Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 485705, 19 pages
doi:10.1155/2010/485705

Research Article

On Weighted L^p Integrability of Functions Defined by Trigonometric Series

Bogdan Szal

Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, ul. Szafrana 4a, 65-516 Zielona Góra, Poland

Correspondence should be addressed to Bogdan Szal, b.szal@wmie.uz.zgora.pl

Received 26 January 2010; Accepted 13 April 2010

Academic Editor: László Losonczi

We introduce a new class of sequences called $\overline{GM}_{\theta}^{r}$ and give a sufficient and necessary condition for weighted L^p integrability of trigonometric series with coefficients to belong to the above class. This is a generalization of the result proved by M. Dyachenko and S. Tikhonov (2009). Then we discuss the relations among the weighted best approximation and the coefficients of trigonometric series. Moreover, we extend the results of B. Wei and D. Yu (2009) to the class $\overline{GM}_{\theta}^{r}$.

1. Introduction

Let L^p, $1 \leq p < \infty$, be the space of all p-power integrable functions f of period 2π equipped with the norm

$$\|f\|_{L^p} = \left(\int_{-\pi}^{\pi} |f(x)|^p dx\right)^{1/p}. \tag{1.1}$$

Write

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx, \qquad g(x) = \sum_{k=1}^{\infty} b_k \sin kx \tag{1.2}$$

for those x's where the series converge. Denote by ϕ either f or g, and let λ_n be its associated coefficients, that is, λ_n is either a_n or b_n.

For $r \in \mathbb{N}$ and a sequence (c_k), let

$$\Delta_r c_k = c_k - c_{k+r}. \tag{1.3}$$

In Subsection 2.1 we generalize the following result.

Theorem 1.1. *Let a nonnegative sequence* $(\lambda_n) \in \mathfrak{R}$, $1 < p < \infty$ *and* $1 - p < \alpha < 1$. *Then*

$$x^{-\alpha}|\phi(x)|^p \in L^1 \Longleftrightarrow \sum_{n=1}^{\infty} n^{\alpha+p-2}\lambda_n^p < \infty. \tag{1.4}$$

In the case when $\mathfrak{R}$ denotes the class M of all decreasing sequences, this theorem was proved in [1–4]; for $\mathfrak{R} \equiv QM$, the class of quasimonotone sequences, in [5]; for $\mathfrak{R} \equiv \overline{GM}(\overline{\beta})$ in [6, 7]; for $\mathfrak{R} \equiv GM(\overline{\beta})$ in [8]; and for $\mathfrak{R} \equiv GM(\beta^*)$ in [9], where

$$\overline{GM}(\beta) := \left\{(c_n) : \sum_{k=n}^{\infty}|\Delta_1 c_k| \le C\beta_n\right\},$$

$$GM(\beta) := \left\{(c_n) : \sum_{k=n}^{2n}|\Delta_1 c_k| \le C\beta_n\right\}, \tag{1.5}$$

$$\overline{\beta}_n = |c_n|, \quad \beta_n^* = \sum_{k=[n/c]}^{[cn]}\frac{|c_k|}{k} \quad \text{for some } c > 1.$$

Note that (see [6, 8, 10–14])

$$M \subsetneq QM \cup \overline{GM}\left(\overline{\beta}\right) \subsetneq GM\left(\overline{\beta}\right) \subsetneq GM(\beta^*). \tag{1.6}$$

In [15] Dyachenko and Tikhonov extended Theorem 1.1 to the class $\overline{GM_\theta} :\equiv \overline{GM}(\beta^\#)$, where $\theta \in (0,1]$ and

$$\beta_n^\# = n^{\theta-1}\sum_{k=[n/c]}^{\infty}\frac{|c_k|}{k^\theta} < \infty \quad \text{for some } c > 1. \tag{1.7}$$

We have (see [15])

$$GM(\beta^*) \subsetneq \overline{GM_1} \subseteq \overline{GM_{\theta 2}} \subseteq \overline{GM_{\theta 1}} \quad \text{for } 0 < \theta_1 \le \theta_2 \le 1. \tag{1.8}$$

Let γ be a nonnegative function defined on the interval $[0,\pi]$. Denote by $E_n(\varphi,\gamma)_p$ the best approximation of φ by trigonometric polynomials of degree at most n in the weighted L^p-norm, that is,

$$E_n(\varphi,\gamma)_p := \inf_{P_n \in \Pi_n}\left\{\int_0^\pi \gamma(x)|\varphi(x) - P_n(x)|^p dx\right\}^{1/p}, \tag{1.9}$$

where Π_n denotes the set of all trigonometric polynomials of degree at most n.

A sequence (c_n) of nonnegative terms is called almost increasing (decreasing) if there exists a constant $C > 0$ such that

$$C \cdot c_n \geq c_m \quad (c_n \leq C \cdot c_m) \text{ for } n \geq m. \tag{1.10}$$

We say that a weight function $\gamma \in \Phi(\alpha, \beta)$, $(\alpha, \beta$ be fixed constants), if γ is defined by the sequence γ_n as follows: $\gamma(\pi/n) := \gamma_n$, $n \in \mathbb{N}$, and there exist positive constants A and B such that

$$A\gamma_n \leq \gamma(x) \leq B\gamma_{n+1} \tag{1.11}$$

for all $x \in (\pi/(n+1), \pi/n]$, and the sequences $(\gamma_n n^{\alpha})$, $(\gamma_n n^{\beta})$ are almost decreasing and almost increasing, respectively.

In Subsection 2.2 we generalize and extend the following results [16].

Theorem 1.2. *Assume that $(b_n) \in GM(\beta^*)$. If $\gamma \in \Phi(-p-1+\alpha, p-1+\beta)$ for some $\alpha, \beta > 0$ and $\sum_{k=1}^{\infty} \gamma_n n^{p-2} \lambda_n^p < \infty$, then for $1 \leq p < \infty$*

$$E_n(g, \gamma)_p \leq C\left(\gamma_{n+1}^{1/p} n^{1-1/p} \sum_{k=n+1}^{[c(n+1)]} |\Delta_1 \lambda_k| + \left(\sum_{k=n+1}^{\infty} \gamma_k k^{p-2} \lambda_k^p\right)^{1/p}\right). \tag{1.12}$$

Theorem 1.3. *Assume that $(a_n) \in GM(\beta^*)$. If $\gamma \in \Phi(-1+\alpha, p-1+\beta)$ for some $\alpha, \beta > 0$ and $\sum_{k=1}^{\infty} \gamma_n n^{p-2} \lambda_n^p < \infty$, then for $1 \leq p < \infty$*

$$E_n(f, \gamma)_p \leq C\left(\gamma_{n+1}^{1/p} n^{1-1/p} \sum_{k=n+1}^{[c(n+1)]} |\Delta_1 \lambda_k| + \left(\sum_{k=n+1}^{\infty} \gamma_k k^{p-2} \lambda_k^p\right)^{1/p}\right). \tag{1.13}$$

If $\gamma \equiv 1$ and $(\lambda_n) \in M$ or $(\lambda_n) \in \overline{GM}(\overline{\beta})$ the above theorem has been obtained by Konyushkov [17] and Leindler [18] for $p > 1$, respectively.

In order to formulate our new results we define the next class of sequences.

Definition 1.4. Let $r \in \mathbb{N}$ and $\theta \in (0, 1]$. One says that a sequence (c_n) belongs to $\overline{GM}_{\theta}^{r}$, if the relation

$$\sum_{k=n}^{\infty} |\Delta_r c_k| \leq C n^{\theta-1} \sum_{k=[n/c]}^{\infty} \frac{|c_k|}{k^{\theta}} < \infty \tag{1.14}$$

holds for all $n \in \mathbb{N}$.

Note that for $r \geq 2$ and $\theta \in (0, 1]$ (see Theorem 2.1(i))

$$\overline{GM_{\theta}} \equiv \overline{GM_{\theta}}^{1} \subsetneq \overline{GM_{\theta}}^{r}. \tag{1.15}$$

Throughout this paper, we use C to denote a positive constant independent of the integer n; C may depend on the parameters such as p, α, r, θ and λ, and it may have different values in different occurrences.

2. Statement of the Results

We formulate our results as follows.

Theorem 2.1. *Suppose that $\theta \in (0,1]$. The following properties are true.*

(i) *For any $r \geq 2$, and $\theta \in (0,1]$ there exists a sequence $(c_n) \in \overline{GM}_\theta^r$, which does not belong to the class $\overline{GM}_\theta \equiv \overline{GM}_\theta^1$.*

(ii) *Let $r_1, r_2 \in \mathbb{N}$, $r_1 \leq r_2$ and $\theta \in (0,1]$. If $r_1 \mid r_2$, then $\overline{GM}_\theta^{r_1} \subseteq \overline{GM}_\theta^{r_2}$.*

(iii) *Let $r_1, r_2 \in \mathbb{N}$ and $\theta \in (0,1]$. If $r_1 \nmid r_2$ and $r_2 \nmid r_1$, then the classes $\overline{GM}_\theta^{r_1}$ and $\overline{GM}_\theta^{r_2}$ are not comparable.*

2.1. Weighted L^p-Integrability

Let $r \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. We define on the interval $[-\pi, \pi]$ an even function $\omega_{\alpha,r}$, which is given on the interval $[0, \pi]$ by the formula

$$\omega_{\alpha,r}(x) := \begin{cases} \left(x - \dfrac{2l\pi}{r}\right)^{-\alpha} & \text{for } x \in \left(\dfrac{2l\pi}{r}, \dfrac{(2l+1)\pi}{r}\right] \text{ and } l \in U_1, \\ \left(\dfrac{2(l+1)\pi}{r} - x\right)^{-\alpha} & \text{for } x \in \left(\dfrac{(2l+1)\pi}{r}, \dfrac{2(l+1)\pi}{r}\right) \text{ and } l \in U_2, \\ 0 & \text{for } x = \dfrac{2l\pi}{r} \text{ and } l \in U_3, \end{cases} \tag{2.1}$$

where $U_1 = \{0, 1, \ldots, [r/2]\}$ if r is an odd number, and $U_1 = \{0, 1, \ldots, [r/2] - 1\}$ if r is an even number; $U_2 = \{0, 1, \ldots, [r/2] - 1\}$ for $r \geq 2$, and $U_3 = \{0, 1, \ldots, [r/2]\}$ for $r \geq 1$.

Theorem 2.2. *Let a nonnegative sequence $(\lambda_n) \in \overline{GM}_\theta^r$, where $r \in \mathbb{N}$, $\theta \in (0,1]$ and $1 \leq p < \infty$. If*

$$1 - \theta p < \alpha < 1, \tag{2.2}$$

then $\omega_{\alpha,r}|\phi|^p \in L^1$ if and only if

$$\sum_{n=1}^{\infty} n^{\alpha+p-2} \lambda_n^p < \infty. \tag{2.3}$$

Theorem 2.3. *Let a nonnegative sequence $(b_n) \in \overline{GM}_\theta^r$ $(r = 1, 2)$, $\theta \in (0,1]$, and $1 \leq p < \infty$. If*

$$1 - \theta p < \alpha < p + 1, \tag{2.4}$$

then $\omega_{\alpha,r}|g|^p \in L^1$ *if and only if*

$$\sum_{n=1}^{\infty} n^{\alpha+p-2} b_n^p < \infty. \tag{2.5}$$

Remark 2.4. If we take $r = 1$ (and $\lambda_n = a_n$ in Theorem 2.2), then the result of Dyachenko and Tikhonov [15, Theorems 4.2 and 4.3] follows from Theorems 2.2 and 2.3. By the embedding relations (1.8) and (1.15) we can also derive from Theorem 2.2 the result of You, Zhou, and Zhou [9].

2.2. Relations between The Best Approximation and Fourier Coefficients

Theorem 2.5. *Let a nonnegative sequence* $(\lambda_n) \in \overline{GM}_\theta^r$, *where* $r \in \mathbb{N}$, $\theta \in (0,1]$, *and* $1 \le p < \infty$. *If*

$$1 - \theta p < \alpha < 1 \tag{2.6}$$

and (2.3) *holds, then*

$$E_n(\phi, \omega_{\alpha,r}) \le C\left(n^{\alpha/p+1-1/p} \sum_{k=n+1}^{[c(n+1)]} |\Delta_r \lambda_k| + \left(\sum_{k=n+1}^{\infty} k^{\alpha+p-2} \lambda_k^p \right)^{1/p} \right), \tag{2.7}$$

where $c > 1$.

Theorem 2.6. *Let a nonnegative sequence* $(b_n) \in \overline{GM}_\theta^r (r = 1,2)$, $\theta \in (0,1]$ *and* $1 \le p < \infty$. *If*

$$1 - \theta p < \alpha < p + 1, \tag{2.8}$$

and (2.5) *holds, then*

$$E_n(g, \omega_{\alpha,r})_p \le C\left(n^{\alpha/p+1-1/p} \sum_{k=n+1}^{[c(n+1)]} |\Delta_r b_k| + \left(\sum_{k=n+1}^{\infty} k^{\alpha+p-2} b_k^p \right)^{1/p} \right), \tag{2.9}$$

where $c > 1$.

Remark 2.7. If we restrict our attention to the class $GM(\beta^*)$, then by (1.8) and (1.15) Wei and Yu's result [16] follows from Theorems 2.5 and 2.6.

3. Auxiliary Results

Denote, for $r \in \mathbb{N}$,

$$\begin{aligned} D_{k,r}(x) &= \frac{\sin(k + r/2)x}{2\sin(rx/2)}, \\ \widetilde{D}_{k,r}(x) &= \frac{\cos(k + r/2)x}{2\sin(rx/2)}. \end{aligned} \tag{3.1}$$

Lemma 3.1 (see [19]). *Let $r \in \mathbb{N}$, $l \in \mathbb{Z}$, and $(c_n) \in \mathbb{C}$. If $x \neq 2l\pi/r$, then for all $m \geq n$*

$$\sum_{k=n}^{m} c_k \cos kx = \sum_{k=n}^{m} \Delta_r c_k D_{k,r}(x) - \sum_{k=m+1}^{m+r} c_k D_{k,-r}(x) + \sum_{k=n}^{n+r-1} c_k D_{k,-r}(x),$$
$$\sum_{k=n}^{m} c_k \sin kx = \sum_{k=m+1}^{m+r} c_k \widetilde{D}_{k,-r}(x) - \sum_{k=n}^{n+r-1} c_k \widetilde{D}_{k,-r}(x) - \sum_{k=n}^{m} \Delta_r c_k \widetilde{D}_{k,r}(x). \tag{3.2}$$

Lemma 3.2 (see [20]). *Let $p \geq 1$, $\gamma_n > 0$ and $a_n \geq 0$, then*

$$\sum_{n=1}^{\infty} \gamma_n \left(\sum_{k=1}^{n} \alpha_k \right)^p \leq p^p \sum_{n=1}^{\infty} \gamma_n^{1-p} \alpha_n^p \left(\sum_{k=n}^{\infty} \gamma_k \right)^p,$$
$$\sum_{n=1}^{\infty} \gamma_n \left(\sum_{k=n}^{\infty} \alpha_k \right)^p \leq p^p \sum_{n=1}^{\infty} \gamma_n^{1-p} \alpha_n^p \left(\sum_{k=1}^{n} \gamma_k \right)^p. \tag{3.3}$$

4. Proofs of The Main Results

4.1. Proof of Theorem 2.1

(i) Let $r \geq 2$, $\theta \in (0,1]$ and

$$c_n := \begin{cases} 0 & \text{if } r \mid n, \\ \dfrac{1}{n} & \text{if } r \nmid n. \end{cases} \tag{4.1}$$

First, we prove that $(c_n) \in \overline{GM}_\theta^r$. Let

$$A(r,k,n) := \{k : n \leq k \text{ and } r \mid k\},$$
$$B(r,k,n) := \{k : n \leq k \text{ and } r \nmid k\}. \tag{4.2}$$

Then for all n

$$\sum_{k=n}^{\infty} |\Delta_r c_k| = \sum_{k \in B(r,k,n)} \frac{r}{k(k+r)} \leq r \sum_{k \in B(r,k,n)}^{\infty} \frac{1}{k^2}$$
$$\leq r n^{\theta-1} \sum_{k \in B(r,k,n)}^{\infty} \frac{1}{k^{1+\theta}} \leq r n^{\theta-1} \sum_{k=n}^{\infty} \frac{c_k}{k^\theta} \tag{4.3}$$

and $(c_n) \in \overline{GM}_\theta^r$. If $r \geq 2$ then

$$\sum_{k=n}^{\infty} |\Delta_1 c_k| \geq \sum_{k \in A(r,k,n)}^{\infty} |\Delta_1 c_k| = \sum_{k \in A(r,k,n)}^{\infty} \frac{1}{k+1} \geq C \ln(n+1) \tag{4.4}$$

and since

$$n^{\theta-1}\sum_{k=[n/c]}^{\infty}\frac{c_k}{k^{\theta}}=n^{\theta-1}\sum_{k\in B(r,k,[n/c])}^{\infty}\frac{1}{k^{1+\theta}}\le n^{\theta-1}\sum_{k=[n/c]}^{\infty}\frac{1}{k^{1+\theta}}\le C\frac{1}{n} \tag{4.5}$$

the inequality

$$\sum_{k=n}^{\infty}|\Delta_1 c_k|\le Cn^{\theta-1}\sum_{k=[n/c]}^{\infty}\frac{c_k}{k^{\theta}} \tag{4.6}$$

does not hold, that is, (c_n) does not belong to $\overline{GM}_{\theta}^{1}$.

(ii) Let $r_1, r_2 \in \mathbb{N}$, $r_1 \le r_2$ and $\theta \in (0,1]$. If $r_1 \mid r_2$, then exists a natural number p such that $r_2 = p \cdot r_1$. Supposing that $(c_n) \in \overline{GM}_{\theta}^{r_1}$, we have for all n

$$\begin{aligned}\sum_{k=n}^{\infty}|\Delta_{r_2}c_k| &= \sum_{k=n}^{\infty}\left|\sum_{l=0}^{p-1}\Delta_{r_1}c_{k+l\cdot r_1}\right| \le \sum_{l=0}^{p-1}\sum_{k=n+l\cdot r_1}^{\infty}|\Delta_{r_1}c_k| \\ &\le \frac{r_2}{r_1}\sum_{k=n}^{\infty}|\Delta_{r_1}c_k| \le Cn^{\theta-1}\sum_{k=[n/c]}^{\infty}\frac{c_k}{k^{\theta}},\end{aligned} \tag{4.7}$$

whence $(c_n) \in \overline{GM}_{\theta}^{r_2}$. Thus $\overline{GM}_{\theta}^{r_1} \subseteq \overline{GM}_{\theta}^{r_2}$.

(iii) Let $r_1, r_2 \in \mathbb{N}$ and $\theta \in (0,1]$ and let

$$c_n^1 := \begin{cases} 0 & \text{if } r_1 \mid n, \\ \dfrac{1}{n} & \text{if } r_1 \nmid n, \end{cases} \qquad c_n^2 := \begin{cases} 0 & \text{if } r_2 \mid n, \\ \dfrac{1}{n} & \text{if } r_2 \nmid n. \end{cases} \tag{4.8}$$

Supposing that $r_1 \nmid r_2$ and $r_2 \nmid r_1$, we can prove, similarly as in (i), that $(c_n^1) \in \overline{GM}_{\theta}^{r_1}$, $(c_1) \notin \overline{GM}_{\theta}^{r_2}$, $(c_2) \in \overline{GM}_{\theta}^{r_2}$ and $(c_2) \notin \overline{GM}_{\theta}^{r_1}$. Therefore the classes $\overline{GM}_{\theta}^{r_1}$ and $\overline{GM}_{\theta}^{r_2}$ are not comparable.

4.2. Proof of Theorem 2.2

We prove the theorem for the case when $\phi(x) = g(x)$. The case when $\phi(x) = f(x)$ can be proved similarly.

Sufficiency. Suppose that (2.3) holds. Then

$$\left\|\omega_{\alpha,r}|g|^p\right\|_{L^1} = 2\int_0^{\pi}\omega_{\alpha,r}(x)\left|g(x)\right|^p dx. \tag{4.9}$$

It is clear that for an odd r

$$\int_0^{\pi} \omega_{\alpha,r}(x)\left|g(x)\right|^p dx = \sum_{l=0}^{[r/2]} \int_{2l\pi/r}^{2l\pi/r+\pi/r} \omega_{\alpha,r}(x)\left|\sum_{k=1}^{\infty} b_k \sin kx\right|^p dx + \sum_{l=0}^{[r/2]-1} \int_{2l\pi/r+\pi/r}^{2(l+1)\pi/r} \omega_{\alpha,r}(x)\left|\sum_{k=1}^{\infty} b_k \sin kx\right|^p dx \tag{4.10}$$

(for $r = 1$ the last sum should be omitted), and for an even r

$$\int_0^{\pi} \omega_{\alpha,r}(x)\left|g(x)\right|^p dx = \sum_{l=0}^{[r/2]} \left(\int_{2l\pi/r}^{2l\pi/r+\pi/r} + \int_{2l\pi/r+\pi/r}^{2(l+1)\pi/r}\right)\omega_{\alpha,r}(x)\left|\sum_{k=1}^{\infty} b_k \sin kx\right|^p dx. \tag{4.11}$$

First, we estimate the following integral:

$$\begin{aligned}&\int_{2l\pi/r}^{2l\pi/r+\pi/r} \omega_{\alpha,r}(x)\left|\sum_{k=1}^{\infty} b_k \sin kx\right|^p dx \\ &\quad \le C\left(\int_{2l\pi/r}^{2l\pi/r+\pi/r} \omega_{\alpha,r}(x)\left|\sum_{k=1}^{n} b_k \sin kx\right|^p dx + \int_{2l\pi/r}^{2l\pi/r+\pi/r} \omega_{\alpha,r}(x)\left|\sum_{k=n+1}^{\infty} b_k \sin kx\right|^p dx\right) \\ &\quad := C(I_1 + I_2).\end{aligned} \tag{4.12}$$

By (3.3), for $\alpha < 1$, we have

$$\begin{aligned} I_1 &= \sum_{n=r}^{\infty} \int_{2l\pi/r\ +\pi/(n+1)}^{2l\pi/r+\pi/n} \left(x - \frac{2l\pi}{r}\right)^{-\alpha} \left|\sum_{k=1}^{n} b_k \sin kx\right|^p dx \\ &\le C\sum_{n=r}^{\infty} n^{\alpha-2}\left(\sum_{k=1}^{n} b_k\right)^p \le C\sum_{n=1}^{\infty} n^{\alpha-2}\left(\sum_{k=1}^{n} b_k\right)^p \le C\sum_{n=1}^{\infty} n^{\alpha+p-2} b_n^p. \end{aligned} \tag{4.13}$$

Using (3.2) with $m \to \infty$ and the inequality

$$\frac{r}{\pi}x - 2l \le \left|\sin\frac{rx}{2}\right| \quad \text{for } x \in \left(\frac{2l\pi}{r}, \frac{2l\pi}{r}, \frac{\pi}{r}\right), \tag{4.14}$$

we get

$$
\begin{aligned}
I_2 &= \sum_{n=r}^{\infty} \int_{2l\pi/r+\pi/(n+1)}^{2l\pi/r+\pi/n} \left(x - \frac{2l\pi}{r}\right)^{-\alpha} \left| \sum_{k=n+1}^{\infty} b_k \sin kx \right|^p dx \\
&= \sum_{n=r}^{\infty} \int_{2l\pi/r+\pi/(n+1)}^{2l\pi/r+\pi/n} \left(x - \frac{2l\pi}{r}\right)^{-\alpha} \left| \sum_{k=n+1}^{\infty} \Delta_r b_k \widetilde{D}_{k,r}(x) + \sum_{k=n+1}^{n+r} b_k \widetilde{D}_{k,-r}(x) \right|^p dx \\
&\le C \sum_{n=r}^{\infty} n^{\alpha} \int_{2l\pi/r+\pi/(n+1)}^{2l\pi/r+\pi/n} \frac{1}{2|\sin rx/2|^p} \left(\sum_{k=n+1}^{\infty} |\Delta_r b_k| + \sum_{k=n+1}^{n+r} b_k \right)^p dx \\
&\le C \sum_{n=r}^{\infty} n^{\alpha} \int_{2l\pi/r+\pi/(n+1)}^{2l\pi/r+\pi/n} \frac{1}{(r/\pi x - 2l)^p} \left(\sum_{k=n+1}^{\infty} |\Delta_r b_k| \right)^p dx \\
&\le C \sum_{n=r}^{\infty} n^{\alpha+p-2} \left(\sum_{k=n}^{\infty} |\Delta_r b_k| \right)^p .
\end{aligned}
\tag{4.15}
$$

If $(b_n) \in \overline{GM}_{\theta}^{r}$, then by (3.3), for $1 - \theta p < \alpha < p + 1$, we obtain

$$
\begin{aligned}
I_2 &\le C \sum_{n=r}^{\infty} n^{\alpha+\theta p-2} \left(\sum_{k=[n/c]}^{\infty} \frac{b_k}{k^{\theta}} \right)^p \\
&\le C \left(\sum_{n=r}^{\infty} n^{\alpha+\theta p-2} \left(\sum_{k=[n/c]}^{n} \frac{b_k}{k^{\theta}} \right)^p + \sum_{n=r}^{\infty} n^{\alpha+\theta p-2} \left(\sum_{k=n}^{\infty} \frac{b_k}{k^{\theta}} \right)^p \right) \\
&\le C \left(\sum_{n=1}^{\infty} n^{\alpha-p-2} \left(\sum_{k=1}^{n} k b_k \right)^p + \sum_{n=1}^{\infty} n^{\alpha+\theta p-2} \left(\sum_{k=n}^{\infty} \frac{b_k}{k^{\theta}} \right)^p \right) \\
&\le C \sum_{n=1}^{\infty} n^{\alpha+p-2} b_n^p .
\end{aligned}
\tag{4.16}
$$

Now, we estimate the following integral:

$$
\begin{aligned}
&\int_{2l\pi/r+\pi/r}^{2(l+1)\pi/r} \omega_{\alpha,r}(x) \left| \sum_{k=1}^{\infty} b_k \sin kx \right|^p dx \\
&\le C \left(\int_{2(l+1)\pi/r-\pi/r}^{2(l+1)\pi/r} \omega_{\alpha,r}(x) \left| \sum_{k=1}^{n} b_k \sin kx \right|^p dx + \int_{2(l+1)\pi/r-\pi/r}^{2(l+1)\pi/r} \omega_{\alpha,r}(x) \left| \sum_{k=n+1}^{\infty} b_k \sin kx \right|^p dx \right) \\
&:= C(I_3 + I_4).
\end{aligned}
\tag{4.17}
$$

By (3.3), for $\alpha < 1$, we have

$$
\begin{aligned}
I_3 &= \sum_{n=r}^{\infty} \int_{2(l+1)\pi/r-\pi/n}^{2(l+1)\pi/r-\pi/(n+1)} \left(\frac{2(l+1)\pi}{r} - x\right)^{-\alpha} \left|\sum_{k=1}^{n} b_k \sin kx\right|^p dx \\
&\le C\sum_{n=1}^{\infty} n^{\alpha-2} \left(\sum_{k=1}^{n} b_k\right)^p \le C\sum_{n=1}^{\infty} n^{\alpha+p-2} b_n^p.
\end{aligned}
\tag{4.18}
$$

Using (3.2) with $m \to \infty$ and the inequality

$$
2(l+1) - \frac{r}{\pi}x \le \left|\sin\frac{rx}{2}\right| \quad \text{for } x \in \left(\frac{(2l+1)\pi}{r}, \frac{2(l+1)\pi}{r}\right),
\tag{4.19}
$$

we obtain

$$
\begin{aligned}
I_4 &= \sum_{n=r}^{\infty} \int_{2(l+1)\pi/r-\pi/n}^{2(l+1)\pi/r-\pi/(n+1)} \left(\frac{2(l+1)\pi}{r} - x\right)^{-\alpha} \left|\sum_{k=n+1}^{\infty} b_k \sin kx\right|^p dx \\
&\le C\sum_{n=r}^{\infty} n^{\alpha} \int_{2(l+1)\pi/r-\pi/n}^{2(l+1)\pi/r-\pi/(n+1)} \left|\sum_{k=n+1}^{\infty} \Delta_r b_k \widetilde{D}_{k,r}(x) + \sum_{k=n+1}^{n+r} b_k \widetilde{D}_{k,-r}(x)\right|^p dx \\
&\le C\sum_{n=r}^{\infty} n^{\alpha} \int_{2(l+1)\pi/r-\pi/n}^{2(l+1)\pi/r-\pi/(n+1)} \frac{1}{2\left|\sin\frac{rx}{2}\right|^p} \left(\sum_{k=n+1}^{\infty} |\Delta_r b_k| + \sum_{k=n+1}^{n+r} b_k\right)^p dx \\
&\le C\sum_{n=r}^{\infty} n^{\alpha} \int_{2(l+1)\pi/r-\pi/n}^{2(l+1)\pi/r-\pi/(n+1)} \frac{1}{(2(l+1) - r/\pi x)^p} \left(\sum_{k=n+1}^{\infty} |\Delta_r b_k|\right)^p dx \\
&\le C\sum_{n=r}^{\infty} n^{\alpha+p-2} \left(\sum_{k=n}^{\infty} |\Delta_r b_k|\right)^p.
\end{aligned}
\tag{4.20}
$$

If $(b_n) \in \overline{GM}_{\theta}^{r}$, then by (3.3), for $1 - \theta p < \alpha < p + 1$, we obtain

$$
\begin{aligned}
I_4 &\le C\sum_{n=r}^{\infty} n^{\alpha+\theta p-2} \left(\sum_{k=[n/c]}^{\infty} \frac{b_k}{k^{\theta}}\right)^p \\
&\le C\left(\sum_{n=1}^{\infty} n^{\alpha-p-2} \left(\sum_{k=1}^{n} k b_k\right)^p + \sum_{n=1}^{\infty} n^{\alpha+\theta p-2} \left(\sum_{k=n}^{\infty} \frac{b_k}{k^{\theta}}\right)^p\right) \\
&\le C\sum_{n=1}^{\infty} n^{\alpha+p-2} b_n^p.
\end{aligned}
\tag{4.21}
$$

Thus, combining (4.9), (4.12)–(4.13), (4.16)–(4.18), (4.21), and (4.10) or (4.11), we obtain that

$$\int_{-\pi}^{\pi} \omega_{\alpha,r}(x)\left|g(x)\right|^{p} dx \leq C\sum_{n=1}^{\infty} n^{\alpha+p-2} b_n^p. \tag{4.22}$$

Necessity. We follow the method adopted by S. Tikhonov [15]. Note that if $1 - p < \alpha$, then $g \in L^1$. Integrating g, we have

$$F(x) := \int_0^x g(t)dt = \sum_{n=1}^{\infty} \frac{b_n}{n}(1 - \cos nx) = 2\sum_{n=1}^{\infty} \frac{b_n}{n}\sin^2\frac{nx}{2} \tag{4.23}$$

and consequently

$$F\left(\frac{\pi}{k}\right) \geq \sum_{n=[k/2]}^{k} \frac{b_n}{n}. \tag{4.24}$$

If $(b_n) \in \overline{GM}_\theta^r$, then using (4.24),

$$\begin{aligned}
b_v &\leq \sum_{l=v}^{v+r-1} b_l = \sum_{l=v}^{\infty}\Delta_r b_l \leq \sum_{l=v}^{\infty}|\Delta_r b_l| \leq Cv^{\theta-1}\sum_{l=[v/c]}^{\infty}\frac{b_l}{l^\theta}. \\
&= Cv^{\theta-1}\sum_{s=0}^{\infty}\sum_{l=2^s[v/c]}^{2^{s+1}[v/c]-1}\frac{b_l}{l^\theta} \leq Cv^{\theta-1}\sum_{s=0}^{\infty}(2^s[v/c])^{1-\theta}\sum_{l=2^s[v/c]}^{2^{s+1}[v/c]}\frac{b_l}{l} \\
&\leq Cv^{\theta-1}\sum_{s=0}^{\infty}(2^s[v/c])^{1-\theta}F\left(\frac{\pi}{2^{s+1}[v/c]}\right) \\
&\leq Cv^{\theta-1}\sum_{s=0}^{\infty}(2^s[v/c])^{-\theta}\sum_{l=2^s[v/c]}^{2^{s+1}[v/c]-1}F\left(\frac{\pi}{l}\right) \\
&\leq Cv^{\theta-1}\sum_{s=0}^{\infty}\sum_{l=2^s[v/c]}^{2^{s+1}[v/c]-1}\frac{1}{l^\theta}F\left(\frac{\pi}{l}\right) \leq Cv^{\theta-1}\sum_{l=[v/c]}^{\infty}\frac{1}{l^\theta}F\left(\frac{\pi}{l}\right).
\end{aligned} \tag{4.25}$$

Using this and (3.3), for $1-\theta p<\alpha<p+1$, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}k^{\alpha+p-2}b_k^p &\le C\sum_{k=1}^{\infty}k^{\alpha+p-2+(\theta-1)p}\left(\sum_{v=[k/c]}^{\infty}\frac{1}{v^{\theta}}F\left(\frac{\pi}{v}\right)\right)^p\\ &\le C\left(\sum_{k=1}^{\infty}k^{\alpha+p-2+(\theta-1)p}\left(\sum_{v=[k/c]}^{k}\frac{1}{v^{\theta}}F\left(\frac{\pi}{v}\right)\right)^p\right.\\ &\qquad\left.+\sum_{k=1}^{\infty}k^{\alpha+p-2+(\theta-1)p}\left(\sum_{v=k}^{\infty}\frac{1}{v^{\theta}}F\left(\frac{\pi}{v}\right)\right)^p\right)\\ &\le C\left(\sum_{k=1}^{\infty}k^{\alpha-2-p}\left(\sum_{v=1}^{k}vF\left(\frac{\pi}{v}\right)\right)^p\right.\\ &\qquad\left.+\sum_{k=1}^{\infty}k^{\alpha+\theta p-2}\left(\sum_{v=k}^{\infty}\frac{1}{v^{\theta}}F\left(\frac{\pi}{v}\right)\right)^p\right)\\ &\le C\sum_{k=1}^{\infty}k^{\alpha+p-2}\left(F\left(\frac{\pi}{k}\right)\right)^p.\end{aligned}\tag{4.26}$$

Defining $d_v:=\int_{\pi/(v+1)}^{\pi/v}|g(x)|dx$ we get

$$\sum_{k=1}^{\infty}k^{\alpha+p-2}b_k^p\le C\sum_{k=1}^{\infty}k^{\alpha+p-2}\left(\sum_{v=k}^{\infty}d_v\right)^p\tag{4.27}$$

and by (3.3), for $\alpha>1-\theta p\ge 1-p$, we obtain

$$\sum_{k=1}^{\infty}k^{\alpha+p-2}b_k^p\le\sum_{k=1}^{\infty}k^{\alpha+2p-2}d_k^p.\tag{4.28}$$

Applying Hölder's inequality, for $p>1$, we have

$$d_k^p=\left(\int_{\pi/(k+1)}^{\pi/k}|g(x)|dx\right)^p\le C\frac{1}{k^{2(p-1)}}\int_{\pi/(k+1)}^{\pi/k}|g(x)|^p dx.\tag{4.29}$$

Finally,

$$\begin{aligned}
\sum_{k=1}^{\infty} k^{\alpha+p-2} b_k^p &\le C\left(\sum_{k=1}^{r} k^{\alpha+2p-2} d_k^p + \sum_{k=r}^{\infty} k^{\alpha+2p-2} d_k^p\right) \\
&\le C\left(\sum_{k=1}^{r} k^{\alpha+2p-2}\left(\int_{\pi/(k+1)}^{\pi/k} |g(x)|dx\right)^p + \sum_{k=r}^{\infty} k^{\alpha} \int_{\pi/(k+1)}^{\pi/k} |g(x)|^p dx\right) \\
&\le C\left(\left(\int_0^{\pi} |g(x)|dx\right)^p + \sum_{k=r}^{\infty} \int_{\pi/(k+1)}^{\pi/k} x^{-\alpha} |g(x)|^p dx\right) \\
&\le C\left(\left(\int_0^{\pi} |g(x)|dx\right)^p + \int_0^{\pi} \omega_{\alpha,r}(x) |g(x)|^p dx\right),
\end{aligned} \tag{4.30}$$

which completes the proof.

4.3. *Proof of Theorem 2.3*

The proof of Theorem 2.3 goes analogously as the proof of Theorem 2.2. The only difference is that instead of (4.13) (for $r = 1, 2$) and (4.18) (for $r = 2$) we use the below estimations.

Applying the inequalities $|\sin kx| \le kx$ for $x \in (0, \pi)$, $|\sin kx| \le k(\pi - x)$ for $x \in (0, \pi)$ and using (3.3), for $\alpha < 1 + p$, we have

$$\begin{aligned}
I_1 &= \sum_{n=r}^{\infty} \int_{\pi/(n+1)}^{\pi/n} \left(x - \frac{2l\pi}{r}\right)^{-\alpha} \left|\sum_{k=1}^{n} b_k \sin kx\right|^p dx \le C \sum_{n=r}^{\infty} n^{\alpha} \int_{\pi/(n+1)}^{\pi/n} \left(x \sum_{k=1}^{n} k b_k\right)^p dx \\
&\le C \sum_{n=1}^{\infty} n^{\alpha-p-2} \left(\sum_{k=1}^{n} k b_k\right)^p \le C \sum_{n=1}^{\infty} n^{\alpha+p-2} b_n^p \\
I_3 &= \sum_{n=2}^{\infty} \int_{\pi-\pi/n}^{\pi-\pi/(n+1)} (\pi - x)^{-\alpha} \left|\sum_{k=1}^{n} b_k \sin kx\right|^p dx \le C \sum_{n=2}^{\infty} n^{\alpha} \int_{\pi-\pi/n}^{\pi-\pi/(n+1)} \left((\pi - x) \sum_{k=1}^{n} k b_k\right)^p dx \\
&\le C \sum_{n=1}^{\infty} n^{\alpha-p-2} \left(\sum_{k=1}^{n} k b_k\right)^p \le C \sum_{n=1}^{\infty} n^{\alpha+p-2} b_n^p.
\end{aligned} \tag{4.31}$$

This ends our proof.

4.4. *Proof of Theorem 2.5*

We prove the theorem for the case when $\phi(x) = g(x)$. The case when $\phi(x) = f(x)$ can be proved similarly.

If $n \le r$ then by (2.3) we obtain that (2.7) obviously holds. Let $n \ge r$. It is clear that if r is an odd number, then

$$E_n(g,\omega_{\alpha,r})_p \le \left\{ \int_0^{\pi} \omega_{\alpha,r}(x)\left|g(x)-S_n(g,x)\right|^p dx \right\}^{1/p}$$
$$= \left\{ \sum_{l=0}^{[r/2]} \int_{2l\pi/r}^{2l\pi/r+\pi/r} \omega_{\alpha,r}(x)\left|g(x)-S_n(g,x)\right|^p dx \right. \tag{4.32}$$
$$\left. + \sum_{l=0}^{[r/2]-1} \int_{2l\pi/r+\pi/r}^{2(l+1)\pi/r} \omega_{\alpha,r}(x)\left|g(x)-S_n(g,x)\right|^p dx \right\}^{1/p}$$

(for $r = 1$ the last sum should be omitted), and if r is an even number, then

$$E_n(g,\omega_{\alpha},r)_p \le \left\{ \sum_{l=0}^{[r/2]} \left(\int_{2l\pi/r}^{2l\pi/r+\pi/r} + \int_{2l\pi/r+\pi/r}^{2(l+1)\pi/r} \right) \omega_{\alpha,r}(x)\left|g(x)-S_n(g,x)\right|^p dx \right\}^{1/p}. \tag{4.33}$$

Let

$$\frac{2l\pi}{r} + \frac{\pi}{m+1} < x \le \frac{2l\pi}{r} + \frac{\pi}{r}, \tag{4.34}$$

where $m := m(x) \ge r$ and $l = 0, 1, \dots, [r/2]-1$ if r is an even number, and $l = 0, 1, \dots, [r/2]$ if r is an odd number.

Then, for $n \ge m$, by (3.2) and (4.14), we get

$$\begin{aligned}
&\int_{2l\pi/r+\pi/(n+1)}^{2l\pi/r+\pi/r} \omega_{\alpha,r}(x)\left|g(x)-S_n(g,x)\right|^p dx \\
&\quad= \sum_{m=r}^{n} \int_{2l\pi/r+\pi/(m+1)}^{2l\pi/r+\pi/m} \left(x-\frac{2l\pi}{r}\right)^{-\alpha} \left|g(x)-S_n(g,x)\right|^p dx \\
&\quad\le C\sum_{m=r}^{n} m^{\alpha} \int_{2l\pi/r+\pi/(m+1)}^{2l\pi/r+\pi/m} \left| \sum_{k=n+1}^{\infty} \Delta_r b_k \widetilde{D}_{k,r}(x) + \sum_{k=n+1}^{n+r} b_k \widetilde{D}_{k,-r}(x) \right|^p dx \\
&\quad\le C\sum_{m=r}^{n} m^{\alpha} \int_{2l\pi/r+\pi/(m+1)}^{2l\pi/r+\pi/m} \frac{1}{2\left|\sin \dfrac{rx}{2}\right|^p} \left(\sum_{k=n+1}^{\infty} |\Delta_r b_k| + \sum_{k=n+1}^{n+r} b_k \right)^p dx \\
&\quad\le C\sum_{m=r}^{n} m^{\alpha} \int_{2l\pi/r+\pi/(m+1)}^{2l\pi/r+\pi/m} \frac{1}{(r/\pi x - 2l)^p} \left(\sum_{k=n+1}^{\infty} |\Delta_r b_k| \right)^p dx \\
&\quad\le C\sum_{m=r}^{n} m^{\alpha+p-2} \left(\sum_{k=n+1}^{\infty} |\Delta_r b_k| \right)^p dx \le C\left(\sum_{m=r}^{n} m^{\alpha+p-2} \left(\sum_{k=n+1}^{[c(n+1)]} |\Delta_r b_k| \right)^p dx \right. \\
&\qquad \left. + \sum_{m=r}^{n} m^{\alpha+p-2} \left(\sum_{k=[c(n+1)]}^{\infty} |\Delta_r b_k| \right)^p dx \right) \\
&\quad := C(\Sigma_1 + \Sigma_2).
\end{aligned} \tag{4.35}$$

We immediately have for $\alpha > 1 - \theta p \geq 1 - p$

$$\Sigma_1 \leq C n^{\alpha+p-1} \left(\sum_{k=n+1}^{[c(n+1)]} |\Delta_r b_k| \right)^p. \tag{4.36}$$

If $(b_n) \in \overline{GM}_\theta^r$ and $p > 1$, then by Hölder's inequality we have for $\alpha > 1 - \theta p$

$$\begin{aligned} \Sigma_2 &\leq C \sum_{m=r}^{n} m^{\alpha+p-2} \left(n^{\theta-1} \sum_{k=n+1}^{\infty} \frac{b_k}{k^\theta} \right)^p \leq C n^{\alpha+\theta p-1} \left(\sum_{k=n+1}^{\infty} \frac{b_k}{k^\theta} \right)^p \\ &\leq C n^{\alpha+\theta p-1} \sum_{k=n+1}^{\infty} k^{\alpha+p-2} b_k^p \left(\sum_{k=n+1}^{\infty} k^{(-p-\theta p-\alpha+2)/(p-1)} \right)^{p-1} \leq \dot{C} \sum_{k=n+1}^{\infty} k^{\alpha+p-2} b_k^p. \end{aligned} \tag{4.37}$$

When $(b_n) \in \overline{GM}_\theta^r$ and $p = 1$, an elementary calculation gives for $\alpha > 1 - \theta$

$$\Sigma_2 \leq C \sum_{m=r}^{n} m^{\alpha-1} \left(n^{\theta-1} \sum_{k=n+1}^{\infty} \frac{b_k}{k^\theta} \right) \leq C n^{\alpha+\theta p-1} \sum_{k=n+1}^{\infty} \frac{b_k}{k^\theta} \leq \dot{C} \sum_{k=n+1}^{\infty} k^{\alpha-1} b_k. \tag{4.38}$$

If $m \geq n + 1 \geq r + 1$, then

$$\begin{aligned} &\int_{2l\pi/r}^{2l\pi/r+\pi/(n+1)} \omega_{\alpha,r}(x) \left| g(x) - S_n(g,x) \right|^p dx \\ &\quad= \sum_{m=n+1}^{\infty} \int_{2l\pi/r+\pi/(m+1)}^{2l\pi/r+\pi/m} \left(x - \frac{2l\pi}{r} \right)^{-\alpha} \left| g(x) - S_n(g,x) \right|^p dx \\ &\quad\leq C \Bigg(\sum_{m=n+1}^{\infty} m^\alpha \int_{2l\pi/r+\pi/(m+1)}^{2l\pi/r+\pi/m} \left| \sum_{k=n+1}^{m} b_k \sin kx \right|^p dx \\ &\qquad+ \sum_{m=n+1}^{\infty} m^\alpha \int_{2l\pi/r+\pi/(m+1)}^{2l\pi/r+\pi/m} \left| \sum_{k=m+1}^{[c(m+1)]} b_k \sin kx \right|^p dx \\ &\qquad+ \sum_{m=n+1}^{\infty} m^\alpha \int_{2l\pi/r+\pi/(m+1)}^{2l\pi/r+\pi/m} \left| \sum_{k=[c(m+1)]+1}^{\infty} b_k \sin kx \right|^p dx \Bigg) \\ &\quad:= C(\Sigma_3 + \Sigma_4 + \Sigma_5). \end{aligned} \tag{4.39}$$

We have

$$\Sigma_3 \leq C \sum_{m=n+1}^{\infty} m^{\alpha-2} \left(\sum_{k=n+1}^{m} b_k \right)^p, \tag{4.40}$$

and taking $\gamma_m = m^{\alpha-2}$ and $\alpha_k = 0$ for $k < n+1$, $\alpha_k = b_k$ for $k \geq n+1$ in (3.3), we get for $\alpha < 1$

$$\Sigma_3 \leq C \sum_{m=n+1}^{\infty} m^{(\alpha-2)(1-p)} b_m^p \left(\sum_{k=m}^{\infty} k^{\alpha-2} \right)^p \leq C \sum_{m=n+1}^{\infty} m^{\alpha+p-2} b_m^p. \tag{4.41}$$

If $(b_n) \in \overline{GM_\theta^r}$, then using (3.2) and (4.14), we have

$$\begin{aligned}
\Sigma_4+\Sigma_5 &\leq C\left(\sum_{m=n+1}^{\infty} m^{\alpha-2} \left(\sum_{k=m+1}^{[c(m+1)]} b_k \right)^p + \sum_{m=n+1}^{\infty} m^{\alpha} \int_{2l\pi/r+\pi/(m+1)}^{2l\pi/r+\pi/m} \frac{1}{\left(\dfrac{r}{\pi}x - 2l \right)^p} \left(\sum_{k=[c(m+1)]+1}^{\infty} |\Delta_r b_k| \right)^p dx \right) \\
&\leq C\left(\sum_{m=n+1}^{\infty} m^{\alpha+\theta p-2} \left(\sum_{k=m+1}^{[c(m+1)]} \frac{b_k}{k^\theta} \right)^p + \sum_{m=n+1}^{\infty} m^{\alpha+p-2} \left(m^{\theta-1} \sum_{k=m+1}^{\infty} \frac{b_k}{k^\theta} \right)^p \right) \\
&\leq C \sum_{m=n+1}^{\infty} m^{\alpha+\theta p-2} \left(\sum_{k=m}^{\infty} \frac{b_k}{k^\theta} \right).
\end{aligned} \tag{4.42}$$

Set $\gamma_m = m^{\alpha+\theta p-2}$, $\alpha_k = 0$ for $k < n+1$ and $\alpha_k = k^{-\theta} b_k$ for $k \geq n+1$. Then by (3.3), we have for $\alpha > 1-\theta p$

$$\Sigma_4 + \Sigma_5 \leq C \sum_{m=n+1}^{\infty} m^{(\alpha+\theta p-2)(1-p)} m^{-\theta p} b_m^p \left(\sum_{k=1}^{m} k^{\alpha+\theta p-2} \right) \leq C \sum_{\mathrm{m=n+1}}^{\infty} m^{\alpha+p-2} b_m^p. \tag{4.43}$$

Let

$$\frac{2(l+1)\pi}{r} - \frac{\pi}{r} \leq x < \frac{2(l+1)\pi}{r} - \frac{\pi}{m+1}, \tag{4.44}$$

where $m := m(x) \geq r$ and $l = 0, 1, \ldots, [r/2] - 1 (r \geq 2)$. Then, for $n \geq m$, using (3.2) and (4.19), we get

$$\begin{aligned}
&\int_{2(l+1)\pi/r-\pi/r}^{2(l+1)\pi/r-\pi/(n+1)} \omega_{\alpha,r}(x) \left| g(x) - S_n(g,x) \right|^p dx \\
&\quad = \sum_{m=r}^{n} \int_{2(l+1)\pi/r-\pi/m}^{2(l+1)\pi/r-\pi/(m+1)} \left(\frac{2(l+1)\pi}{r} - x \right)^{-\alpha} \left| g(x) - S_n(g,x) \right|^p dx \\
&\quad \leq C \sum_{m=r}^{n} m^{\alpha} \int_{2(l+1)\pi/r-\pi/m}^{2(l+1)\pi/r-\pi/(m+1)} \frac{1}{2|\sin rx/2|^p} \left(\sum_{k=n+1}^{\infty} |\Delta_r b_k| + \sum_{k=n+1}^{n+r} b_k \right)^p dx \\
&\quad \leq C \sum_{m=r}^{n} m^{\alpha} \int_{2(l+1)\pi/r-\pi/m}^{2(l+1)\pi/r-\pi/(m+1)} \frac{1}{(2(l+1) - r/\pi x)^p} \left(\sum_{k=n+1}^{\infty} |\Delta_r b_k| \right)^p \\
&\quad \leq C \sum_{m=r}^{n} m^{\alpha+p-2} \left(\sum_{k=n+1}^{\infty} |\Delta_r b_k| \right)^p \\
&\quad \leq C(\Sigma_1 + \Sigma_2).
\end{aligned} \tag{4.45}$$

Therefore,

$$\int_{2(l+1)\pi/r-\pi/r}^{2(l+1)\pi/r-\pi/(n+1)} \omega_{\alpha,r}(x)\left|g(x)-S_n(g,x)\right|^p dx \le C\left(n^{\alpha+p-1}\left(\sum_{k=n+1}^{[c(n+1)]}|\Delta_r b_k|\right)^p + \sum_{k=n+1}^{\infty} k^{\alpha+p-2} b_k^p\right). \tag{4.46}$$

If $m \ge n+1 \ge r+1$, then

$$\begin{aligned}
&\int_{2(l+1)\pi/r-\pi/(n+1)}^{2(l+1)\pi/r} \omega_{\alpha,r}(x)\left|g(x)-S_n(g,x)\right|^p dx \\
&\quad= \sum_{m=n+1}^{\infty}\int_{2(l+1)\pi/r-\pi/m}^{2(l+1)\pi/r-\pi/(m+1)}\left(\frac{2(l+1)\pi}{r}-x\right)^{-\alpha}\left|g(x)-S_n(g,x)\right|^p dx \\
&\quad\le C\left(\sum_{m=n+1}^{\infty} m^{\alpha}\int_{2(l+1)\pi/r-\pi/m}^{2(l+1)\pi/r-\pi/(m+1)}\left|\sum_{k=n+1}^{m} b_k \sin kx\right|^p dx\right. \\
&\qquad+\sum_{m=n+1}^{\infty} m^{\alpha}\int_{2(l+1)\pi/r-\pi/m}^{2(l+1)\pi/r-\pi/(m+1)}\left|\sum_{k=m+1}^{[c(m+1)]} b_k \sin kx\right|^p dx \\
&\qquad\left.+\sum_{m=n+1}^{\infty} m^{\alpha}\int_{2(l+1)\pi/r-\pi/m}^{2(l+1)\pi/r-\pi/(m+1)}\left|\sum_{k=[c(m+1)]+1}^{\infty} b_k \sin kx\right|^p dx\right) \\
&\quad:= C(\Sigma_6+\Sigma_7+\Sigma_8).
\end{aligned} \tag{4.47}$$

Similarly as in the estimation of the quantity Σ_3 using (3.3) for $\alpha < 1$, we have

$$\Sigma_6 \le C\sum_{m=n+1}^{\infty} m^{\alpha-2}\left(\sum_{k=n+1}^{m} b_k\right)^p \le C\sum_{m=n+1}^{\infty} m^{\alpha+p-2} b_m^p. \tag{4.48}$$

If $(b_n) \in \overline{GM}_{\theta}^{r}$, then using (3.2) and (4.19), we have

$$\begin{aligned}
\Sigma_7+\Sigma_8 &\le C\left(\sum_{m=n+1}^{\infty} m^{\alpha-2}\left(\sum_{k=m+1}^{[c(m+1)]} b_k\right)^p\right. \\
&\qquad\left.+\sum_{m=n+1}^{\infty} m^{\alpha}\int_{2(l+1)\pi/r-\pi/m}^{2(l+1)\pi/r-\pi/(m+1)}\frac{1}{(2(l+1)-r/\pi x)^p}\left(\sum_{k=[c(m+1)]+1}^{\infty}|\Delta_r b_k|\right)^p dx\right) \\
&\le C\left(\sum_{m=n+1}^{\infty} m^{\alpha+\theta p-2}\left(\sum_{k=m+1}^{[c(m+1)]}\frac{b_k}{k^{\theta}}\right)^p + \sum_{m=n+1}^{\infty} m^{\alpha+p-2}\left(m^{\theta-1}\sum_{k=m+1}^{\infty}\frac{b_k}{k^{\theta}}\right)^p\right) \\
&\le C\sum_{m=n+1}^{\infty} m^{\alpha+\theta p-2}\left(\sum_{k=m}^{\infty}\frac{b_k}{k^{\theta}}\right).
\end{aligned} \tag{4.49}$$

Further, by (3.3), we have for $\alpha > 1 - \theta p$

$$\Sigma_7 + \Sigma_8 \le C \sum_{m-n+1}^{\infty} m^{\alpha+p-2} b_m^p. \tag{4.50}$$

Combining (4.32) or (4.33), (4.35)–(4.43) and (4.45)–(4.50) we complete the proof of Theorem 2.5.

4.5. Proof of Theorem 2.6

The proof of Theorem 2.6 goes analogously as the proof of Theorem 2.5. The only difference is that instead of (4.41) (for $r = 1, 2$) and (4.48) (for $r = 2$) we use the below estimations.

Applying the inequalities $|\sin kx| \le kx$ for $x \in (0, \pi)$ and $|\sin kx| \le k(\pi - x)$ for $x \in (0, \pi)$ and using (3.3), for $\alpha < 1 + p$, we have

$$\begin{aligned}
\Sigma_3 &= \sum_{m=n+1}^{\infty} m^{\alpha} \int_{\pi/(m+1)}^{\pi/m} \left| \sum_{k=n+1}^{m} b_k \sin kx \right|^p dx \le \sum_{m=n+1}^{\infty} m^{\alpha} \int_{\pi/(m+1)}^{\pi/m} \left(x \sum_{k=n+1}^{m} k b_k \right)^p dx \\
&\le C \sum_{m=n+1}^{\infty} m^{\alpha-p-2} \left(\sum_{k=n+1}^{m} k b_k \right)^p \le C \sum_{m=n+1}^{\infty} m^{\alpha+p-2} b_m^p, \\
\Sigma_6 &= \sum_{m=n+1}^{\infty} m^{\alpha} \int_{\pi-\pi/m}^{\pi-\pi/(m+1)} \left| \sum_{k=n+1}^{m} b_k \sin kx \right|^p dx \\
&\le \sum_{m=n+1}^{\infty} m^{\alpha} \int_{\pi-\pi/m}^{\pi-\pi/(m+1)} \left((\pi - x) \sum_{k=n+1}^{m} k b_k \right)^p dx \\
&\le C \sum_{m=n+1}^{\infty} m^{\alpha-p-2} \left(\sum_{k=n+1}^{m} k b_k \right)^p \le C \sum_{m=n+1}^{\infty} m^{\alpha+p-2} b_m^p.
\end{aligned} \tag{4.51}$$

This completes the proof.

References

[1] P. Heywood, "On the integrability of functions defined by trigonometric series," *The Quarterly Journal of Mathematics*, vol. 5, pp. 71–76, 1954.

[2] Y.-M. Chen, "On the integrability of functions defined by trigonometrical series," *Mathematische Zeitschrift*, vol. 66, pp. 9–12, 1956.

[3] A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, Mass, USA, 1977.

[4] R. P. Boas Jr., *Integrability Theorems for Trigonometric Transforms*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 38, Springer, New York, NY, USA, 1967.

[5] R. Askey and S. Wainger, "Integrability theorems for Fourier series," *Duke Mathematical Journal*, vol. 33, pp. 223–228, 1966.

[6] L. Leindler, "A new class of numerical sequences and its applications to sine and cosine series," *Analysis Mathematica*, vol. 28, no. 4, pp. 279–286, 2002.

[7] S. Yu. Tikhonov, "On the integrability of trigonometric series," *Matematicheskie Zametki*, vol. 78, no. 3-4, pp. 437–442, 2005.

[8] S. Tikhonov, "Trigonometric series with general monotone coefficients," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 721–735, 2007.
[9] D. S. Yu, P. Zhou, and S. P. Zhou, "On L^p integrability and convergence of trigonometric series," *Studia Mathematica*, vol. 182, no. 3, pp. 215–226, 2007.
[10] S. Tikhonov, "Trigonometric series of Nikol'skii classes," *Acta Mathematica Hungarica*, vol. 114, no. 1-2, pp. 61–78, 2007.
[11] S. Tikhonov, "Best approximation and moduli of smoothness: computation and equivalence theorems," *Journal of Approximation Theory*, vol. 153, no. 1, pp. 19–39, 2008.
[12] R. J. Le and S. P. Zhou, "A new condition for the uniform convergence of certain trigonometric series," *Acta Mathematica Hungarica*, vol. 108, no. 1-2, pp. 161–169, 2005.
[13] L. Leindler, "Embedding results regarding strong approximation of sine series," *Acta Scientiarum Mathematicarum*, vol. 71, no. 1-2, pp. 91–103, 2005.
[14] D. S. You, P. Zhou, and S. P. Zhou, "Ultimate generalization to monotonicity for uniform convergence of trigonometric series," preprint, http://arxiv.org/abs/math/0611805.
[15] M. Dyachenko and S. Tikhonov, "Integrability and continuity of functions represented by trigonometric series: coefficients criteria," *Studia Mathematica*, vol. 193, no. 3, pp. 285–306, 2009.
[16] B. Wei and D. Yu, "On weighted L^p integrability and approximation of trigonometric series," *Analysis in Theory and Applications*, vol. 25, no. 1, pp. 40–54, 2009.
[17] A. A. Konyushkov, "Best approximation by trigonometric polynomials and Fourier coefficients," *Sbornik Mathematics*, vol. 44, pp. 53–84, 1958.
[18] L. Leindler, "Best approximation and Fourier coefficients," *Analysis Mathematica*, vol. 31, no. 2, pp. 117–129, 2005.
[19] B. Szal, "On L-convergence of trigonometric series," to appear, http://arxiv.org/abs/0908.4578.
[20] L. Leindler, "Generalization of inequalities of Hardy and Littlewood," *Acta Scientiarum Mathematicarum*, vol. 31, pp. 279–285, 1970.