3(n+1)}-f(k^n x)/k^{3n}}(t) \geq \psi_{(k^n x, k^n x)}(k^{3(n+1)}t), \quad \forall x \in X, t > 0, n \in \mathbb{N}. \quad (3.30)$$

Thus we have

$$\mu_{f(k^{n+1}x)/k^{3(n+1)}-f(k^n x)/k^{3n}}\left(\frac{t}{k^{n+1}}\right) \geq \psi_{(k^n x, k^n x)}(k^{2(n+1)}t), \quad \forall x \in X, t > 0, n \in \mathbb{N}. \quad (3.31)$$

Since $1 > 1/2 + 1/2^2 + \dots + 1/2^m$, by the triangle inequality, it follows that

$$\begin{aligned} \mu_{f(k^m x)/k^{3m}-f(x)}(t) &\geq T_{n=0}^{m-1} \left(\mu_{f(k^{n+1}x)/k^{3(n+1)}-f(k^n x)/k^{3n}} \left(\sum_{n=0}^{m-1} \frac{t}{k^{n+1}} \right) \right) \\ &\geq T_{n=0}^{m-1} \left(\psi_{k^n x, k^n x} (k^{2(n+1)}t) \right) \\ &= T_{j=1}^m \left(\psi_{k^{j-1}x, k^{j-1}x} (k^{2j}t) \right), \quad \forall x \in X, t > 0. \end{aligned} \quad (3.32)$$

In order to prove the convergence of the sequence $\{f(k^m x)/k^{3m}\}$, we replace x with $k^{m'}x$ in (3.32) to find that

$$\mu_{f(k^{m+m'}x)/k^{3(m+m')} - f(k^{m'}x)/k^{3m'}}(t) \geq T_{j=1}^m \left(\psi_{k^{j+m'-1}x, k^{j+m'-1}x} (k^{2j+3m'}t) \right). \quad (3.33)$$

Since the right hand side of inequality (3.33) tends to 1 as $m', m \rightarrow \infty$, the sequence $\{f(k^m x)/k^{3m}\}$ is a Cauchy sequence. Therefore, we can define

$$C(x) = \lim_{m \rightarrow \infty} \frac{f(k^m x)}{k^{3m}}, \quad \forall x \in X. \quad (3.34)$$

Now, we show that C is a cubic mapping. In fact, replacing x, y with $k^m x$ and $k^m y$ in (3.15), respectively, and then taking the limit $m \rightarrow \infty$, we find that C satisfies (1.10) for all $x, y \in X$. Therefore, the mapping $C : X \rightarrow Y$ is cubic.

Letting the limit $m \rightarrow \infty$ in (3.32), we get inequality (3.18) by (3.26).

Finally, to prove the uniqueness of the cubic function C subject to inequality (3.19), assume that there exists a cubic function C' which satisfies inequality (3.19). Since $C(k^m x) = k^{3m}C(x)$ and $C'(k^m x) = k^{3m}C'(x)$ for all $x \in X$ and $m \in \mathbb{N}$, it follows from (3.19) that

$$\begin{aligned} &\mu_{C(x)-C'(x)}(t) \\ &= \mu_{C(k^m x)-C'(k^m x)}(k^{3m}t) \geq T\left(\mu_{C(k^m x)-f(k^m x)}(k^{3m-1}t), \mu_{f(k^m x)-C'(k^m x)}(k^{3m-1}t)\right) \\ &\geq T\left[T_{j=1}^\infty\left(T\left(\rho_{0,k^{m+j-1}x}\left(\frac{2k^{2j+3m}}{k^2(k-1)}t\right), \rho_{k^{m+j-1}x,0}\left(\frac{k^{(3m+2j)}(1-k^2)}{k^2(k-2)}t\right)\right)\right), \right. \\ &\quad \left. T_{j=1}^\infty\left(T\left(\rho_{0,k^{m+j-1}x}\left(\frac{2k^{2j+3m}}{k^2(k-1)}t\right), \rho_{k^{m+j-1}x,0}\left(\frac{k^{(3m+2j)}(1-k^2)}{k^2(k-2)}t\right)\right)\right)\right] \end{aligned} \tag{3.35}$$

for all $x \in X$ and $t > 0$. By letting $m \rightarrow \infty$ in inequality (3.35), we know that $C = C'$. This completes the proof. \square

Theorem 3.3. *If*

$$\begin{aligned} &\lim_{m \rightarrow \infty} T_{j=1}^\infty \left[T\left(\rho_{0,k^{m+j-1}x}(k^{j+2m}t), \rho_{0,-k^{m+j-1}x}(k^{j+2m}t)\right) \right] \\ &= 1 \\ &= \lim_{m \rightarrow \infty} T_{j=1}^\infty \left[T\left(\rho_{0,k^{m+j-1}x}\left(\frac{2k^{2j+3m}}{k^2(k-1)}t\right), \rho_{k^{m+j-1}x,0}\left(\frac{k^{2j+3m}(1-k^2)}{k^2(k-2)}t\right)\right), \right. \\ &\quad \left. T\left(\rho_{0,-k^{m+j-1}x}\left(\frac{2k^{2j+3m}}{k^2(k-1)}t\right), \rho_{-k^{m+j-1}x,0}\left(\frac{k^{2j+3m}(1-k^2)}{k^2(k-2)}t\right)\right) \right] \end{aligned} \tag{3.36}$$

for all $x \in X$ and $t > 0$ and

$$\begin{aligned} &\lim_{m \rightarrow \infty} T\left(\rho_{k^m x, k^m y}(k^{2n-1}t), \rho_{k^m x, k^m y}(k^{2n-1}t)\right) \\ &= 1 \\ &= \lim_{m \rightarrow \infty} T\left(\rho_{k^m x, k^m y}(k^{3n-1}t), \rho_{k^m x, k^m y}(k^{3n-1}t)\right), \quad \forall x, y \in X, t > 0, \end{aligned} \tag{3.37}$$

then there exist a unique quadratic mapping $Q : X \rightarrow Y$ and a unique cubic mapping $C : X \rightarrow Y$ such that

$$\begin{aligned}
& \mu_{f(x)-C(x)-Q(x)}(t) \\
& \geq T \left[T_{j=1}^{\infty} \left[T \left(\rho_{0,k^{j-1}x} \left(\frac{k^j}{2} t \right), \rho_{0,-k^{j-1}x} \left(\frac{k^j}{2} t \right) \right) \right], \right. \\
& \quad T_{j=1}^{\infty} \left[T \left(\rho_{0,k^{j-1}x} \left(\frac{k^{2j}}{k^2(k-1)} t \right), \rho_{k^{j-1}x,0} \left(\frac{k^{2j}(1-k^2)}{2k^2(k-2)} t \right) \right), \right. \\
& \quad \left. \left. T \left(\rho_{0,-k^{j-1}x} \left(\frac{k^{2j}}{k^2(k-1)} t \right), \rho_{-k^{j-1}x,0} \left(\frac{k^{2j}(1-k^2)}{2k^2(k-2)} t \right) \right) \right] \right], \quad \forall x \in X, t > 0.
\end{aligned} \tag{3.38}$$

Proof. If $f_e(x) = (1/2)[f(x) + f(-x)]$ for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$, and

$$\begin{aligned}
& \mu_{f_e(x+ky)+f_e(x-ky)-k^2[f_e(x+y)+f_e(x-y)]-(2(k^2-1)/k^2(k-2))f_e(kx)+((k^3-k^2-k+1)/2(k-2))f_e(2x)-\tilde{f}_e(2y)+8\tilde{f}_e(y)}(t) \\
& \geq T(\rho_{x,y}(2t), \rho_{-x,-y}(2t)) \\
& \geq T(\rho_{x,y}(t), \rho_{-x,-y}(t)), \quad \forall x, y \in X, t > 0,
\end{aligned} \tag{3.39}$$

where $\tilde{f}_e(y) := f_e(y) - f_e(-y)$ for all $x, y \in X$. Hence, in view of Theorem 3.1, there exists a unique quadratic $Q : X \rightarrow Y$ such that

$$\mu_{Q(x)-f_e(x)}(t) \geq T_{j=1}^{\infty} \left[T \left(\rho_{0,k^{j-1}x} \left(k^j t \right), \rho_{0,-k^{j-1}x} \left(k^j t \right) \right) \right], \quad \forall x \in X, t > 0. \tag{3.40}$$

Let $f_o(x) = (1/2)[f(x) - f(-x)]$ for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$ and

$$\begin{aligned}
& \mu_{f_o(x+ky)+f_o(x-ky)-k^2[f_o(x+y)+f_o(x-y)]-(2(k^2-1)/k^2(k-2))f_o(kx)+((k^3-k^2-k+1)/2(k-2))f_o(2x)-\tilde{f}_o(2y)+8\tilde{f}_o(y)}(t) \\
& \geq T(\rho_{x,y}(2t), \rho_{-x,-y}(2t)) \\
& \geq T(\rho_{x,y}(t), \rho_{-x,-y}(t)), \quad \forall x, y \in X, t > 0,
\end{aligned} \tag{3.41}$$

where $\tilde{f}_o(y) := f_o(y) - f_o(-y)$ for all $x, y \in X$. From Theorem 3.2, it follows that there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\begin{aligned}
& \mu_{C(x)-f_o(x)}(t) \geq T_{j=1}^{\infty} \left[T \left(\rho_{0,k^{j-1}x} \left(\frac{2k^{2j}}{k^2(k-1)} t \right), \rho_{k^{j-1}x,0} \left(\frac{k^{2j}(1-k^2)}{k^2(k-2)} t \right) \right), \right. \\
& \quad \left. T \left(\rho_{0,-k^{j-1}x} \left(\frac{2k^{2j}}{k^2(k-1)} t \right), \rho_{-k^{j-1}x,0} \left(\frac{k^{2j}(1-k^2)}{k^2(k-2)} t \right) \right) \right], \quad \forall x \in X, t > 0.
\end{aligned} \tag{3.42}$$

Obviously, (3.38) follows from (3.40) and (3.42). This completes the proof. \square

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