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Research Article

A New Method to Study Analytic Inequalities

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We present a new method to study analytic inequalities involving n variables. Regarding its applications, we proved some well-known inequalities and improved Carleman's inequality.

1. Monotonicity Theorems

Throughout this paper, we denote $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+$ the set of strictly positive real numbers, $n \in \mathbb{N}$, $n \geq 2$.

In this section, we present the main results of this paper.

Theorem 1.1. *Suppose that $a, b \in \mathbb{R}$ with $a < b$ and $c \in [a, b]$, $f : [a, b]^n \to \mathbb{R}$ has continuous partial derivatives and*

$$D_m = \left\{ (x_1, x_2, \ldots, x_{n-1}, c) \mid \min_{1 \leq k \leq n-1} \{x_k\} \geq c,\ x_m = \max_{1 \leq k \leq n-1} \{x_k\} \neq c \right\}, \quad m = 1, 2, \ldots, n-1. \tag{1.1}$$

If $\partial f(\mathbf{x})/\partial x_m > 0$ for all $\mathbf{x} \in D_m$ $(m = 1, 2, \ldots, n-1)$, then

$$f(y_1, y_2, \ldots, y_{n-1}, c) \geq f(c, c, \ldots c, c), \tag{1.2}$$

for all $y_m \in [c, b]$ $(m = 1, 2, \ldots, n-1)$.

Proof. Without loss of generality, since we assume that $n = 3$ and $y_1 > y_2 > c$.

For $x_1 \in [y_2, y_1]$, we clearly see that $(x_1, y_2, c) \in D_1$, then

$$\left.\frac{\partial f(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=(x_1,y_2,c)} > 0. \tag{1.3}$$

From the continuity of the partial derivatives of f and

$$\left.\frac{\partial f(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=(y_2,y_2,c)} > 0, \tag{1.4}$$

we know that there exists $\varepsilon > 0$ such that $y_2 - \varepsilon \geq c$ and

$$\left.\frac{\partial f(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=(x_1,y_2,c)} > 0, \tag{1.5}$$

for any $x_1 \in [y_2 - \varepsilon, y_2]$. Hence, since $f(\cdot, y_2, c) : x_1 \in [y_2 - \varepsilon, y_1] \to f(x_1, y_2, c)$ is strictly monotone increasing, then we have

$$f(y_1, y_2, c) > f(y_2, y_2, c) > f(y_2 - \varepsilon, y_2, c). \tag{1.6}$$

Next, for $x_2 \in [y_2 - \varepsilon, y_2]$, then $(y_2 - \varepsilon, x_2, c) \in D_2$ and

$$\left.\frac{\partial f(\mathbf{x})}{\partial x_2}\right|_{\mathbf{x}=(y_2-\varepsilon,x_2,c)} > 0. \tag{1.7}$$

Hence, we get

$$f(y_1, y_2, c) > f(y_2, y_2, c) > f(y_2 - \varepsilon, y_2, c) > f(y_2 - \varepsilon, y_2 - \varepsilon, c). \tag{1.8}$$

If $y_2 - \varepsilon = c$, then Theorem 1.1 is true. Otherwise, we repeat the above process and we clearly see that the first and second variables in f are decreasing and no less than c. Let s, t be their limit values, respectively, then $f(y_1, y_2, c) > f(s, t, c)$ and $s, t \geq c$. If $s = c, t = c$, then Theorem 1.1 is also true; otherwise, we repeat the above process again and denote p and q the greatest lower bounds for the first and the second variables , respectively. We clearly see that $p = q = c$; therefore, $f(y_1, y_2, c) > f(c, c, c)$ and Theorem 1.1 is true. □

Similarly, we have the following theorem.

Theorem 1.2. *Suppose that $a, b \in \mathbb{R}$ with $a < b$ and $c \in [a, b]$, $f : [a, b]^n \to \mathbb{R}$ has continuous partial derivatives and*

$$E_m = \left\{ (x_1, x_2, \ldots, x_{n-1}, c) \mid \max_{1\le k\le n-1} \{x_k\} \le c,\ x_m = \min_{1\le k\le n-1} \{x_k\} \ne c \right\}, \quad m = 1, 2, \ldots, n{-}1. \tag{1.9}$$

If $\partial f(\mathbf{x})/\partial x_m < 0$ for all $\mathbf{x} \in E_m$ $(m = 1, 2, \ldots, n-1)$, then

$$f(y_1, y_2, \ldots, y_{n-1}, c) \ge f(c, c, \ldots c, c), \tag{1.10}$$

for all $y_m \in [a, c]$ $(m = 1, 2, \ldots, n-1)$.

It follows from Theorems 1.1 and 1.2 that we get the following Corollaries 1.3–1.6.

Corollary 1.3. *Suppose that $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b]^n \to \mathbb{R}$ has continuous partial derivatives and*

$$D_m = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_n) \mid a \le \min_{1\le k\le n} \{x_k\} < x_m = \max_{1\le k\le n} \{x_k\} \le b \right\}, \quad m = 1, 2, \ldots, n. \tag{1.11}$$

If $\partial f(\mathbf{x})/\partial x_m > 0$ for all $\mathbf{x} \in D_m$ and $m = 1, 2, \ldots, n$, then

$$f(x_1, x_2, \ldots, x_n) \ge f(x_{\min}, x_{\min}, \ldots, x_{\min}), \tag{1.12}$$

for all $x_m \in [a, b]$ $(m = 1, 2, \ldots, n)$ with $x_{\min} = \min_{1\le k\le n}\{x_k\}$.

Corollary 1.4. *Suppose $a, b \in \mathbb{R}$ with $a < b$, then*

$$D_1 = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_n) \mid a \le \min_{1\le k\le n} \{x_k\} < x_1 = \max_{1\le k\le n} \{x_k\} \le b \right\}, \tag{1.13}$$

and $f : [a, b]^n \to \mathbb{R}$ is symmetric with continuous partial derivatives. If $\partial f(\mathbf{x})/\partial x_1 > 0$ for all $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in D_1$, then

$$f(x_1, x_2, \ldots, x_n) \ge f(x_{\min}, x_{\min}, \ldots, x_{\min}), \tag{1.14}$$

where $x_{\min} = \min_{1\le k\le n}\{x_k\}$. Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Corollary 1.5. *Suppose $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b]^n \to \mathbb{R}$ has continuous partial derivatives and*

$$E_m = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_n) \mid a \le x_m = \min_{1\le k\le n} \{x_k\} < \max_{1\le k\le n} \{x_k\} \le b \right\}. \tag{1.15}$$

If $\partial f(\mathbf{x})/\partial x_m < 0$ *for all* $\mathbf{x} \in E_m$ *and* $m = 1, 2, \ldots, n$*, then*

$$f(x_1, x_2, \ldots, x_n) \geq f(x_{\max}, x_{\max}, \ldots, x_{\max}), \tag{1.16}$$

where $x_{\max} = \max_{1\leq k\leq n}\{x_k\}$*. Equality holds if and only if* $x_1 = x_2 = \cdots = x_n$*.*

Corollary 1.6. *Suppose* $a, b \in \mathbb{R}$ *with* $a < b$*, then*

$$E_n = \left\{\mathbf{x} = (x_1, x_2, \ldots, x_n) \mid a \leq x_n = \min_{1\leq k\leq n}\{x_k\} < \max_{1\leq k\leq n}\{x_k\} \leq b\right\}, \tag{1.17}$$

and $f : [a,b]^n \to \mathbb{R}$ *is symmetric with continuous partial derivatives . If* $\partial f(\mathbf{x})/\partial x_n < 0$ *for all* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in E_n$*, then*

$$f(x_1, x_2, \ldots, x_n) \geq f(x_{\max}, x_{\max}, \ldots, x_{\max}), \tag{1.18}$$

where $x_{\max} = \max_{1\leq k\leq n}\{x_k\}$*. Equality holds if and only if* $x_1 = x_2 = \cdots = x_n$*.*

2. Unifying Proof of Some Well-Known Inequality

In this section, we denote $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $a_{\min} = \min_{1\leq k\leq n}\{a_k\}$, $a_{\max} = \max_{1\leq k\leq n}\{a_k\}$, and

$$D_m = \{\mathbf{a} \mid a_m = a_{\max} > a_{\min} > 0\}, \quad m = 1, 2, \ldots, n. \tag{2.1}$$

Proposition 2.1 (Power Mean Inequality). *If the power mean* $M_r(\mathbf{a})$ *of order* r *is defined by* $M_r(\mathbf{a}) = ((1/n)\sum_{i=1}^n a_i^r)^{1/r}$ *for* $r \neq 0$ *and* $M_0(\mathbf{a}) = \prod_{i=1}^n a_i^{1/n}$*, then* $M_r(\mathbf{a}) \geq M_s(\mathbf{a})$ *for* $r > s$*; equality holds if and only if* $a_1 = a_2 = \cdots = a_n$*.*

Proof. It is well known that $M_r(\mathbf{a})$ is symmetric with respect to $a_1, a_2, \ldots, a_n$ and $r \mapsto M_r(\mathbf{a})$ is continuous. Without loss of generality, we assume that $r, s \neq 0$. Then

$$\begin{aligned} f(\mathbf{a}) &= \frac{1}{r}\ln\left(\frac{\sum_{i=1}^n a_i^r}{n}\right) - \frac{1}{s}\ln\left(\frac{\sum_{i=1}^n a_i^s}{n}\right), \quad \mathbf{a} \in \mathbb{R}_+^n, \\ \frac{\partial f(\mathbf{a})}{\partial a_1} &= \frac{a_1^{r-1}}{\sum_{i=1}^n a_i^r} - \frac{a_1^{s-1}}{\sum_{i=1}^n a_i^s} \\ &= \frac{\sum_{i=2}^n \left(a_1^{r-1}a_i^s - a_1^{s-1}a_i^r\right)}{\sum_{i=1}^n a_i^r \cdot \sum_{i=1}^n a_i^s} = \frac{\sum_{i=2}^n a_1^{s-1}a_i^r\left[(a_1/a_i)^{r-s} - 1\right]}{\sum_{i=1}^n a_i^r \cdot \sum_{i=1}^n a_i^s}. \end{aligned} \tag{2.2}$$

If $\mathbf{a} \in D_1$, then $\partial f(\mathbf{a})/\partial a_1 > 0$. It follows from Corollary 1.4 that we get

$$
\begin{gathered}
f(a_1, a_2, \ldots, a_n) \geq f(a_{\min}, a_{\min}, \ldots, a_{\min}), \\
\left(\frac{\sum_{i=1}^{n} a_i^r}{n}\right)^{1/r} \geq \left(\frac{\sum_{i=1}^{n} a_i^s}{n}\right)^{1/s}, \quad M_r(\mathbf{a}) \geq M_s(\mathbf{a}).
\end{gathered} \tag{2.3}
$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$. □

Proposition 2.2 (Holder Inequality). *Suppose that* $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n$ $(p, q > 1)$. *If* $1/p + 1/q = 1$, *then*

$$
\left(\sum_{k=1}^{n} x_k^p\right)^{1/p} \left(\sum_{k=1}^{n} y_k^q\right)^{1/q} \geq \sum_{k=1}^{n} x_k y_k. \tag{2.4}
$$

Proof. Let $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}_+^n$ and

$$
f : \mathbf{a} \in \mathbb{R}_+^n \longrightarrow \left(\sum_{k=1}^{n} b_k\right)^{1/p} \left(\sum_{k=1}^{n} b_k a_k\right)^{1/q} - \sum_{k=1}^{n} b_k a_k^{1/q}, \quad \mathbf{a} \in \mathbb{R}_+^n. \tag{2.5}
$$

If $\mathbf{a} \in D_1$, then

$$
\begin{aligned}
\frac{\partial f(\mathbf{a})}{\partial a_1} &= \frac{1}{q} b_1 \left(\sum_{k=1}^{n} b_k\right)^{1/p} \left(\sum_{k=1}^{n} b_k a_k\right)^{1/q-1} - \frac{1}{q} b_1 a_1^{1/q-1} \\
&= \frac{1}{q} b_1 a_1^{-1/p} \left(\sum_{k=1}^{n} b_k a_k\right)^{-1/p} \left[\left(\sum_{k=1}^{n} b_k\right)^{1/p} a_1^{1/p} - \left(\sum_{k=1}^{n} b_k a_k\right)^{1/p}\right] \\
&> \frac{1}{q} b_1 a_1^{-1/p} \left(\sum_{k=1}^{n} b_k a_k\right)^{-1/p} \left[\left(\sum_{k=1}^{n} b_k\right)^{1/p} a_1^{1/p} - \left(\sum_{k=1}^{n} b_k a_1\right)^{1/p}\right] \\
&= 0.
\end{aligned} \tag{2.6}
$$

Similarly, if $\mathbf{a} \in D_m$ $(m = 2, 3, \ldots, n)$, then $\partial f(\mathbf{a})/\partial a_m > 0$. From Theorem 1.1, we get

$$
\begin{gathered}
f(a_1, a_2, \ldots, a_n) \geq f(a_{\min}, a_{\min}, \ldots, a_{\min}), \\
\left(\sum_{k=1}^{n} b_k\right)^{1/p} \left(\sum_{k=1}^{n} b_k a_k\right)^{1/q} \geq \sum_{k=1}^{n} b_k a_k^{1/q}.
\end{gathered} \tag{2.7}
$$

Therefore, Proposition 2.2 follows from $a_k = y_k^q / x_k^p$ and $b_k = x_k^p$. □

Proposition 2.3 (Minkowski Inequality). *Suppose that* $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n$. *If* $p > 1$, *then*

$$\left(\sum_{k=1}^{n} x_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} y_k^p\right)^{1/p} \geq \left(\sum_{k=1}^{n} (x_k + y_k)^p\right)^{1/p}. \tag{2.8}$$

Proof. Let $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}_+^n$ and

$$f : \mathbf{a} \in \mathbb{R}_+^n \longrightarrow \left(\sum_{k=1}^{n} b_k a_k\right)^{1/p} - \left(\sum_{k=1}^{n} b_k \left(a_k^{1/p} + 1\right)^p\right)^{1/p}, \quad \mathbf{a} \in \mathbb{R}_+^n. \tag{2.9}$$

If $\mathbf{a} \in D_1$, then

$$\begin{aligned}
\frac{\partial f(\mathbf{a})}{\partial a_1} &= \frac{1}{p} b_1 \left(\sum_{k=1}^{n} b_k a_k\right)^{1/p-1} - \frac{1}{p} b_1 a_1^{1/p-1} \left(a_1^{1/p} + 1\right)^{p-1} \left(\sum_{k=1}^{n} b_k \left(a_k^{1/p} + 1\right)^p\right)^{1/p-1} \\
&= \frac{1}{p} b_1 \left(\sum_{k=1}^{n} b_k a_k\right)^{1/p-1} \left(\sum_{k=1}^{n} b_k \left(a_k^{1/p} + 1\right)^p\right)^{1/p-1} \\
&\quad \cdot \left[\left(\sum_{k=1}^{n} b_k \left(a_k^{1/p} + 1\right)^p\right)^{1-1/p} - \left(1 + a_1^{-1/p}\right)^{p-1} \left(\sum_{k=1}^{n} b_k a_k\right)^{1-1/p}\right] \\
&= \frac{1}{p} b_1 \left(\sum_{k=1}^{n} b_k a_k\right)^{1/p-1} \left(\sum_{k=1}^{n} b_k \left(a_k^{1/p} + 1\right)^p\right)^{1/p-1} \\
&\quad \cdot \left[\left(\sum_{k=1}^{n} b_k \left(a_k^{1/p} + 1\right)^p\right)^{1-1/p} - \left(\sum_{k=1}^{n} b_k \left(a_k^{1/p} + a_k^{1/p} a_1^{-1/p}\right)^p\right)^{1-1/p}\right] \\
&> \frac{1}{p} b_1 \left(\sum_{k=1}^{n} b_k a_k\right)^{1/p-1} \left(\sum_{k=1}^{n} b_k \left(a_k^{1/p} + 1\right)^p\right)^{1/p-1} \\
&\quad \cdot \left[\left(\sum_{k=1}^{n} b_k \left(a_k^{1/p} + 1\right)^p\right)^{1-1/p} - \left(\sum_{k=1}^{n} b_k \left(a_k^{1/p} + a_1^{1/p} a_1^{-1/p}\right)^p\right)^{1-1/p}\right] \\
&= 0.
\end{aligned} \tag{2.10}$$

Similarly, If $\mathbf{a} \in D_m$ $(m = 2, 3, \ldots, n)$, then $\partial f(\mathbf{a}) / \partial a_m > 0$. It follows from Theorem 1.1 that we get

$$\begin{gathered}
f(a_1, a_2, \ldots, a_n) \geq f(a_{\min}, a_{\min}, \ldots, a_{\min}), \\
\left(\sum_{k=1}^{n} b_k a_k\right)^{1/p} \geq \left(\sum_{k=1}^{n} b_k \left(a_k^{1/p} + 1\right)^p\right)^{1/p} - \left(\sum_{k=1}^{n} b_k\right)^{1/p}.
\end{gathered} \tag{2.11}$$

Therefore, Proposition 2.3 follows from $a_k = y_k^p / x_k^p$ and $b_k = x_k^p$. □

3. A Brief Proof for Hardy's Inequality

If $a_n \geq 0$ $(n \in \mathbb{N},\ n \geq 1)$ with $\sum_{n=1}^{\infty} a_n^p < +\infty$, then the well-known Hardy's inequality (see [1, Theorem 326]) is

$$\left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p \geq \sum_{n=1}^{\infty} \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p. \tag{3.1}$$

In this section, we establish the following result involving Hardy's inequality.

Theorem 3.1. *Let $n \in \mathbb{N}$, $n \geq 1$, and $a_k \geq 0$ $(k \in \mathbb{N}, k \geq 1)$. If*

$$B_n = \min_{1\leq k\leq n}\left\{\left(k - \frac{1}{2}\right)^{1/p} a_k\right\}, \tag{3.2}$$

then

$$\begin{aligned} &\left(\frac{p}{p-1}\right)^p \sum_{k=1}^{n} a_k^p - \sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k} a_j\right)^p \\ &\quad \geq B_n\left[\left(\frac{p}{p-1}\right)^p \sum_{k=1}^{n}\frac{1}{k-1/2} - \sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}\frac{1}{(j-1/2)^{1/p}}\right)^p\right]. \end{aligned} \tag{3.3}$$

Proof. Let $b_k = (k - 1/2)^{1/p} a_k$, then inequality (3.3) is equivalent to

$$\begin{aligned} &\left(\frac{p}{p-1}\right)^p \sum_{k=1}^{n} \frac{b_k^p}{k-1/2} - \sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k} \frac{b_j}{(j-1/2)^{1/p}}\right)^p \\ &\quad \geq B_n\left[\left(\frac{p}{p-1}\right)^p \sum_{k=1}^{n}\frac{1}{k-1/2} - \sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}\frac{1}{(j-1/2)^{1/p}}\right)^p\right], \end{aligned} \tag{3.4}$$

and $B_n = \min_{1\leq k\leq n}\{b_k\}$. Let

$$\begin{aligned} &D_m = \left\{\mathbf{b} \mid b_m = \max_{1\leq k\leq n}\{b_k\} > \min_{1\leq k\leq n}\{b_k\} > 0\right\}, \quad m = 1, 2, \ldots, n, \\ &f : \mathbf{b} \in [0, +\infty)^n \longrightarrow \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{n} \frac{b_k^p}{k-1/2} - \sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k} \frac{b_j}{(j-1/2)^{1/p}}\right)^p. \end{aligned} \tag{3.5}$$

If $\mathbf{b} \in D_m$ $(m = 1, 2, \ldots, n)$, then

$$\begin{aligned}
\frac{\partial f(\mathbf{b})}{\partial b_m} &= p\left(\frac{p}{p-1}\right)^p \frac{b_m^{p-1}}{m-1/2} - \sum_{k=m}^{n}\left[\frac{p}{k^p(m-1/2)^{1/p}}\left(\sum_{j=1}^{k}\frac{b_j}{(j-1/2)^{1/p}}\right)^{p-1}\right] \\
&> \frac{pb_m^{p-1}}{(m-1/2)^{1/p}} \cdot \left[\left(\frac{p}{p-1}\right)^p \frac{1}{(m-1/2)^{1-1/p}} - \sum_{k=m}^{\infty}\frac{1}{k^p}\left(\sum_{j=1}^{k}\frac{1}{(j-1/2)^{1/p}}\right)^{p-1}\right].
\end{aligned} \tag{3.6}$$

Making use of the well-known Hadamard's inequality of convex functions, we get

$$\begin{aligned}
\frac{\partial f(\mathbf{b})}{\partial b_m} &> \frac{pb_m^{p-1}}{(m-1/2)^{1/p}}\left[\left(\frac{p}{p-1}\right)^p \frac{1}{(m-1/2)^{1-1/p}} - \sum_{k=m}^{\infty}\frac{1}{k^p}\left(\int_{1/2}^{k+1/2}\frac{1}{(x-1/2)^{1/p}}dx\right)^{p-1}\right] \\
&= \frac{pb_m^{p-1}}{(m-1/2)^{1/p}}\left[\left(\frac{p}{p-1}\right)^p \frac{1}{(m-1/2)^{1-1/p}} - \left(\frac{p}{p-1}\right)^{p-1}\sum_{k=m}^{\infty}k^{(-2p+1)/p}\right] \\
&> \frac{pb_m^{p-1}}{(m-1/2)^{1/p}}\left[\left(\frac{p}{p-1}\right)^p \frac{1}{(m-1/2)^{1-1/p}} - \left(\frac{p}{p-1}\right)^{p-1}\int_{m-1/2}^{+\infty}x^{(-2p+1)/p}dx\right] \\
&= 0.
\end{aligned} \tag{3.7}$$

Then Theorem 1.1 leads to

$$f(b_1, b_2, \ldots, b_n) \ge f(B_n, B_n, \ldots, B_n), \tag{3.8}$$

and we clearly see that inequalities (3.4) and (3.3) are true. □

Corollary 3.2. *Let $n \in \mathbb{N}$, $n \ge 1$, and $a_k \ge 0$ $(k \in \mathbb{N},\ k \ge 1)$. If*

$$B_n = \min_{1 \le k \le n}\left\{\left(k - \frac{1}{2}\right)^{1/p} a_k\right\}, \tag{3.9}$$

then

$$\left(\frac{p}{p-1}\right)^p \sum_{k=1}^{n} a_k^p - \sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}a_j\right)^p > B_n\left(\frac{p}{p-1}\right)^p \sum_{k=1}^{n}\frac{1}{k(2k-1)} \ge B_n\left(\frac{p}{p-1}\right)^p. \tag{3.10}$$

Proof. From inequality (3.3), we clearly see that

$$\begin{aligned}
&\left(\frac{p}{p-1}\right)^p\sum_{k=1}^{n}\frac{b_k^p}{k-1/2}-\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k}\frac{b_j}{(j-1/2)^{1/p}}\right)^p\\
&\quad> B_n\left[\left(\frac{p}{p-1}\right)^p\sum_{k=1}^{n}\frac{1}{k-1/2}-\sum_{k=1}^{n}\left(\frac{1}{k}\int_{1/2}^{k+1/2}\frac{1}{(x-1/2)^{1/p}}dx\right)^p\right]\\
&\quad= B_n\left(\frac{p}{p-1}\right)^p\sum_{k=1}^{n}\left(\frac{1}{k-1/2}-\frac{1}{k}\right)\\
&\quad= B_n\left(\frac{p}{p-1}\right)^p\sum_{k=1}^{n}\frac{1}{k(2k-1)}\\
&\quad\geq B_n\left(\frac{p}{p-1}\right)^p.
\end{aligned} \tag{3.11}$$

□

Remark 3.3. If $n \to +\infty$, then inequality (3.1) follows from inequality (3.10).

4. A Refinement of Carleman's Inequality

If $a_n \geq 0$ $(n \in \mathbb{N},\ n \geq 1)$ with $0 < \sum_{n=1}^{\infty} a_n < \infty$, then the well-known Carleman's inequality is

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k\right)^{1/n} < e\sum_{n=1}^{\infty}a_n, \tag{4.1}$$

with the best possible constant factor e (see [2]).

Recently, Yang and Debnath [3] gave a strengthened version of (4.1) as follows:

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}a_k\right)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\frac{1}{2n+2}\right)a_n. \tag{4.2}$$

Some other strengthened versions of (4.1) were given in [4–9]. In this section, we give a refinement for Carleman's inequality (see Corollary 4.4).

Lemma 4.1. *If $m \in \mathbb{N}$ and $m \geq 1$, then*

$$e\left(1-\frac{2}{3m+7}\right)\frac{1}{m} > \sum_{k=m}^{\infty}\frac{1}{k(k!)^{1/k}}, \tag{4.3}$$

$$e\left(1-\frac{2}{3m+10}\right)\frac{1}{m+1} > \frac{1}{((m+1)!)^{1/(m+1)}}. \tag{4.4}$$

Proof. Let $\psi(m) = e(1-2/(3m+7))(1/m)-\sum_{k=m}^{\infty}(1/k(k!)^{1/k})$, then inequality $\psi(m) > \psi(m+1)$ is equivalent to inequality

$$1-\frac{2m+2}{3m+7}+\frac{2m}{3m+10} > \frac{m+1}{e(m!)^{1/m}}. \tag{4.5}$$

If $1 \le m \le 16$, then simple computation leads to inequality (4.5).

If $m \ge 17$, then it is not difficult to verify that $\sqrt{2\pi m} \ge e^{7/3}$ and

$$\sqrt{2\pi m} \ge e^{(21m^2+71m+70)/(9m^2+39m+50)}. \tag{4.6}$$

If $x > 0$, then $e > (1+1/x)^x$; this implies that

$$e > \left(1+\frac{21m^2+71m+70}{(9m^2+39m+50)m}\right)^{(9m^2+39m+50)m/(21m^2+71m+70)}. \tag{4.7}$$

From inequalities (4.6) and (4.7), we get

$$\begin{aligned} \sqrt{2\pi m} &> \left(1+\frac{21m^2+71m+70}{(9m^2+39m+50)m}\right)^m, \\ (2\pi m)^{1/(2m)} &> \frac{(m+1)(3m+7)(3m+10)}{m(9m^2+39m+50)}, \\ \frac{m+5}{3m+7}+\frac{2m}{3m+10} &> \frac{m+1}{m(2\pi m)^{1/(2m)}}. \end{aligned} \tag{4.8}$$

From the well-known Stirling Formula $m! = \sqrt{2\pi m}(m/e)^m \exp(\theta_m/12m)$ $(0 < \theta_m < 1)$, we get

$$m! > \sqrt{2\pi m}\left(\frac{m}{e}\right)^m. \tag{4.9}$$

Therefore, inequality (4.5) follows from inequalities (4.8) and (4.9).

From the monotonicity of sequence $\{\psi(m)\}_{m=1}^{\infty}$ and $\lim_{m\to+\infty}\psi(m) = 0$, we get $\psi(m) > 0$; therefore, inequality (4.3) is proved.

Meanwhile, we have

$$\begin{gathered}
\sqrt{2\pi(m+1)} > e^{2/3}, \\
\sqrt{2\pi(m+1)} > e^{(2m+2)/(3m+8)}, \\
\sqrt{2\pi(m+1)} > \left(1+\frac{2}{3m+8}\right)^{(3m+8)/2\cdot(2m+2)/(3m+8)}, \\
(2\pi(m+1))^{1/(2m+2)} > \frac{3m+10}{3m+8}, \\
e\left(1-\frac{2}{3m+10}\right)\frac{1}{m+1} > \frac{e}{(m+1)(2\pi(m+1))^{1/(2m+2)}}.
\end{gathered} \tag{4.10}$$

Therefore, inequality (4.4) follows from inequalities (4.10) and

$$(m+1)! > \sqrt{2\pi(m+1)}\left(\frac{m+1}{e}\right)^{m+1}. \tag{4.11}$$

□

Theorem 4.2. *Let $n \in \mathbb{N}$, $n \geq 1$, and $a_k > 0$ $(k = 1, 2, \ldots, n)$. If $B_n = \min_{1\leq k\leq n}\{ka_k\}$, then*

$$e\sum_{k=1}^{n}\left(1-\frac{2}{3k+7}\right)a_k - \sum_{k=1}^{n}\left(\prod_{j=1}^{k}a_j\right)^{1/k} \geq B_n\left[e\sum_{k=1}^{n}\left(1-\frac{2}{3k+7}\right)\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{(k!)^{1/k}}\right]. \tag{4.12}$$

Proof. Let $b_k = ka_k$, $k = 1, 2, \ldots, n$, and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$,

$$\begin{gathered}
D_m = \left\{\mathbf{b} \mid b_m = \max_{1\leq k\leq n}\{b_k\} > \min_{1\leq k\leq n}\{b_k\} > 0\right\}, \quad m = 1, 2, \ldots, n, \\
f : \mathbf{b} \in \mathbb{R}_+^n \longrightarrow e\sum_{k=1}^{n}\left(1-\frac{2}{3k+7}\right)\frac{b_k}{k} - \sum_{k=1}^{n}\left(\frac{1}{k!}\prod_{j=1}^{k}b_j\right)^{1/k}, \quad \mathbf{b} \in \mathbb{R}_+^n.
\end{gathered} \tag{4.13}$$

Then inequality (4.12) is equivalent to the following inequality:

$$e\sum_{k=1}^{n}\left(1-\frac{2}{3k+7}\right)\frac{b_k}{k} - \sum_{k=1}^{n}\left(\frac{1}{k!}\prod_{j=1}^{k}b_j\right)^{1/k} \geq B_n\left[e\sum_{k=1}^{n}\left(1-\frac{2}{3k+7}\right)\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{(k!)^{1/k}}\right], \tag{4.14}$$

where $B_n = \min_{1\leq k\leq n}\{b_k\}$.

If $\mathbf{b} \in D_m$ $(m = 1, 2, \ldots, n)$, then

$$\begin{aligned}\frac{\partial f(\mathbf{b})}{\partial b_m} &= e\left(1 - \frac{2}{3m+7}\right)\frac{1}{m} - \sum_{k=m}^{n}\frac{1}{kb_m}\left(\frac{1}{k!}\prod_{j=1}^{k} b_j\right)^{1/k} \\ &> e\left(1 - \frac{2}{3m+7}\right)\frac{1}{m} - \sum_{k=m}^{n}\frac{1}{k(k!)^{1/k}} \\ &> e\left(1 - \frac{2}{3m+7}\right)\frac{1}{m} - \sum_{k=m}^{\infty}\frac{1}{k(k!)^{1/k}}.\end{aligned} \tag{4.15}$$

From inequality (4.3) and $\partial f(\mathbf{b})/\partial b_m > 0$ together with Theorem 1.1, we clearly see that

$$f(b_1, b_2, \ldots, b_n) \geq f(B_n, B_n, \ldots, B_n). \tag{4.16}$$

Therefore, inequality (4.14) is proved. □

Corollary 4.3. *Let $n \in \mathbb{N}$, $n \geq 1$, and $a_k > 0$ $(k = 1, 2, \ldots, n)$. If $B_n = \min_{1 \leq k \leq n}\{ka_k\}$, then*

$$e\sum_{k=1}^{n}\left(1 - \frac{2}{3k+7}\right)a_k - \sum_{k=1}^{n}\left(\prod_{j=1}^{k} a_j\right)^{1/k} \geq B_n\left(\frac{4}{5}e - 1\right). \tag{4.17}$$

Proof. Let $T(m) = e\sum_{k=1}^{m}(1 - 2/(3k+7))(1/k) - \sum_{k=1}^{m}(1/(k!)^{1/k})$ $(m = 1, 2, \ldots, n)$, then inequality (4.4) implies that $\{T(m)\}_{m=1}^{n}$ is a strictly increasing sequence. Then from inequality (4.12) we get

$$e\sum_{k=1}^{n}\left(1 - \frac{2}{3k+7}\right)a_k - \sum_{k=1}^{n}\left(\prod_{j=1}^{k} a_j\right)^{1/k} \geq B_n T(n) \geq B_n T(1) = B_n\left(\frac{4}{5}e - 1\right). \tag{4.18}$$

□

Let $n \to +\infty$; thus, we know that Corollary 4.4 is true.

Corollary 4.4. *If $a_n \geq 0$ $(n \in \mathbb{N}, n \geq 1)$ with $0 < \sum_{n=1}^{\infty} a_n < \infty$, then*

$$\sum_{n=1}^{\infty}\left(\prod_{j=1}^{n} a_j\right)^{1/n} \leq e\sum_{n=1}^{\infty}\left(1 - \frac{2}{3n+7}\right)a_n. \tag{4.19}$$

Remark 4.5. Many other applications for Theorem 1.1 appeared in [10].

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