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Research Article

Fejér-Type Inequalities (I)

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We establish some new Fejér-type inequalities for convex functions.

1. Introduction

Throughout this paper, let $f : [a,b] \to \mathbb{R}$ be convex, and let $g : [a,b] \to [0,\infty)$ be integrable and symmetric to $(a+b)/2$. We define the following functions on $[0,1]$ that are associated with the well-known Hermite-Hadamard inequality [1]

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

namely

$$I(t) = \int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right)\right]g(x)dx,$$

$$J(t) = \int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{3a+b}{4}\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{a+3b}{4}\right)\right]g(x)dx,$$

$$
\begin{aligned}
M(t) &= \int_a^{(a+b)/2} \frac{1}{2}\left[f\left(ta+(1-t)\frac{x+a}{2}\right)+f\left(t\frac{a+b}{2}+(1-t)\frac{x+b}{2}\right)\right]g(x)dx \\
&\quad+\int_{(a+b)/2}^{b} \frac{1}{2}\left[f\left(t\frac{a+b}{2}+(1-t)\frac{x+a}{2}\right)+f\left(tb+(1-t)\frac{x+b}{2}\right)\right]g(x)dx, \\
N(t) &= \int_a^b \frac{1}{2}\left[f\left(ta+(1-t)\frac{x+a}{2}\right)+f\left(tb+(1-t)\frac{x+b}{2}\right)\right]g(x)dx.
\end{aligned}
\tag{1.2}
$$

For some results which generalize, improve, and extend the famous integral inequality (1.1), see [2–6].

In [2], Dragomir established the following theorem which is a refinement of the first inequality of (1.1).

Theorem A. *Let f be defined as above, and let H be defined on $[0,1]$ by*

$$
H(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx. \tag{1.3}
$$

Then, H is convex, increasing on $[0,1]$, and for all $t\in[0,1]$, one has

$$
f\left(\frac{a+b}{2}\right)=H(0)\le H(t)\le H(1)=\frac{1}{b-a}\int_a^b f(x)dx. \tag{1.4}
$$

In [6], Yang and Hong established the following theorem which is a refinement of the second inequality in (1.1).

Theorem B. *Let f be defined as above, and let P be defined on $[0,1]$ by*

$$
P(t)=\frac{1}{2(b-a)}\int_a^b\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)\right]dx. \tag{1.5}
$$

Then, P is convex, increasing on $[0,1]$, and for all $t\in[0,1]$, one has

$$
\frac{1}{b-a}\int_a^b f(x)dx=P(0)\le P(t)\le P(1)=\frac{f(a)+f(b)}{2}. \tag{1.6}
$$

In [3], Fejér established the following weighted generalization of the Hermite-Hadamard inequality (1.1).

Theorem C. *Let f, g be defined as above. Then,*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.7}$$

is known as Fejér inequality.

In this paper, we establish some Fejér-type inequalities related to the functions I, J, M, N introduced above.

2. Main Results

In order to prove our main results, we need the following lemma.

Lemma 2.1 (see [4]). *Let f be defined as above, and let $a \le A \le C \le D \le B \le b$ with $A+B = C+D$. Then,*

$$f(C) + f(D) \le f(A) + f(B). \tag{2.1}$$

Now, we are ready to state and prove our results.

Theorem 2.2. *Let f, g, and I be defined as above. Then I is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, one has the following Fejér-type inequality:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx = I(0) \le I(t) \le I(1) = \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right]g(x)dx. \tag{2.2}$$

Proof. It is easily observed from the convexity of f that I is convex on $[0,1]$. Using simple integration techniques and under the hypothesis of g, the following identity holds on $[0,1]$:

$$\begin{aligned}
I(t) &= \int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2} + (1-t)\frac{a+b}{2}\right)g(x) + f\left(t\frac{a+2b-x}{2} + (1-t)\frac{a+b}{2}\right)g(a+b-x)\right]dx \\
&= \int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2} + (1-t)\frac{a+b}{2}\right) + f\left(t\frac{a+2b-x}{2} + (1-t)\frac{a+b}{2}\right)\right]g(x)dx \\
&= \int_a^{(a+b)/2}\left[f\left(tx + (1-t)\frac{a+b}{2}\right) + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)\right]g(2x-a)dx.
\end{aligned} \tag{2.3}$$

Let $t_1 < t_2$ in $[0,1]$. By Lemma 2.1, the following inequality holds for all $x \in [a,(a+b)/2]$:

$$\begin{aligned} &f\left(t_1x+(1-t_1)\frac{a+b}{2}\right)+f\left(t_1(a+b-x)+(1-t_1)\frac{a+b}{2}\right)\\ &\quad\le f\left(t_2x+(1-t_2)\frac{a+b}{2}\right)+f\left(t_2(a+b-x)+(1-t_2)\frac{a+b}{2}\right). \end{aligned} \tag{2.4}$$

Indeed, it holds when we make the choice

$$\begin{aligned} A&=t_2x+(1-t_2)\frac{a+b}{2},\\ C&=t_1x+(1-t_1)\frac{a+b}{2},\\ D&=t_1(a+b-x)+(1-t_1)\frac{a+b}{2},\\ B&=t_2(a+b-x)+(1-t_2)\frac{a+b}{2}, \end{aligned} \tag{2.5}$$

in Lemma 2.1.

Multipling the inequality (2.4) by $g(2x-a)$, integrating both sides over x on $[a,(a+b)/2]$ and using identity (2.3), we derive $I(t_1) \le I(t_2)$. Thus I is increasing on $[0,1]$ and then the inequality (2.2) holds. This completes the proof. □

Remark 2.3. Let $g(x) = 1/(b-a)$ $(x \in [a,b])$ in Theorem 2.2. Then $I(t) = H(t)$ $(t \in [0,1])$ and the inequality (2.2) reduces to the inequality (1.4), where H is defined as in Theorem A.

Theorem 2.4. *Let f, g, J be defined as above. Then J is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, one has the following Fejér-type inequality:*

$$\begin{aligned} \frac{f((3a+b)/4)+f((a+3b)/4)}{2}\int_a^b g(x)dx &= J(0)\le J(t)\le J(1)\\ &=\frac{1}{2}\int_a^b\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)dx. \end{aligned} \tag{2.6}$$

Proof. By using a similar method to that from Theorem 2.2, we can show that J is convex on $[0,1]$, the identity

$$\begin{aligned} J(t)=\int_a^{(3a+b)/4}&\left[f\left(tx+(1-t)\frac{3a+b}{4}\right)+f\left(t\left(\frac{3a+b}{2}-x\right)+(1-t)\frac{3a+b}{4}\right)\right.\\ &\left.+f\left(t\left(x+\frac{b-a}{2}\right)+(1-t)\frac{a+3b}{4}\right)+f\left(t(a+b-x)+(1-t)\frac{a+3b}{4}\right)\right]\\ &\times g(2x-a)dx \end{aligned} \tag{2.7}$$

holds on $[0,1]$, and the inequalities

$$\begin{aligned}
&f\left(t_1x+(1-t_1)\frac{3a+b}{4}\right)+f\left(t_1\left(\frac{3a+b}{2}-x\right)+(1-t_1)\frac{3a+b}{4}\right)\\
&\quad\le f\left(t_2x+(1-t_2)\frac{3a+b}{4}\right)+f\left(t_2\left(\frac{3a+b}{2}-x\right)+(1-t_2)\frac{3a+b}{4}\right),
\end{aligned}\tag{2.8}$$

$$\begin{aligned}
&f\left(t_1\left(x+\frac{b-a}{2}\right)+(1-t_1)\frac{a+3b}{4}\right)+f\left(t_1(a+b-x)+(1-t_1)\frac{a+3b}{4}\right)\\
&\quad\le f\left(t_2\left(x+\frac{b-a}{2}\right)+(1-t_2)\frac{a+3b}{4}\right)+f\left(t_2(a+b-x)+(1-t_2)\frac{a+3b}{4}\right)
\end{aligned}\tag{2.9}$$

hold for all $t_1 < t_2$ in $[0,1]$ and $x \in [a,(3a+b)/4]$.

By (2.7)–(2.9) and using a similar method to that from Theorem 2.2, we can show that J is increasing on $[0,1]$ and (2.6) holds. This completes the proof. □

The following result provides a comparison between the functions I and J.

Theorem 2.5. *Let f, g, I, and J be defined as above. Then $I(t) \le J(t)$ on $[0,1]$.*

Proof. By the identity

$$J(t)=\int_a^{(a+b)/2}\left[f\left(tx+(1-t)\frac{3a+b}{4}\right)+f\left(t(a+b-x)+(1-t)\frac{a+3b}{4}\right)\right]g(2x-a)dx,\tag{2.10}$$

on $[0,1]$, (2.3) and using a similar method to that from Theorem 2.2, we can show that $I(t) \le J(t)$ on $[0,1]$. The details are omited. □

Further, the following result incorporates the properties of the function M.

Theorem 2.6. *Let f, g, M be defined as above. Then M is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, one has the following Fejér-type inequality:*

$$\begin{aligned}
&\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)dx\\
&\quad= M(0)\le M(t)\le M(1)=\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)dx.
\end{aligned}\tag{2.11}$$

Proof. Follows by the identity

$$M(t) = \int_a^{(3a+b)/4} \left[f(ta + (1-t)x) + f\left(t\frac{a+b}{2} + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \right.$$
$$\left. + f\left(t\frac{a+b}{2} + (1-t)\left(x + \frac{b-a}{2}\right)\right) + f(tb + (1-t)(a+b-x)) \right] \tag{2.12}$$
$$\times g(2x-a)dx,$$

on $[0,1]$. The details are left to the interested reader. □

We now present a result concerning the properties of the function N.

Theorem 2.7. *Let f, g, N be defined as above. Then N is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, one has the following Fejér-type inequality:*

$$\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right] g(x)dx = N(0) \leq N(t) \leq N(1) = \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{2.13}$$

Proof. By the identity

$$N(t) = \int_a^{(a+b)/2} \left[f(ta + (1-t)x) + f(tb + (1-t)(a+b-x))\right] g(2x-a)dx \tag{2.14}$$

on $[0,1]$ and using a similar method to that for Theorem 2.2, we can show that N is convex, increasing on $[0,1]$ and (2.13) holds. □

Remark 2.8. Let $g(x) = 1/(b-a)$ $(x \in [a,b])$ in Theorem 2.7. Then $N(t) = P(t)$ $(t \in [0,1])$ and the inequality (2.13) reduces to (1.6), where P is defined as in Theorem B.

Theorem 2.9. *Let f, g, M, and N be defined as above. Then $M(t) \leq N(t)$ on $[0,1]$.*

Proof. By the identity

$$N(t) = \int_a^{(3a+b)/4} \left[f(ta + (1-t)x) + f\left(ta + (1-t)\left(\frac{3a+b}{2} - x\right)\right) \right.$$
$$\left. + f(tb + (1-t)(a+b-x)) + f\left(tb + (1-t)\left(x + \frac{b-a}{2}\right)\right)\right] g(2x-a)dx, \tag{2.15}$$

on $[0,1]$, (2.12) and using a similar method to that for Theorem 2.2, we can show that $M(t) \leq N(t)$ on $[0,1]$. This completes the proof. □

The following Fejér-type inequality is a natural consequence of Theorems 2.2–2.9.

Corollary 2.10. *Let f, g be defined as above. Then one has*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx &\le \frac{f((3a+b)/4)+f((a+3b)/4)}{2}\int_a^b g(x)dx \\
&\le \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)dx \\
&\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)dx \\
&\le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx.
\end{aligned} \tag{2.16}$$

Remark 2.11. Let $g(x) = 1/(b-a)$ $(x \in [a,b])$ in Corollary 2.10. Then the inequality (2.16) reduces to

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{f((3a+b)/4)+f((a+3b)/4)}{2} \le \frac{1}{b-a}\int_a^b f(x)dx \\
&\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \le \frac{f(a)+f(b)}{2},
\end{aligned} \tag{2.17}$$

which is a refinement of (1.1).

Remark 2.12. In Corollary 2.10, the third inequality in (2.16) is the weighted generalization of Bullen's inequality [5]

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]. \tag{2.18}$$

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