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Review Article

Complete Convergence for Negatively Dependent Sequences of Random Variables

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We study the complete convergence for negatively dependent sequences of random variables. As a result, we extend some complete convergence theorems for independent random variables to the case of negatively dependent random variables without necessarily imposing any extra conditions.

1. Introduction and Lemmas

Definition 1.1. Random variables X and Y are said to negatively dependent (ND) if

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y) \tag{1.1}$$

for all $x, y \in$ R. A collection of random variables is said to be pairwise negatively dependent (PND) if every pair of random variables in the collection satisfies (1.1).

It is important to note that (1.1) implies

$$P(X > x, Y > y) \le P(X > x)P(Y > y) \tag{1.2}$$

for all $x, y \in$ R. Moreover, it follows that (1.2) implies (1.1), and hence, (1.1) and (1.2) are equivalent. Ebrahimi and Ghosh [1] showed that (1.1) and (1.2) are not equivalent for a collection of 3 or more random variables. They considered random variables X_1, X_2, and X_3 where (X_1, X_2, X_3) assumed the values $(0,1,1), (1,0,1), (1,1,0)$, and $(0,0,0)$ each with probability $1/4$. The random variables X_1, X_2, and X_3 are pairwise independent, and hence, they satisfy both (1.1) and (1.2) for all pairs. However,

$$P(X_1 > x_1, X_2 > x_2, X_3 > x_3) \le P(X_1 > x_1)P(X_2 > x_2)P(X_3 > x_3) \tag{1.3}$$

for all $x_1,\ x_2$, and x_3, but

$$P(X_1 \le 0, X_2 \le 0, X_3 \le 0) = \frac{1}{4} \not\le \frac{1}{8} = P(X_1 \le 0) P(X_2 \le 0) P(X_3 \le 0). \tag{1.4}$$

Placing probability $1/4$ on each of the other vertices $\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$ provides the converse example of pairwise independent random variables which will not satisfy (1.3) with $x_1 = 0,\ \ x_2 = 0$, and $x_3 = 0$ but where the desired "$\le$" in $P(X_1 \le x_1, X_2 \le x_2, X_3 \le x_3) \le \prod_{i=1}^{3} P(X_i \le x_i)$ hold for all $x_1,\ x_2$, and x_3. Consequently, the following definition is needed to define sequences of negatively dependent random variables.

Definition 1.2. Random variables $X_1, \ldots, X_n$ are said to be negatively dependent (ND) if for all real $x_1, \ldots, x_n$,

$$\begin{gathered} P\left(\bigcap_{j=1}^{n} (X_j \le x_j)\right) \le \prod_{j=1}^{n} P[X_j \le x_j], \\ P\left(\bigcap_{j=1}^{n} (X_j > x_j)\right) \le \prod_{j=1}^{n} P(X_j > x_j). \end{gathered} \tag{1.5}$$

An infinite sequence of random variables $\{X_n;\ n \ge 1\}$ is said to be ND if every finite subset $X_1, \ldots, X_n$ is ND.

Definition 1.3. Random variables $X_1, X_2, \ldots, X_n,\ n \ge 2$ are said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{cov}\big(f_1(X_i;\ i \in A_1), f_2(X_j;\ j \in A_2)\big) \le 0, \tag{1.6}$$

where f_1 and f_2 are increasing for every variable (or decreasing for every variable), such that this covariance exists. An infinite sequence of random variables $\{X_n;\ n \ge 1\}$ is said to be NA if every finite subfamily is NA.

The definition of PND is given by Lehmann [2]. The concept of ND is given by Bozorgnia et al. [3], and the definition of NA is introduced by Joag-Dev and Proschan [4]. These conceptions of dependence random variables have been very useful in reliability theory and applications.

It is easy to see that NA implies ND from the definitions. But in the following example, we will show that ND does not imply NA.

Example 1.4. Let X_i be a binary random variable such that $P(X_i = 0) = P(X_i = 1) = 0.5$ for $i = 1, 2, 3$. Let (X_1, X_2, X_3) take the values $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, and $(1,1,1)$, each with probability $1/4$.

It can be verified that all the ND conditions hold. However,

$$P(X_1 + X_3 \le 1, X_2 \le 0) = \frac{4}{8} \not\le \frac{3}{8} = P(X_1 + X_3 \le 1) P(X_2 \le 0). \tag{1.7}$$

Thus, $X_1,\ X_2,\ X_3$ are not NA.

From the above example, it is shown that ND is much weaker than NA. Because of the wide applications of ND random variables, the notions of ND random variables have received more and more attention recently. A series of useful results have been established (cf. [3, 5–11]). Hence, it is highly desirable and of considerable significance to extend the limit properties of independent or NA random variables to the case of ND random variables theorems and applications.

Complete convergence is one of the most important problems in probability theory. Recent results of the complete convergence can be found in Wu [11, 12] and Sung [13, 14]. In this paper we study the complete convergence for negatively dependent random variables. As a result, we extend some complete convergence theorems for independent random variables to the negatively dependent random variables without necessarily imposing any extra conditions.

Lemma 1.5 (see [3]). *Let $X_1, \dots, X_n$ be ND random variables and let $\{f_n;\ n \geq 1\}$ be a sequence of Borel functions all of which are monotone increasing (or all are monotone decreasing). Then $\{f_n(X_n);\ n \geq 1\}$ is a sequence of ND r.v.'s.*

Lemma 1.6 (see [3]). *Let $X_1, \dots, X_n$ be nonnegative r.v.'s which are ND. Then*

$$E\left(\prod_{j=1}^{n} X_j\right) \leq \prod_{j=1}^{n} EX_j. \tag{1.8}$$

In particular, let $X_1, \dots, X_n$ be ND and let $t_1, \dots, t_n$ be all nonnegative (or nonpositive) real numbers. Then

$$E\left(\exp\left(\sum_{j=1}^{n} t_j X_j\right)\right) \leq \prod_{j=1}^{n} E\left(\exp\left(t_j X_j\right)\right). \tag{1.9}$$

Lemma 1.7. *Let $\{X_n;\ n \geq 1\}$ be a sequence of ND random variables with $EX_i = 0,\ EX_i^2 < \infty$. Then for $n \geq 1$,*

$$ES_n^2 \leq \sum_{k=1}^{n} EX_k^2, \tag{1.10}$$

where $S_n = \sum_{k=1}^{n} X_k$.

Proof. Obviously, ND implies PND from their definitions. Thus, by Lemma 2 of Wu [12], Lemma 1.7 holds. □

2. Main Results and the Proof

In the following, let $a_n \ll b_n$ denote that there exists a constant $c > 0$ such that $a_n \leq cb_n$ for sufficiently large n, $\log x$ mean $\ln(\max(x, \mathrm{e}))$, and $S_n = \sum_{j=1}^{n} X_j$.

Theorem 2.1. *Let* $\{X_n;\ n \geq 1\}$ *be a sequence of ND identically distributed random variables. Let for some* $0 < \alpha \leq 1$

$$E|X_1|^{2/\alpha} < \infty, \quad EX_1 = 0, \tag{2.1}$$

$$|a_{nk}| \leq cn^{-\alpha} \quad \text{for } k \leq n \text{ and some } 0 < c < \infty, \quad a_{nk} = 0 \quad \text{for } k > n, \tag{2.2}$$

$$c_n \widehat{=} \sum_{k=1}^{n} a_{nk}^2 = o\left(\log^{-1} n\right). \tag{2.3}$$

Then

$$T_n \widehat{=} \sum_{k=1}^{n} a_{nk} X_k \xrightarrow{c} 0, \tag{2.4}$$

where $\xrightarrow{c}$ *denotes complete convergence.*

Theorem 2.2. *Let* $\{X_n;\ n \geq 1\}$ *be a sequence of ND identically distributed random variables with*

$$E|X_1|^{2/\alpha} < \infty, \quad \text{for some } \alpha > 1. \tag{2.5}$$

Then

$$\frac{S_n}{n^{\alpha}} \xrightarrow{c} 0. \tag{2.6}$$

Remark 2.3. Theorems 2.1 and 2.2 extend corresponding results for independent r.v.s. to ND r.v.s. without necessarily adding any extra conditions.

Proof of Theorem 2.1. By $2/\alpha \geq 2$ and (2.1), we have $EX_1^2 < \infty$. Let $\varepsilon > 0$ be given. By $T_n \widehat{=} \sum_{k=1}^{n} a_{nk}^{+} X_k - \sum_{k=1}^{n} a_{nk}^{-} X_k$, where $a_{nk}^{+} = \max(a_{nk}, 0) \geq 0$, and $a_{nk}^{-} = \max(-a_{nk}, 0) \geq 0$, without loss of generality, we can assume that $a_{nk} > 0$ for all $n \geq 1$, $k \leq n$, and $EX_1^2 = 1$. For $k \leq n$, let

$$\begin{aligned}
X_{nk}^{(1)} &= -n^{-\alpha/3} a_{nk}^{-1} I_{(X_k < -n^{-\alpha/3} a_{nk}^{-1})} + X_k I_{(|X_k| \leq n^{-\alpha/3} a_{nk}^{-1})} + n^{-\alpha/3} a_{nk}^{-1} I_{(X_k > n^{-\alpha/3} a_{nk}^{-1})}, \\
X_{nk}^{(2)} &= \left(X_k + n^{-\alpha/3} a_{nk}^{-1}\right) I_{(X_k < -(\varepsilon a_{nk}^{-1})/4)} + \left(X_k - n^{-\alpha/3} a_{nk}^{-1}\right) I_{(X_k > (\varepsilon a_{nk}^{-1})/4)}, \\
X_{nk}^{(3)} &= X_k - X_k^{(1)} - X_k^{(2)} \\
&= \left(X_k + n^{-\alpha/3} a_{nk}^{-1}\right) I_{(-(\varepsilon a_{nk}^{-1})/4 < X_k < -n^{-\alpha/3} a_{nk}^{-1})} + \left(X_k - n^{-\alpha/3} a_{nk}^{-1}\right) I_{(n^{-\alpha/3} a_{nk}^{-1} < X_k < (\varepsilon a_{nk}^{-1})/4)}, \\
& \qquad T_n^{(i)} = \sum_{k=1}^{n} a_{nk} X_{nk}^{(i)}, \quad i = 1, 2, 3.
\end{aligned} \tag{2.7}$$

Thus,

$$T_n = T_n^{(1)} + T_n^{(2)} + T_n^{(3)}. \tag{2.8}$$

Note that

$$(|T_n| > 3\varepsilon) \subset \left(\left|T_n^{(1)}\right| > \varepsilon\right) \cup \left(\left|T_n^{(2)}\right| > \varepsilon\right) \cup \left(\left|T_n^{(3)}\right| > \varepsilon\right). \tag{2.9}$$

We shall prove that $\sum_{n=1}^{\infty} P(|T_n| > 3\varepsilon) < \infty$ by proving that

$$\sum_{n=1}^{\infty} P\left(\left|T_n^{(i)}\right| > \varepsilon\right) < \infty, \quad i = 1, 2, 3. \tag{2.10}$$

By $EX_k = 0$ and (2.2),

$$\begin{aligned}
\left|ET_n^{(1)}\right| &= \left|\sum_{k=1}^{n} a_{nk} EX_{nk}^{(1)}\right| = \left|\sum_{k=1}^{n} a_{nk} E\left(X_k - X_{nk}^{(1)}\right)\right| \\
&\leq \sum_{k=1}^{n} a_{nk} \left(E\left|X_k + n^{-\alpha/3} a_{nk}^{-1}\right| I_{(X_k < -n^{-\alpha/3} a_{nk}^{-1})} + E\left|X_k - n^{-\alpha/3} a_{nk}^{-1}\right| I_{(X_k > n^{-\alpha/3} a_{nk}^{-1})}\right) \\
&\leq \sum_{k=1}^{n} a_{nk} E|X_1| I_{(|X_1| > n^{-\alpha/3} a_{nk}^{-1})} \\
&\ll n^{-\alpha+1} E|X_1| I_{(|X_1| > c^{-1} n^{2\alpha/3})} \\
&\leq n^{-(1+\alpha)/3} E|X_1|^{2/\alpha} \longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{2.11}$$

So to prove $\sum_{n=1}^{\infty} P(|T_n^{(1)}| > \varepsilon) < \infty$ it suffices to show that

$$\sum_{n=1}^{\infty} P\left(\left|T_n^{(1)} - ET_n^{(1)}\right| > \frac{\varepsilon}{2}\right) < \infty. \tag{2.12}$$

Let $\widetilde{X}_{nk}^{(1)} = X_{nk}^{(1)} - EX_{nk}^{(1)}$, $\widetilde{T}_n^{(1)} = T_n^{(1)} - ET_n^{(1)}$. Fix $n \geq 1$. Let $u = \min(\varepsilon/4c_n, n^{\alpha/3}/2)$. Since $|ua_{nk}\widetilde{X}_{nk}^{(1)}| \leq 1$, $E\widetilde{X}_{nk}^{(1)} = 0$, and $E(\widetilde{X}_{nk}^{(1)})^2 \leq EX_1^2 = 1$, it follows that

$$\begin{aligned}
E\exp\left(ua_{nk}\widetilde{X}_{nk}^{(1)}\right) &= 1 + \sum_{j=1}^{\infty} \frac{E\left(ua_{nk}\widetilde{X}_{nk}^{(1)}\right)^j}{j!} \\
&= 1 + E\left(ua_{nk}\widetilde{X}_{nk}^{(1)}\right) + \frac{E\left(ua_{nk}\widetilde{X}_{nk}^{(1)}\right)^2}{2}\left(1 + 2\sum_{j=3}^{\infty}\frac{1}{j!}\right) \\
&\leq 1 + \frac{u^2 a_{nk}^2 E\left(\widetilde{X}_{nk}^{(1)}\right)^2}{2}(1 + 2(\mathrm{e} - 2.5)) \\
&\leq 1 + u^2 a_{nk}^2 \leq \exp\left(u^2 a_{nk}^2\right).
\end{aligned} \tag{2.13}$$

Since $X_{nk}^{(1)}$ and $-X_{nk}^{(1)}$ are nondecreasing and nonincreasing functions of X_k, respectively, thus $\{ua_{nk}\widetilde{X}_{nk}^{(1)},\ n \geq 1,\ k \leq n\}$ and $\{ua_{nk}(-\widetilde{X}_{nk}^{(1)}),\ n \geq 1,\ k \leq n\}$ are also ND by Lemma 1.5. It follows from Lemma 1.6 and (2.13) that

$$\begin{aligned} E\exp\left(u\widetilde{T}_n^{(1)}\right) &= E\left(\prod_{k=1}^{n}\exp\left(ua_{nk}\widetilde{X}_{nk}^{(1)}\right)\right) \\ &\leq \prod_{k=1}^{n}E\left(\exp\left(ua_{nk}\widetilde{X}_{nk}^{(1)}\right)\right) \\ &\leq \prod_{k=1}^{n}\exp\left(u^2a_{nk}^2\right) = \exp\left(u^2c_n\right). \end{aligned} \tag{2.14}$$

By the Markov inequality,

$$P\left(\widetilde{T}_n^{(1)} > \varepsilon\right) \leq \exp(-\varepsilon u)E\exp\left(uT_n^{(1)}\right) \leq \exp\left(-\varepsilon u + u^2c_n\right). \tag{2.15}$$

Since $\{-X_{nk}^{(1)}\}$ is also satisfying the conditions: $|ua_{nk}(-\widetilde{X}_{nk}^{(1)})| \leq 1$, $E(-\widetilde{X}_{nk}^{(1)}) = 0$, and $E(-\widetilde{X}_{nk}^{2}) \leq 1$, replacing the X_k by $-X_k$ in (2.15) the argument will then establish

$$P\left(\widetilde{T}_n^{(1)} < -\varepsilon\right) \leq \exp\left(-\varepsilon u + u^2c_n\right). \tag{2.16}$$

Thus,

$$P\left(\left|\widetilde{T}_n^{(1)}\right| > \frac{\varepsilon}{2}\right) \leq 2\exp\left(-\frac{\varepsilon u}{2} + u^2c_n\right). \tag{2.17}$$

If $\varepsilon/2c_n > n^{\alpha/3}$, then by the definition of u, we have $u = n^{\alpha/3}/2$, $-\varepsilon u/2 + u^2c_n \leq -\varepsilon n^{\alpha/3}/8$. Hence

$$\sum_{n=1}^{\infty}P\left(\left|T_n^{(1)}\right| > \frac{\varepsilon}{2}\right) \leq 2\sum_{n=1}^{\infty}\exp\left(-\frac{\varepsilon n^{\alpha/3}}{8}\right) < \infty. \tag{2.18}$$

If $\varepsilon/2c_n \leq n^{\alpha/3}$, then by the definition of u, we have $u = \varepsilon/4c_n$, and $-\varepsilon u/2 + u^2c_n = -\varepsilon^2/16c_n$. And $c_n < \varepsilon^2/32\log n$ for sufficiently large n from $c_n = o(\log^{-1}n)$. Hence

$$\sum_{n=1}^{\infty}P\left(\left|T_n^{(1)}\right| > \frac{\varepsilon}{2}\right) \leq 2\sum_{n=1}^{\infty}\exp\left(-\frac{\varepsilon^2}{16c_n}\right) \ll \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty. \tag{2.19}$$

Now we prove that $\sum_{n=1}^{\infty}P(|T_n^{(2)}| > \varepsilon) < \infty$. Since

$$\left|T_n^{(2)}\right| = \left|\sum_{k=1}^{n}a_{nk}X_{nk}^{(2)}\right| \leq \sum_{k=1}^{n}a_{nk}\left|X_{nk}^{(2)}\right| \leq \sum_{k=1}^{n}a_{nk}|X_k|I_{(|X_k|>(\varepsilon a_{nk}^{-1}/4))}, \tag{2.20}$$

therefore,

$$\left(\left|T_n^{(2)}\right| > \varepsilon\right) \subset \bigcup_{k=1}^{n}\left(|X_k| > \frac{\varepsilon a_{nk}^{-1}}{4}\right) \subset \bigcup_{k=1}^{n}\left(|X_k| > \varepsilon n^{\alpha}(4c)^{-1}\right). \tag{2.21}$$

Since $0 < a_{nk} \le cn^{-\alpha}$ for $k \le n$, thus

$$\begin{aligned} P\left(\left|T_n^{(2)}\right| > \varepsilon\right) &\le \sum_{k=1}^{n} P\left(|X_k| > \varepsilon n^{\alpha}(4c)^{-1}\right) \\ &= nP\left(|X_1| > \varepsilon n^{\alpha}(4c)^{-1} \widehat{=} (Bn)^{\alpha}\right), \end{aligned} \tag{2.22}$$

where $B = \varepsilon^{1/\alpha}(4c)^{-1/\alpha} > 0$. Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} P\left(\left|T_n^{(2)}\right| > \varepsilon\right) &\le \sum_{n=1}^{\infty} nP\left(|X_1|^{1/\alpha} > Bn\right) \\ &= \sum_{n=1}^{\infty}\sum_{j=n}^{\infty} nP\left(Bj < |X_1|^{1/\alpha} \le (j+1)B\right) \\ &= \sum_{j=1}^{\infty}\sum_{n=1}^{j} nP\left(Bj < |X_1|^{1/\alpha} \le (j+1)B\right) \\ &\le \sum_{j=1}^{\infty} j^2 P\left(Bj < |X_1|^{1/\alpha} \le (j+1)B\right) \\ &\le \sum_{j=1}^{\infty} B^{-2} E|X_1|^{2/\alpha} I_{(Bj < |X_1|^{1/\alpha} \le (j+1)B)} \\ &\ll E|X_1|^{2/\alpha} < \infty. \end{aligned} \tag{2.23}$$

Lastly, we prove that $\sum_{n=1}^{\infty} P(|T_n^{(3)}| > \varepsilon) < \infty$. Since

$$\left(\left|T_n^{(3)}\right| > \varepsilon\right) \subset \left(\text{there exist at least 4 indices } k \text{ such that } |X_k| > a_{nk}^{-1} n^{-\alpha/3}\right), \tag{2.24}$$

we have

$$\begin{aligned} &P\left(\left|T_n^{(3)}\right| > \varepsilon\right) \\ &\quad\le P\left(\text{there exist at least 4 indices } k \text{ such that } a_{nk}|X_k| > n^{-\alpha/3}\right) \\ &\quad\le \sum_{1 \le i_1 < \cdots < i_4 \le n} P\left(a_{ni_1}|X_{i_1}| > n^{-\alpha/3}, a_{ni_2}|X_{i_2}| > n^{-\alpha/3}, a_{ni_3}|X_{i_3}| > n^{-\alpha/3}, a_{ni_4}|X_{i_4}| > n^{-\alpha/3}\right). \end{aligned} \tag{2.25}$$

By the definition of ND, and the fact that $0 < a_{nk} \le cn^{-\alpha}$ for $k \le n$, we conclude that

$$\begin{aligned} P\left(\left|T_n^{(3)}\right| > \varepsilon\right) &\le \sum_{1\le i_1<\cdots<i_4\le n} \prod_{j=1}^{4} P\left(\left|X_{i_j}\right| > c^{-1}n^{2\alpha/3}\right) \\ &= C_n^4 P^4\left(|X_1| > c^{-1}n^{2\alpha/3}\right) \le n^4 P^4\left(|X_1| > c^{-1}n^{2\alpha/3}\right). \end{aligned} \tag{2.26}$$

Thus, by (2.1)

$$\begin{aligned} \sum_{n=1}^{\infty} P\left(\left|T_n^{(3)}\right| > \varepsilon\right) &\ll \sum_{n=1}^{\infty} n^4 \left(n^{(-2\alpha/3)(2/\alpha)} E|X_1|^{2/\alpha}\right)^4 \\ &\le \left(E|X_1|^{2/\alpha}\right)^4 \sum_{n=1}^{\infty} n^{-4/3} < \infty. \end{aligned} \tag{2.27}$$

Together with (2.19)–(2.27), we get

$$\sum_{n=1}^{\infty} P(|T_n| > \varepsilon) < \infty, \tag{2.28}$$

for all $\varepsilon > 0$ as desired. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Let

$$\begin{gathered} X_{kn} = -n^{\alpha} I_{(X_k<-n^{\alpha})} + X_k I_{(|X_k|\le n^{\alpha})} + n^{\alpha} I_{(X_k>n^{\alpha})}, \quad \forall n \ge 1,\ k \le n, \\ S_{kn} = \sum_{k=1}^{n} X_{kn}. \end{gathered} \tag{2.29}$$

If $2/\alpha \ge 1$, then $E|X_1| < \infty$ and $nP(|X_1| > n^{\alpha}) \to 0$ from (2.5). Thus

$$\begin{aligned} n^{-\alpha}|ES_{kn}| &\le n^{-\alpha} \sum_{k=1}^{n} \left(E|X_k| I_{(|X_k|\le n^{\alpha})} + n^{\alpha} P(|X_k| > n^{\alpha})\right) \\ &\le E|X_1| n^{1-\alpha} + nP(|X_1| > n^{\alpha}) \longrightarrow 0. \end{aligned} \tag{2.30}$$

If $2/\alpha < 1$, from above proof, we have

$$n^{-\alpha}|ES_{kn}| \le n^{1-\alpha} \sum_{k=1}^{n} E|X_1| I_{((k-1)^{\alpha}<|X_1|\le k^{\alpha})} + nP(|X_1| > n^{\alpha}). \tag{2.31}$$

Since

$$\begin{aligned}\sum_{k=1}^{\infty} k^{1-\alpha} E|X_1| I_{((k-1)^{\alpha}<|X_1|\le k^{\alpha})} &= \sum_{k=1}^{\infty} k^{1-\alpha} E|X_1|^{2/\alpha} k^{\alpha(1-2/\alpha)} I_{(k-1<|X_1|^{1/\alpha}\le k)} \\ &= \sum_{k=1}^{\infty} k^{-1} E|X_1|^{2/\alpha} I_{(k-1<|X_1|^{1/\alpha}\le k)} \\ &\le E|X_1|^{2/\alpha} < \infty,\end{aligned} \tag{2.32}$$

by $\alpha > 1$, and the Kronecker lemma, combining with (2.31), we get $n^{-\alpha}|ES_{kn}| \to 0$. Thus, for sufficiently large n,

$$n^{-\alpha}|ES_{kn}| < \frac{\varepsilon}{2}. \tag{2.33}$$

Since X_{nk} is nondecreasing function of X_k, thus $\{X_{nk},\ n \ge 1,\ k \le n\}$ is also ND by Lemma 1.5. It follows from Lemma 1.7, (2.13), $2/\alpha < 2$, the Markov inequality, and $\sum_{n=1}^{\infty} nP(|X_1| > n^{\alpha}) \ll E|X_1|^{2/\alpha} < \infty$, that

$$\begin{aligned}\sum_{n=1}^{\infty} P(|S_n| > \varepsilon n^{\alpha}) &\le \sum_{n=1}^{\infty} P\left(\bigcup_{k=1}^{n} |X_k| > n^{\alpha}\right) + \sum_{n=1}^{\infty} P(|S_{kn} - ES_{kn}| > \varepsilon n^{\alpha} - |ES_{kn}|) \\ &\le \sum_{n=1}^{\infty} nP(|X_1| > n^{\alpha}) + \sum_{n=1}^{\infty} P\left(|S_{kn} - ES_{kn}| > \frac{\varepsilon n^{\alpha}}{2}\right) \\ &\ll E|X_1|^{2/\alpha} + \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} E(S_{kn} - ES_{kn})^2 \ll \sum_{n=1}^{\infty} n^{-2\alpha} \sum_{k=1}^{n} EX_{kn}^2 \\ &\le \sum_{n=1}^{\infty} n^{-2\alpha+1} \left(EX_1^2 I_{(|X_1|<n^{\alpha})} + n^{2\alpha} P(|X_1| > n^{\alpha})\right) \\ &= \sum_{n=1}^{\infty} n^{-2\alpha+1} \sum_{k=1}^{n} EX_1^2 I_{((k-1)^{\alpha}\le|X_1|<k^{\alpha})} + \sum_{n=1}^{\infty} nP(|X_1| > n^{\alpha}) \\ &\ll \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} n^{-2\alpha+1} E|X_1|^{2/\alpha} k^{\alpha(2-2/\alpha)} I_{((k-1)^{\alpha}\le|X_1|<k^{\alpha})} + E|X_1|^{2/\alpha} \\ &\ll \sum_{k=1}^{\infty} k^{-2\alpha+2} E|X_1|^{2/\alpha} k^{\alpha(2-2/\alpha)} I_{((k-1)^{\alpha}\le|X_1|<k^{\alpha})} \\ &= \sum_{k=1}^{\infty} E|X_1|^{2/\alpha} I_{((k-1)^{\alpha}\le|X_1|<k^{\alpha})} \\ &< \infty.\end{aligned} \tag{2.34}$$

Hence,

$$\frac{S_n}{n^\alpha} \xrightarrow{c} 0. \tag{2.35}$$

This completes the proof of Theorem 2.2. □

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