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Research Article

A Note on Mixed-Mean Inequalities

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We give a simpler proof of a result of Holland concerning a mixed arithmetic-geometric mean inequality. We also prove a result of mixed-mean inequality involving the symmetric means.

1. Introduction

Let $M_{n,r}(\mathbf{x})$ be the generalized weighted power means: $M_{n,r}(\mathbf{q},\mathbf{x}) = (\sum_{i=1}^{n} q_i x_i^r)^{1/r}$, where $\mathbf{q} = (q_1, q_2, \ldots, q_n)$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and $q_i > 0, 1 \leq i \leq n$, with $\sum_{i=1}^{n} q_i = 1$. Here $M_{n,0}(\mathbf{q},\mathbf{x})$ denotes the limit of $M_{n,r}(\mathbf{q},\mathbf{x})$ as $r \to 0^+$. Unless specified, we always assume that $x_i > 0, 1 \leq i \leq n$. When there is no risk of confusion, we will write $M_{n,r}$ for $M_{n,r}(\mathbf{q},\mathbf{x})$ and we also define $A_n = M_{n,1}, G_n = M_{n,0}$, and $H_n = M_{n,-1}$.

The celebrated Hardy's inequality (see [1, Theorem 326]) asserts that, for $p > 1$, $a_n \geq 0$,

$$\sum_{n=1}^{\infty} \left(\frac{\sum_{k=1}^{n} a_k}{n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p. \tag{1.1}$$

Among the many different proofs of Hardy's inequality as well as its generalizations and extensions in the literature, one novel approach is via the mixed-mean inequalities (see, e.g., [2, Theorem 7]). By mixed-mean inequalities, we will mean the following inequalities:

$$\left(\sum_{n=1}^{m} a_{m,n} \left(\sum_{k=1}^{m} b_{n,k} x_k \right)^p \right)^{1/p} \leq \sum_{n=1}^{m} b_{m,n} \left(\sum_{k=1}^{m} a_{n,k} x_k^p \right)^{1/p}, \tag{1.2}$$

where $(a_{i,j}), (b_{i,j})$ are two $m \times m$ matrices with nonnegative entries and the above inequality being meant to hold for any vector $\mathbf{x} \in \mathbb{R}^m$ with nonnegative entries. Here $p \geq 1$ and, when $0 < p \leq 1$, we want the inequality above to be reversed.

The meaning of mixed mean becomes more clear when $(a_{i,j}),(b_{i,j})$ are weighted mean matrices. Here we say that a matrix $A=(a_{n,k})$ is a weighted mean matrix if $a_{n,k}=0$ for $n<k$ and

$$a_{n,k}=\frac{w_k}{W_n}, \quad 1\le k\le n, \qquad W_n=\sum_{i=1}^{n} w_i, \quad w_i\ge 0,\ w_1>0. \tag{1.3}$$

Now we focus our attention to the case of (1.2) for $(a_{i,j})=(b_{i,j})$ being weighted mean matrices given in (1.3). In this case, for fixed $\mathbf{x}=(x_1,\dots,x_n)$, $\mathbf{w}=(w_1,\dots,w_n)$, we define $\mathbf{x}_i=(x_1,\dots,x_i)$, $\mathbf{w}_i=(w_1,\dots,w_i)$, $W_i=\sum_{j=1}^{i} w_j$, $M_{i,r}=M_{i,r}(\mathbf{x}_i)=M_{i,r}(\mathbf{w}_i/W_i,\mathbf{x}_i)$, and $\mathbf{M}_{i,r}=(M_{1,r},\dots,M_{i,r})$. Then we have the following mixed-mean inequalities of Nanjundiah [3] (see also [4]).

Theorem 1.1. *Let $r>s$ and $n\ge 2$. If, for $2\le k\le n-1$, $W_nw_k-W_kw_n>0$, then*

$$M_{n,s}(\mathbf{M}_{n,r})\ge M_{n,r}(\mathbf{M}_{n,s}), \tag{1.4}$$

with equality holding if and only if $x_1=\cdots=x_n$.

A very elegant proof of Theorem 1.1 for the case $r=1, s=0$ is given by Kedlaya in [5]. In fact, the following Popoviciu-type inequalities were established in [5] (see also [4, Theorem 9]).

Theorem 1.2. *Let $n\ge 2$. If, for $2\le k\le n-1$, $W_nw_k-W_kw_n>0$, then*

$$W_{n-1}(\ln M_{n-1,0}(\mathbf{M}_{n-1,1})-\ln M_{n-1,1}(\mathbf{M}_{n-1,0}))\le W_n(\ln M_{n,0}(\mathbf{M}_{n,1})-\ln M_{n,1}(\mathbf{M}_{n,0})), \tag{1.5}$$

with equality holding if and only if $x_n=M_{n-1,0}=M_{n-1,1}(M_{n-1,0})$.

It is easy to see that the case $r=1, s=0$ of Theorem 1.1 follows from Theorem 1.2. As was pointed out by Kedlaya that the method used in [5] can be applied to establish both Popoviciu-type and Rado-type inequalities for mixed means for a general pair $r>s$, the details were worked out in [6] and the following Rado-type inequalities were established in [6].

Theorem 1.3. *Let $1>s$ and $n\ge 2$. If, for $2\le k\le n-1$, $W_nw_k-W_kw_n>0$, then*

$$W_{n-1}(M_{n-1,s}(\mathbf{M}_{n-1,1})-M_{n-1,1}(\mathbf{M}_{n-1,s}))\le W_n(M_{n,s}(\mathbf{M}_{n,1})-M_{n,1}(\mathbf{M}_{n,s})), \tag{1.6}$$

with equality holding if and only if $x_1=\cdots=x_n$ and the above inequality reverses when $s>1$.

A different proof of Theorem 1.1 for the case $r=1, s=0$ was given in [7] and Bennett used essentially the same approach in [8, 9] to study (1.2) for the cases $(a_{i,j}),(b_{i,j})$ being lower triangular matrices, namely, $a_{i,j}=b_{i,j}=0$ if $j>i$. Among other things, he showed [8] that inequalities (1.2) hold when $(a_{i,j}),(b_{i,j})$ are Hausdorff matrices.

In [10], Holland further improved the condition in Theorem 1.3 for the case $s=0$ by proving the following.

Theorem 1.4. *Let $n \geq 2$. If, for $2 \leq k \leq n-1$, $W_k^2 \geq w_{k+1}\sum_{i=1}^{k-1} W_i$, then*

$$W_{n-1}(M_{n-1,0}(\mathbf{M}_{n-1,1}) - M_{n-1,1}(\mathbf{M}_{n-1,0})) \leq W_n(M_{n,0}(\mathbf{M}_{n,1}) - M_{n,1}(\mathbf{M}_{n,0})), \tag{1.7}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

It is our goal in this paper to first give a simpler proof of the above result by modifying Holland's own approach. This is done in the next section and, in Section 3, we will prove a result of mixed-mean inequality involving the symmetric means.

2. A Proof of Theorem 1.4

First, we recast (1.7) as

$$G_n(\mathbf{A}_n) - \frac{W_{n-1}}{W_n}G_{n-1}(\mathbf{A}_{n-1}) - \frac{w_n}{W_n}G_n \geq 0. \tag{2.1}$$

We now note that

$$\begin{aligned} G_n(\mathbf{A}_n) &= (G_{n-1}(\mathbf{A}_{n-1}))^{W_{n-1}/W_n} A_n^{w_n/W_n}, \\ G_{n-1}(\mathbf{A}_{n-1}) &= A_n \prod_{i=1}^{n-1}\left(\frac{A_i}{A_{i+1}}\right)^{W_i/W_{n-1}}. \end{aligned} \tag{2.2}$$

We may assume that $x_k > 0, 1 \leq k \leq n$, and the case $x_k = 0$ for some k will follow by continuity. Thus on dividing $G_n(\mathbf{A}_n)$ on both sides of (2.1) and using (2.2), we can recast (2.1) as

$$\frac{W_{n-1}}{W_n}\prod_{i=1}^{n-1}\left(\frac{A_i}{A_{i+1}}\right)^{W_i w_n/(W_{n-1}W_n)} + \frac{w_n}{W_n}\prod_{i=1}^{n}\left(\frac{x_i}{A_i}\right)^{w_i/W_n} \leq 1. \tag{2.3}$$

We now express that $x_i = (W_i A_i - W_{i-1}A_{i-1})/w_i, 1 \leq i \leq n$, with $W_0 = A_0 = 0$ to recast (2.3) as

$$\frac{W_{n-1}}{W_n}\prod_{i=1}^{n-1}\left(\frac{A_i}{A_{i+1}}\right)^{W_i w_n/(W_{n-1}W_n)} + \frac{w_n}{W_n}\prod_{i=1}^{n}\left(\frac{W_i A_i - W_{i-1}A_{i-1}}{w_i A_i}\right)^{w_i/W_n} \leq 1. \tag{2.4}$$

We now set $y_i = A_i/A_{i+1}$, $1 \leq i \leq 2$, to further recast the above inequality as

$$\frac{W_{n-1}}{W_n}\prod_{i=1}^{n-1} y_i^{W_i w_n/(W_{n-1}W_n)} + \frac{w_n}{W_n}\prod_{i=1}^{n-1}\left(\frac{W_{i+1}}{w_{i+1}} - \frac{W_i}{w_{i+1}}y_i\right)^{w_{i+1}/W_n} \leq 1. \tag{2.5}$$

It now follows from the assumption of Theorem 1.4 that

$$c_n = 1 - \sum_{i=1}^{n-1}\frac{W_i w_n}{W_{n-1}W_n} \geq 0, \tag{2.6}$$

so that by the arithmetic-geometric mean inequality we have

$$\prod_{i=1}^{n-1} y_i^{W_i w_n/(W_{n-1}W_n)} = 1^{c_n} \prod_{i=1}^{n-1} y_i^{W_i w_n/(W_{n-1}W_n)} \leq \sum_{i=1}^{n-1} \frac{W_i w_n y_i}{W_{n-1}W_n} + 1 - \sum_{i=1}^{n-1} \frac{W_i w_n}{W_{n-1}W_n}. \tag{2.7}$$

Similarly, we have

$$\prod_{i=1}^{n-1} \left(\frac{W_{i+1}}{w_{i+1}} - \frac{W_i}{w_{i+1}} y_i \right)^{w_{i+1}/W_n} \leq \sum_{i=1}^{n-1} \frac{w_{i+1}}{W_n} \left(\frac{W_{i+1}}{w_{i+1}} - \frac{W_i}{w_{i+1}} y_i \right) + \frac{w_1}{W_n}. \tag{2.8}$$

Now it is easy to see that inequality (2.5) follows on adding inequalities (2.7) and (2.8), and this completes the proof of Theorem 1.4.

3. A Discussion on Symmetric Means

Let $0 \leq r \leq n$; we recall that the rth symmetric function $E_{n,r}$ of $\mathbf{x}$ and its mean $P_{n,r}$ are defined by

$$E_{n,r}(\mathbf{x}) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \prod_{j=1}^{r} x_{i_j}, \; P_{n,r}^r(\mathbf{x}) = \frac{E_{n,r}(\mathbf{x})}{\binom{n}{r}}, \quad 1 \leq r \leq n, E_{n,0} = P_{n,0} = 1. \tag{3.1}$$

It is well known that, for fixed $\mathbf{x}$ of dimension n, $P_{n,r}$ is a nonincreasing function of r for $1 \leq r \leq n$ with $P_{n,1} = A_n, P_{n,n} = G_n$ (with weights $w_i = 1$, $1 \leq i \leq n$). In view of the mixed-mean inequalities for the generalized weighted power means (Theorem 1.1), it is natural to ask whether similar results hold for the symmetric means. Of course one may have to adjust the notion of such mixed means in order for this to make sense for all n. For example, when $r = 3$, $n = 2$, the notion of $P_{2,3}$ is not even defined. From now on, we will only focus on the extreme cases of the symmetric means; namely, $r = 2$ or $r = n - 1$. In these cases it is then natural to define $P_{1,2} = x_1$, and, on recasting $P_{n,n-1} = G_n^{n/(n-1)}/H_n^{1/(n-1)}$, we see that it is also natural for us to define $P_{1,0} = x_1$ (note that this is not consistent with our definition of $P_{n,0}$ above).

We now prove a mixed-mean inequality involving $P_{n,2}$ and A_n. We first note the following result of Marcus and Lopes [11] (see also [12, pages 33–35]).

Theorem 3.1. *Let $0 < r \leq n$ and $x_i, y_i > 0$ for $i = 1, 2, \ldots, n$. Then*

$$P_{n,r}(\mathbf{x} + \mathbf{y}) \geq P_{n,r}(\mathbf{x}) + P_{n,r}(\mathbf{y}), \tag{3.2}$$

with equality holding if and only if $r = 1$ or there exists a constant λ such that $\mathbf{x} = \lambda \mathbf{y}$.

We also need the following lemma of C. D. Tarnavas and D. D. Tarnavas [6].

Lemma 3.2. *Let $f : R^1 \to R^1$ be a convex function and suppose that for $n \geq 2, 1 \leq k \leq n-1$, that $W_n w_k - W_k w_n > 0$. Then*

$$\frac{1}{W_{n-1}}\sum_{k=1}^{n-1} w_k f(W_{n-1}A_k) \geq \frac{1}{W_n}\sum_{k=1}^{n} w_k f(W_n A_k - w_n x_k). \tag{3.3}$$

The equality holds if and only if $n = 2$ or $x_1 = \cdots = x_n$ when $f(x)$ is strictly convex. When $f(x)$ is concave, then the above inequality is reversed.

We now apply Lemma 3.2 to obtain the following.

Lemma 3.3. *For $n \geq 2$ and $w_i = 1, 1 \leq i \leq n$,*

$$P_{n-1,2}((n-1)\mathbf{A}_{n-1}) \leq P_{n,2}(n\mathbf{A}_n - \mathbf{x}_n), \tag{3.4}$$

with equality holding in both cases if and only if $n = 2$ or $x_1 = \cdots = x_n$.

Proof. The case $n = 2$ yields an identity, so we may assume that $n \geq 3$ here. Write $a_i = (n-1)A_i, 1 \leq i \leq n-1; b_j = nA_j - x_j, 1 \leq j \leq n$. Note that $n\sum_{i=1}^{n-1} a_i = (n-1)\sum_{i=1}^{n} b_i$, and now Lemma 3.2 with $f(x) = x^2$ implies that $(n-1)\sum_{i=1}^{n} b_i^2 \leq n\sum_{i=1}^{n-1} a_i^2$. On expanding $(n\sum_{i=1}^{n-1} a_i)^2 = ((n-1)\sum_{i=1}^{n} b_i)^2$, we obtain

$$\begin{aligned} n^2\sum_{i=1}^{n-1} a_i^2 + 2n^2 \sum_{1\leq i\neq j\leq n-1} a_i a_j &= (n-1)^2\sum_{i=1}^{n} b_i^2 + 2(n-1)^2 \sum_{1\leq i\neq j\leq n} b_i b_j \\ &\leq n(n-1)\sum_{i=1}^{n-1} a_i^2 + 2(n-1)^2 \sum_{1\leq i\neq j\leq n} b_i b_j. \end{aligned} \tag{3.5}$$

Hence,

$$n\sum_{i=1}^{n-1} a_i^2 + 2n^2 \sum_{1\leq i\neq j\leq n-1} a_i a_j \leq 2(n-1)^2 \sum_{1\leq i\neq j\leq n} b_i b_j. \tag{3.6}$$

Using $M_{n,2} \geq A_n = P_{n,1} \geq P_{n,2}$, we obtain

$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i^2 \geq \frac{1}{\binom{n-1}{2}} \sum_{1\leq i\neq j\leq n-1} a_i a_j. \tag{3.7}$$

So by (3.6),

$$\frac{1}{\binom{n-1}{2}} \sum_{1\leq i\neq j\leq n-1} a_i a_j \leq \frac{1}{\binom{n}{2}} \sum_{1\leq i\neq j\leq n} b_i b_j, \tag{3.8}$$

which is just what we want. □

We now prove the following mixed-mean inequality involving the symmetric means.

Theorem 3.4. *Let $n \geq 1$ and define $\mathbf{P}_{n,2} = (P_{1,2}, \ldots, P_{n,2})$. Then*

$$(n-1)\left(P_{n-1,2}(\mathbf{P}_{n-1,1}) - P_{n-1,1}(\mathbf{P}_{n-1,2})\right) \leq n\left(P_{n,2}(\mathbf{P}_{n,1}) - P_{n,1}(\mathbf{P}_{n,2})\right), \tag{3.9}$$

with equality holding if and only if $x_1 = \cdots = x_n$. It follows that

$$P_{n,1}(\mathbf{P}_{n,2}) \leq P_{n,2}(\mathbf{P}_{n,1}), \tag{3.10}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Proof. It suffices to prove (3.9) here. We may assume that $n \geq 2$ here and we will use the idea in [6]. Lemma 3.3 implies that

$$\begin{aligned} P_{n,2} + (n-1)P_{n-1,2}(\mathbf{P}_{n-1,1}) &\leq P_{n,2} + P_{n,2}(n\mathbf{A}_n - \mathbf{x}_n) \\ &\leq P_{n,2}(n\mathbf{A}_n - \mathbf{x}_n + \mathbf{x}_n) = nP_{n,2}(\mathbf{P}_{n,1}), \end{aligned} \tag{3.11}$$

where the last inequality follows from Theorem 3.1 for the case $r = 2$. It is easy to see that the above inequality is equivalent to (3.9) and this completes the proof. □

Now we let $n \geq 1$ and define $\mathbf{P}_{n,n-1} = (P_{1,0}, \ldots, P_{n,n-1})$ with $P_{1,0} = x_1$ here. Then it is interesting to see whether the following inequality holds or not:

$$P_{n,1}(\mathbf{P}_{n,n-1}) \leq P_{n,n-1}(\mathbf{P}_{n,1}). \tag{3.12}$$

We note here that, if the above inequality holds, then it is easy to deduce from it via the approach in [2] the following Hardy-type inequality:

$$\sum_{i=1}^{n} \frac{G_i^{i/(i-1)}}{H_i^{1/(i-1)}}(\mathbf{x}_i) \leq e\sum_{i=1}^{n} x_i, \tag{3.13}$$

where we define $G_1^{1/0}/H_1^{1/0} = x_1$. We now end this paper by proving the following result.

Theorem 3.5. *Let $n \geq 1$ and $\mathbf{x} \geq \mathbf{0}$. Then*

$$\sum_{i=1}^{n} \frac{G_i^{i/(i-1)}}{H_i^{1/(i-1)}}(\mathbf{x}_i) \leq 3\sum_{i=1}^{n} x_i, \tag{3.14}$$

where one defines $G_1^{1/0}/H_1^{1/0} = x_1$.

Proof. We follow an approach of Knopp [13, 14] here (see also [15]). For $i \geq 1$, we define

$$a_i = \sum_{k=1}^{i} \frac{kx_k}{i(i+1)}. \tag{3.15}$$

It is easy to check by partial summation that

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} x_i. \tag{3.16}$$

Certainly, we have $a_1 = x_1/2 = P_{1,0}(\mathbf{x}_1)/2$ and, for $i \geq 2$, we apply the inequality $P_{i,1} \geq P_{i,i-1}$ to the numbers $x_1/(i+1), 2x_2/(i+1), \ldots, ix_i/(i+1)$ to see that

$$a_i \geq \left(\frac{(i-1)!}{(i+1)^{i-1}} \right)^{1/(i-1)} P_{i,i-1}(\mathbf{x}_i) := \gamma_i P_{i,i-1}(\mathbf{x}_i) \ . \tag{3.17}$$

We now show by induction that $\gamma_i \geq 1/3$ for $i \geq 2$; equivalently, this is

$$3^{i-1}(i-1)! \geq (i+1)^{i-1}. \tag{3.18}$$

Note first that the above inequality holds when $i = 2, 3$ and suppose now that it holds for some $i = k \geq 3$. Then by induction,

$$3^k k! \geq 3k(k+1)^{k-1}. \tag{3.19}$$

Now using $(1 + 1/n)^n < e$, we have

$$\frac{3k(k+1)^{k-1}}{(k+2)^k} = \frac{3k(k+2)}{(k+1)^2} \left(\frac{k+1}{k+2} \right)^{k+1} \geq \frac{3k(k+2)}{e(k+1)^2}. \tag{3.20}$$

It is easy to see that the last expression above is no less than 1 when $k \geq 3$ and this proves inequality (3.18) for the case $i = k + 1$. This completes the proof of the theorem. □

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