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Research Article

Global Estimates for Singular Integrals of the Composition of the Maximal Operator and the Green's Operator

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We establish the Poincaré type inequalities for the composition of the maximal operator and the Green's operator in John domains.

1. Introduction

Let Ω be a bounded, convex domain and B a ball in $\mathbb{R}^n$, $n \geq 2$. We use σB to denote the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$, $\sigma > 0$. We do not distinguish the balls from cubes in this paper. We use $|E|$ to denote the n-dimensional Lebesgue measure of the set $E \subseteq \mathbb{R}^n$. We say that w is a weight if $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $w > 0$, a.e.

Differential forms are extensions of functions in $\mathbb{R}^n$. For example, the function $u(x_1, x_2, \ldots, x_n)$ is called a 0-form. Moreover, if $u(x_1, x_2, \ldots, x_n)$ is differentiable, then it is called a differential 0-form. The 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^n u_i(x_1, x_2, \ldots, x_n)dx_i$. If the coefficient functions $u_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, are differentiable, then $u(x)$ is called a differential 1-form. Similarly, a differential k-form $u(x)$ is generated by $\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}\}$, $k = 1, 2, \ldots, n$, that is,

$$u(x) = \sum_I u_I(x)dx_I = \sum u_{i_1 i_2 \cdots i_k}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}, \tag{1.1}$$

where $\wedge$ is the Wedge Product, $I = (i_1, i_2, \ldots, i_k)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Let

$$\wedge^l = \wedge^l(\mathbb{R}^n) \tag{1.2}$$

be the set of all l-forms in $\mathbb{R}^n$,

$$D'\left(\Omega, \wedge^l\right) \tag{1.3}$$

the space of all differential l-forms on Ω, and

$$L^p\left(\Omega, \wedge^l\right) \tag{1.4}$$

the l-forms $u(x) = \sum_I u_I(x) dx_I$ on Ω satisfying $\int_\Omega |u_I|^p dx < \infty$ for all ordered l-tuples I, $l = 1, 2, \ldots, n$. We denote the exterior derivative by

$$d : D'\left(\Omega, \wedge^l\right) \longrightarrow D'\left(\Omega, \wedge^{l+1}\right) \tag{1.5}$$

for $l = 0, 1, \ldots, n-1$, and define the Hodge star operator

$$\star : \wedge^k \longrightarrow \wedge^{n-k} \tag{1.6}$$

as follows. If $u = u_I dx_I$, $i_1 < i_2 < \cdots < i_k$ is a differential k-form, then

$$\star u = (-1)^{\sum(I)} u_I dx_J, \tag{1.7}$$

where $I = (i_1, i_2, \ldots, i_k)$, $J = \{1, 2, \ldots, n\} - I$, and $\sum(I) = k(k+1)/2 + \sum_{j=1}^{k} i_j$. The Hodge codifferential operator

$$d^\star : D'\left(\Omega, \wedge^{l+1}\right) \longrightarrow D'\left(\Omega, \wedge^l\right) \tag{1.8}$$

is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n-1$. We write

$$\|u\|_{s,\Omega} = \left(\int_\Omega |u|^s dx\right)^{1/s}. \tag{1.9}$$

The differential forms can be used to describe various systems of PDEs and to express different geometric structures on manifolds. For instance, some kinds of differential forms are often utilized in studying deformations of elastic bodies, the related extrema for variational integrals, and certain geometric invariance. Differential forms have become invaluable tools for many fields of sciences and engineering; see [1, 2] for more details.

In this paper, we will focus on a class of differential forms satisfying the well-known nonhomogeneous A-harmonic equation

$$d^\star A(x, du) = B(x, du), \tag{1.10}$$

where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions

$$|A(x,\xi)| \le a|\xi|^{p-1}, \qquad A(x,\xi)\cdot\xi \ge |\xi|^p, \qquad |B(x,\xi)| \le b|\xi|^{p-1} \tag{1.11}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1.10). If the operator $B = 0$, (1.10) becomes $d^\star A(x, du) = 0$, which is called the (homogeneous) A-harmonic equation. A solution to (1.10) is an element of the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that $\int_\Omega A(x,du)\cdot d\varphi + B(x,du)\cdot\varphi = 0$ for all $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ with compact support. Let $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ be defined by $A(x,\xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then, A satisfies the required conditions and $d^\star A(x, du) = 0$ becomes the p-harmonic equation

$$d^\star\left(du|du|^{p-2}\right) = 0 \tag{1.12}$$

for differential forms. If u is a function (0-form), (1.12) reduces to the usual p-harmonic equation $\operatorname{div}(\nabla u|\nabla u|^{p-2}) = 0$ for functions. A remarkable progress has been made recently in the study of different versions of the harmonic equations; see [3] for more details.

Let $C^\infty(\Omega, \wedge^l)$ be the space of smooth l-forms on Ω and

$$\mathcal{W}\left(\Omega, \wedge^l\right) = \left\{u \in L^1_{\mathrm{loc}}\left(\Omega, \wedge^l\right) : u \text{ has generalized gradient}\right\}. \tag{1.13}$$

The harmonic l-fields are defined by

$$\mathscr{H}\left(\Omega, \wedge^l\right) = \left\{u \in \mathcal{W}\left(\Omega, \wedge^l\right) : du = d^\star u = 0,\ u \in L^p \text{ for some } 1 < p < \infty\right\}. \tag{1.14}$$

The orthogonal complement of $\mathscr{H}$ in L^1 is defined by

$$\mathscr{H}^\perp = \left\{u \in L^1 :< u, h >= 0 \text{ for all } h \in \mathscr{H}\right\}. \tag{1.15}$$

Then, the Green's operator G is defined as

$$G : C^\infty\left(\Omega, \wedge^l\right) \longrightarrow \mathscr{H}^\perp \cap C^\infty\left(\Omega, \wedge^l\right) \tag{1.16}$$

by assigning $G(u)$ to be the unique element of $\mathscr{H}^\perp \cap C^\infty(\Omega, \wedge^l)$ satisfying Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator that maps $C^\infty(\Omega, \wedge^l)$ onto $\mathscr{H}$ so that $H(u)$ is the harmonic part of u. See [4] for more properties of these operators.

For any locally L^s-integrable form $u(y)$, the Hardy-Littlewood maximal operator $\mathcal{M}_s$ is defined by

$$\mathcal{M}_s(u) = \sup_{r>0}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y)|^s dy\right)^{1/s}, \tag{1.17}$$

where $B(x,r)$ is the ball of radius r, centered at x, $1 \leq s < \infty$. We write $\mathcal{M}(u) = \mathcal{M}_1(u)$ if $s = 1$. Similarly, for a locally L^s-integrable form $u(y)$, we define the sharp maximal operator $\mathcal{M}_s^{\#}$ by

$$\mathcal{M}_s^{\#}(u) = \sup_{r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} \left| u(y) - u_{B(x,r)} \right|^s dy \right)^{1/s}, \tag{1.18}$$

where the l-form $u_B \in D'(B, \wedge^l)$ is defined by

$$u_B = \begin{cases} |B|^{-1} \int_B u(y)dy, & l = 0, \\ d(Tu), & l = 1, 2, \ldots, n \end{cases} \tag{1.19}$$

for all $u \in L^p(B, \wedge^l)$, $1 \leq p < \infty$, and T is the homotopy operator which can be found in [3]. Also, from [5], we know that both $\mathcal{M}_s(u)$ and $\mathcal{M}_s^{\#}(u)$ are L^s-integrable 0-form.

Differential forms, the Green's operator, and maximal operators are widely used not only in analysis and partial differential equations, but also in physics; see [2–4, 6–9]. Also, in real applications, we often need to estimate the integrals with singular factors. For example, when calculating an electric field, we will deal with the integral $E(r) = (1/4\pi\epsilon_0) \int_D \rho(x)((r - x)/\|r - x\|^3)dx$, where $\rho(x)$ is a charge density and x is the integral variable. The integral is singular if $r \in D$. When we consider the integral of the vector field $\mathbf{F} = \nabla f$, we have to deal with the singular integral if the potential function f contains a singular factor, such as the potential energy in physics. It is clear that the singular integrals are more interesting to us because of their wide applications in different fields of mathematics and physics. In recent paper [10], Ding and Liu investigated singular integrals for the composition of the homotopy operator T and the projection operator H and established some inequalities for these composite operators with singular factors. In paper [11], they keep working on the same topic and derive global estimates for the singular integrals of these composite operators in δ-John domains. The purpose of this paper is to estimate the Poincaré type inequalities for the composition of the maximal operator and the Green's operator over the δ-John domain.

2. Definitions and Lemmas

We first introduce the following definition and lemmas that will be used in this paper.

Definition 2.1. A proper subdomain $\Omega \subset \mathbb{R}^n$ is called a δ-John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that

$$d(\xi, \partial\Omega) \geq \delta |x - \xi| \tag{2.1}$$

for each $\xi \in \gamma$. Here $d(\xi, \partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.

Lemma 2.2 (see [12]). *Let ϕ be a strictly increasing convex function on $[0,\infty)$ with $\phi(0) = 0$ and D a domain in $\mathbb{R}^n$. Assume that u is a function in D such that $\phi(|u|) \in L^1(D,\mu)$ and $\mu(\{x \in D : |u - c| > 0\}) > 0$ for any constant c, where μ is a Radon measure defined by $d\mu(x) = w(x)dx$ for a weight $w(x)$. Then, one has*

$$\int_D \phi\left(\frac{a}{2}|u - u_{D,\mu}|\right) d\mu \le \int_D \phi(a|u|) d\mu \tag{2.2}$$

for any positive constant a, where $u_{D,\mu} = (1/\mu(D)) \int_D u d\mu$.

Lemma 2.3 (see [13]). *Each Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that*

$$\bigcup_i Q_i = \Omega, \qquad \sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{5/4}Q_i} \le N\chi_\Omega \tag{2.3}$$

and some $N > 1$, and if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R (this cube need not be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if Ω is δ-John, then there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0 = Q_{j_0}, Q_{j_1}, \ldots, Q_{j_k} = Q$ from $\mathcal{V}$ and such that $Q \subset \rho Q_{j_i}$, $i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$.

Lemma 2.4 (see [14]). *Let u be a smooth differential form satisfying* (1.10) *in a domain D, $\sigma > 10 < s$, and $t < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \le C|B|^{(t-s)/st}\|u\|_{t,\sigma B} \tag{2.4}$$

for all balls B with $\sigma B \subset D$, where $\sigma > 1$ is a constant.

Lemma 2.5 (see [5]). *Let $\mathcal{M}_s$ be the Hardy-Littlewood maximal operator defined in* (1.17)*, G the Green's operator, and $u \in L^t(\Omega, \wedge^l)$, $l = 1, 2, 3, \ldots, n$, $1 \le s < t < \infty$, a smooth differential form in a bounded domain Ω. Then,*

$$\|\mathcal{M}_s(G(u))\|_{t,\Omega} \le C\|u\|_{t,\Omega} \tag{2.5}$$

for some constant C, independent of u.

Lemma 2.6 (see [5]). *Let $u \in L^s(\Omega, \wedge^l)$, $l = 1, 2, 3, \ldots, n$, $1 \le s < \infty$, be a smooth differential form in a bounded domain Ω, $\mathcal{M}_s^{\#}$ the sharp maximal operator defined in* (1.18)*, and G the Green's operator. Then,*

$$\left\|\mathcal{M}_s^{\#}(G(u))\right\|_{s,\Omega} \le C|\Omega|^{1/s}\|u\|_{s,\Omega} \tag{2.6}$$

for some constant C, independent of u.

Lemma 2.7. *Let $u \in L^t_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, be a smooth differential form satisfying the A-harmonic equation* (1.10) *in convex domain Ω, G the Green's operator, and $\mathcal{M}_s$ the Hardy-Littlewood* *maximal operator defined in* (1.17) *with* $1 < s < t < \infty$. *Then, there exists a constant* $C(n, t, \alpha, \lambda, \rho)$, *independent of* u, *such that*

$$\left(\int_B |\mathcal{M}_s(G(u))|^t \frac{1}{d(x,\partial\Omega)^\alpha} dx\right)^{1/t} \leq C(n,t,\alpha,\lambda,\rho)|B|^\gamma \left(\int_{\rho B} |u|^t \frac{1}{|x-x_B|^\lambda} dx\right)^{1/t} \tag{2.7}$$

for all balls B *with* $\rho B \subset \Omega$ *and any real number* α *and* λ *with* $\alpha > \lambda \geq 0$ *and* $\gamma = (\lambda - \alpha)/nt$, *where* x_B *is the center of the ball and* $\rho > 1$ *is a constant.*

Proof. Let $\varepsilon \in (0,1)$ be small enough such that $\varepsilon n < \alpha - \lambda$ and B any ball with $B \subset \Omega$, center x_B and radius r_B. Taking $k = t/(1-\varepsilon)$, we see that $k > t$. Note that $1/t = 1/k + (k-t)/kt$; using Hölder's inequality, we obtain

$$\begin{aligned}
&\left(\int_B |\mathcal{M}_s(G(u))|^t \frac{1}{d(x,\partial\Omega)^\alpha} dx\right)^{1/t} \\
&\quad = \left(\int_B \left(|\mathcal{M}_s(G(u))| \frac{1}{d(x,\partial\Omega)^{\alpha/t}}\right)^t dx\right)^{1/t} \\
&\quad \leq \left(\int_B |\mathcal{M}_s(G(u))|^k dx\right)^{1/k} \left(\int_B \left(\frac{1}{d(x,\partial\Omega)^{\alpha/t}}\right)^{kt/(k-t)} dx\right)^{(k-t)/kt} \\
&\quad \leq \|\mathcal{M}_s(G(u))\|_{k,B} \left(\int_B (d(x,\partial\Omega))^{-\alpha\beta} dx\right)^{1/\beta t},
\end{aligned} \tag{2.8}$$

where $\beta = k/(k-t)$. Since $k > t > s$, using Lemma 2.5, we get

$$\|\mathcal{M}_s(G(u))\|_{k,B} \leq C_1 \|u\|_{k,B}. \tag{2.9}$$

Let $m = ntk/(nt + \alpha k - \lambda k)$, then $0 < m < t < k$. Using Lemma 2.4, we have

$$\|u\|_{k,B} \leq C_2 |B|^{(m-k)/mk} \|u\|_{m,\rho B}, \tag{2.10}$$

where $\rho > 1$ is a constant and $\rho B \subset \Omega$. By Hölder's inequality with $1/m = 1/t + (t-m)/mt$ again, we find

$$\begin{aligned}
\|u\|_{m,\rho B} &= \left(\int_{\rho B} \left(|u||x-x_B|^{-\lambda/t}|x-x_B|^{\lambda/t}\right)^m dx\right)^{1/m} \\
&\leq \left(\int_{\rho B} \left(|u||x-x_B|^{-\lambda/t}\right)^t dx\right)^{1/t} \left(\int_{\rho B} \left(|x-x_B|^{\lambda/t}\right)^{mt/(t-m)} dx\right)^{(t-m)/mt} \\
&\leq \left(\int_{\rho B} |u|^t |x-x_B|^{-\lambda} dx\right)^{1/t} \left(\int_{\rho B} (|x-x_B|^{m\lambda/(t-m)} dx\right)^{(t-m)/mt}.
\end{aligned} \tag{2.11}$$

Note that $d(x, \partial\Omega) \geq (\rho - 1)r_B$ for all $x \in B$, it follows that

$$(d(x, \partial\Omega))^{-\alpha\beta} \leq \left[(\rho - 1)r_B\right]^{-\alpha\beta}. \tag{2.12}$$

Hence, we have

$$\begin{aligned}\left(\int_B (d(x, \partial\Omega))^{-\alpha\beta} dx\right)^{1/\beta t} &\leq \left[(\rho - 1)r_B\right]^{-\alpha/t} |B|^{1/\beta t} \\ &= C_3 (r_B)^{-\alpha/t} |B|^{1/\beta t}.\end{aligned} \tag{2.13}$$

Now, by the elementary integral calculation, we obtain

$$\left(\int_{\rho B} (|x - x_B|^{m\lambda/(t-m)} dx\right)^{(t-m)/mt} \leq C_4 (\rho r_B)^{\lambda/t + n(t-m)/mt}. \tag{2.14}$$

Substituting (2.9)–(2.14) into (2.8), we obtain

$$\begin{aligned}&\left(\int_B |\mathcal{M}_s(G(u))|^t \frac{1}{d(x, \partial\Omega)^\alpha} dx\right)^{1/t} \\ &\quad < C_5 (r_B)^{-\alpha/t + \lambda/t + n(t-m)/mt} |B|^{1/\beta t + (m-k)/mk} \left(\int_{\rho B} |u|^t |x - x_B|^{-\lambda} dx\right)^{1/t} \\ &\quad = C_5 (r_B)^{n/k - n/t} |B|^{1/t - 1/k + (\lambda - \alpha)/nt} \left(\int_{\rho B} |u|^t |x - x_B|^{-\lambda} dx\right)^{1/t} \\ &\quad = C_6 |B|^{1/k - 1/t} |B|^{1/t - 1/k + (\lambda - \alpha)/nt} \left(\int_{\rho B} |u|^t |x - x_B|^{-\lambda} dx\right)^{1/t} \\ &\quad = C_6 |B|^{(\lambda - \alpha)/nt} \left(\int_{\rho B} |u|^t |x - x_B|^{-\lambda} dx\right)^{1/t} \\ &\quad = C(n, t, \alpha, \lambda, \rho) |B|^\gamma \left(\int_{\rho B} |u|^t |x - x_B|^{-\lambda} dx\right)^{1/t}.\end{aligned} \tag{2.15}$$

We have completed the proof. □

Similarly, by Lemma 2.6, we can prove the following lemma.

Lemma 2.8. *Let $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $1 < s < \infty$, $l = 1, 2, \ldots, n$, be a smooth differential form satisfying the A-harmonic equation* (1.10) *in convex domain Ω, $\mathcal{M}^\#_s$ the sharp maximal operator defined in* (1.18)*, and G Green's operator. Then, there exists a constant $C(n, s, \alpha, \lambda, \rho)$, independent of u, such that*

$$\left(\int_B \left|\mathcal{M}^\#_s(G(u))\right|^s \frac{1}{d(x, \partial\Omega)^\alpha} dx\right)^{1/s} \leq C(n, s, \alpha, \lambda, \rho) |B|^\gamma \left(\int_{\rho B} |u|^s \frac{1}{|x - x_B|^\lambda} dx\right)^{1/s} \tag{2.16}$$

for all balls B with $\rho B \subset \Omega$ and any real number α and λ with $\alpha > \lambda \geq 0$ and $\gamma = 1/s - (\lambda - \alpha)/ns$, where x_B is the center of the ball and $\rho > 1$ is a constant.

3. Main Results

Theorem 3.1. *Let $u \in L^t_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, be a smooth differential form satisfying the A-harmonic equation* (1.10)*, G Green's operator, and $\mathcal{M}_s$ the Hardy-Littlewood maximal operator defined in* (1.17) *with $1 < s < t < \infty$. Then, there exists a constant $C(n, \rho, t, \alpha, \lambda, N, Q_0, \Omega)$, independent of u, such that*

$$\begin{aligned}\left(\int_\Omega \left|\mathcal{M}_s(G(u)) - (\mathcal{M}_s(G(u)))_{Q_0}\right|^t \frac{1}{d(x, \partial\Omega)^\alpha} dx\right)^{1/t} \\ \leq C(n, \rho, t, \alpha, \lambda, N, Q_0, \Omega)\left(\int_\Omega |u|^t g(x) dx\right)^{1/t}\end{aligned} \tag{3.1}$$

for any bounded and convex δ-John domain $\Omega \subset \mathbb{R}^n$, where

$$g(x) = \sum_i \chi_{\rho Q_i} \frac{1}{\left|x - x_{Q_i}\right|^\lambda}, \tag{3.2}$$

$\rho > 1$ and $\alpha > \lambda \geq 0$ are constants, the fixed cube $Q_0 \subset \Omega$, the cubes $Q_i \subset \Omega$, the constant $N > 1$ appeared in Lemma 2.3, and x_{Q_i} is the center of Q_i.

Proof. First, we use Lemma 2.3 for the bounded and convex δ-John domain Ω. There is a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ for Ω such that $\Omega = \cup Q_i$, and $\sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{5/4}Q_i} \leq N\chi_\Omega$ for some $N > 1$. Moreover, there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0 = Q_{j_0}, Q_{j_1}, \ldots, Q_{j_k} = Q$ from $\mathcal{V}$ such that $Q \subset \rho Q_{j_i}$, $i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$. Then, by the elementary inequality $(a+b)^s \leq 2^s(|a|^s + |b|^s)$, $s \geq 0$, we have

$$\begin{aligned}&\left(\int_\Omega \left|\mathcal{M}_s(G(u)) - (\mathcal{M}_s(G(u)))_{Q_0}\right|^t \frac{1}{d(x, \partial\Omega)^\alpha} dx\right)^{1/t} \\ &\quad = \left(\int_{\cup Q_i} \left|\mathcal{M}_s(G(u)) - (\mathcal{M}_s(G(u)))_{Q_0}\right|^t d\mu\right)^{1/t} \\ &\quad \leq \left(\sum_{Q_i \in \mathcal{V}} \left(2^t \int_{Q_i} \left|\mathcal{M}_s(G(u)) - (\mathcal{M}_s(G(u)))_{Q_i}\right|^t d\mu \right.\right. \\ &\qquad\qquad \left.\left. +2^t \int_{Q_i} \left|(\mathcal{M}_s(G(u)))_{Q_i} - (\mathcal{M}_s(G(u)))_{Q_0}\right|^t d\mu\right)\right)^{1/t}\end{aligned}$$

$$\le C_1(t)\left(\left(\sum_{Q_i\in\mathcal{V}}\int_{Q_i}\left|\mathcal{M}_s(G(u))-(\mathcal{M}_s(G(u)))_{Q_i}\right|^t d\mu\right)^{1/t} + \left(\sum_{Q_i\in\mathcal{V}}\int_{Q_i}\left|(\mathcal{M}_s(G(u)))_{Q_i}-(\mathcal{M}_s(G(u)))_{Q_0}\right|^t d\mu\right)^{1/t}\right). \tag{3.3}$$

The first sum in (3.3) can be estimated by using Lemma 2.2 with $\varphi = x^t$, $a = 2$, and Lemma 2.7:

$$\begin{aligned}
\sum_{Q_i\in\mathcal{V}}\int_{Q_i}&\left|\mathcal{M}_s(G(u))-(\mathcal{M}_s(G(u)))_{Q_i}\right|^t d\mu \\
&\le \sum_{Q_i\in\mathcal{V}}\int_{Q_i} 2^t|\mathcal{M}_s(G(u))|^t d\mu \\
&\le C_2(n,\rho,t,\alpha,\lambda,\Omega)\sum_{Q_i\in\mathcal{V}}|Q_i|^{\gamma t}\int_{\rho Q_i}|u|^t d\mu_i \\
&\le C_3(n,\rho,t,\alpha,\lambda,\Omega)|\Omega|^{\gamma t}\sum_{Q_i\in\mathcal{V}}\int_{\Omega}\left(|u|^t d\mu_i\right)\chi_{\rho Q_i} \\
&= C_4(n,\rho,t,\alpha,\lambda,N,\Omega)|\Omega|^{\gamma t}\int_{\Omega}|u|^t g(x)dx \\
&= C_5(n,\rho,t,\alpha,\lambda,N,\Omega)\int_{\Omega}|u|^t g(x)dx,
\end{aligned} \tag{3.4}$$

where $\mu(x)$ and $\mu_i(x)$ are the Radon measures defined by $d\mu = (1/d(x,\partial\Omega)^\alpha)dx$ and $d\mu_i(x) = (1/|x-x_{Q_i}|^\lambda)dx$, respectively.

To estimate the second sum in (3.3), we need to use the property of δ-John domain. Fix a cube $Q_i \in \mathcal{V}$ and let $Q_0 = Q_{j_0}, Q_{j_1}, \ldots, Q_{j_k} = Q_i$ be the chain in Lemma 2.3. Then we have

$$\left|(\mathcal{M}_s(G(u)))_{Q_i}-(\mathcal{M}_s(G(u)))_{Q_0}\right| \le \sum_{i=0}^{k-1}\left|(\mathcal{M}_s(G(u)))_{Q_{j_i}}-(\mathcal{M}_s(G(u)))_{Q_{j_{i+1}}}\right|. \tag{3.5}$$

The chain $\{Q_{j_i}\}$ also has property that for each i, $i = 0,1,\ldots,k-1$, $Q_{j_i}\cap Q_{j_{i+1}} \ne \emptyset$. Thus, there exists a cube D_i such that $D_i \subset Q_{j_i}\cap Q_{j_{i+1}}$ and $Q_{j_i}\cup Q_{j_{i+1}} \subset ND_i$, $N > 1$, so,

$$\frac{\max\{|Q_{j_i}|,|Q_{j_{i+1}}|\}}{|Q_{j_i}\cap Q_{j_{i+1}}|} \le \frac{\max\{|Q_{j_i}|,|Q_{j_{i+1}}|\}}{|D_i|} \le C_6(N) \tag{3.6}$$

Note that

$$\begin{aligned}\mu(Q) &= \int_Q \frac{1}{d(x,\partial\Omega)^\alpha}dx \\ &\geq \int_Q \frac{1}{(\operatorname{diam}(\Omega))^\alpha}dx \\ &= C_7(n,\alpha,\Omega)|Q|,\end{aligned} \tag{3.7}$$

where $C_7(n,\alpha,\Omega)$ is a positive constant. By (3.6), (3.7), and Lemma 2.7, we have

$$\begin{aligned}&\left|(\mathcal{M}_s(G(u)))_{Q_{j_i}} - (\mathcal{M}_s(G(u)))_{Q_{j_{i+1}}}\right|^t \\ &\quad= \frac{1}{\mu(Q_{j_i}\cap Q_{j_{i+1}})}\int_{Q_{j_i}\cap Q_{j_{i+1}}}\left|(\mathcal{M}_s(G(u)))_{Q_{j_i}} - (\mathcal{M}_s(G(u)))_{Q_{j_{i+1}}}\right|^t \frac{dx}{d(x,\partial\Omega)^\alpha} \\ &\quad\leq \frac{C_8(n,\alpha,\Omega)}{|Q_{j_i}\cap Q_{j_{i+1}}|}\int_{Q_{j_i}\cap Q_{j_{i+1}}}\left|(\mathcal{M}_s(G(u)))_{Q_{j_i}} - (\mathcal{M}_s(G(u)))_{Q_{j_{i+1}}}\right|^t \frac{dx}{d(x,\partial\Omega)^\alpha} \\ &\quad\leq \frac{C_8(n,\alpha,\Omega)C_6(N)}{\max\{|Q_{j_i}|,|Q_{j_{i+1}}|\}}\int_{Q_{j_i}\cap Q_{j_{i+1}}}\left|(\mathcal{M}_s(G(u)))_{Q_{j_i}} - (\mathcal{M}_s(G(u)))_{Q_{j_{i+1}}}\right|^t d\mu \\ &\quad\leq C_9(n,t,\alpha,N,\Omega)\sum_{k=i}^{i+1}\frac{1}{|Q_{j_k}|}\int_{Q_{j_k}}\left|\mathcal{M}_s(G(u)) - (\mathcal{M}_s(G(u)))_{Q_{j_k}}\right|^t d\mu \\ &\quad\leq C_{10}(n,\rho,t,\alpha,\lambda,N,\Omega)\sum_{k=i}^{i+1}\frac{|Q_{j_k}|^{\gamma t}}{|Q_{j_k}|}\int_{\rho Q_{j_k}}|u|^t d\mu_{j_k} \\ &\quad= C_{10}(n,\rho,t,\alpha,\lambda,N,\Omega)\sum_{k=i}^{i+1}|Q_{j_k}|^{\gamma t-1}\int_{\rho Q_{j_k}}|u|^t d\mu_{j_k} \\ &\quad\leq C_{11}(n,\rho,t,\alpha,\lambda,N,\Omega)\sum_{k=i}^{i+1}|\Omega|^{\gamma t-1}\int_{\Omega}\left(|u|^t d\mu_{j_k}\right)\chi_{\rho Q_{j_k}} \\ &\quad\leq C_{12}(n,\rho,t,\alpha,\lambda,N,\Omega)\sum_{Q_i\in\mathcal{V}}\int_{\Omega}\left(|u|^t d\mu_i\right)\chi_{\rho Q_i} \\ &\quad= C_{12}(n,\rho,t,\alpha,\lambda,N,\Omega)\int_{\Omega}|u|^t g(x)dx.\end{aligned} \tag{3.8}$$

Then, by (3.5), (3.8), and the elementary inequality $|\sum_{i=1}^M t_i|^s \leq M^{s-1}\sum_{i=1}^M |t_i|^s$, we finally obtain

$$\begin{aligned}&\sum_{Q_i\in\mathcal{V}}\int_{Q_i}\left|(\mathcal{M}_s(G(u)))_{Q_i} - (\mathcal{M}_s(G(u)))_{Q_0}\right|^t d\mu \\ &\quad\leq C_{13}(n,\rho,t,\alpha,\lambda,N,\Omega)\sum_{Q_i\in\mathcal{V}}\int_{Q_i}\left(\int_{\Omega}|u|^t g(x)dx\right)d\mu \\ &\quad= C_{13}(n,\rho,t,\alpha,\lambda,N,\Omega)\left(\sum_{Q_i\in\mathcal{V}}\int_{Q_i}d\mu\right)\int_{\Omega}|u|^t g(x)dx\end{aligned}$$

$$
\begin{aligned}
&= C_{13}(n,\rho,t,\alpha,\lambda,N,\Omega)\left(\int_\Omega d\mu\right)\int_\Omega |u|^t g(x)dx \\
&= C_{14}(n,\rho,t,\alpha,\lambda,N,\Omega)\mu(\Omega)\int_\Omega |u|^t g(x)dx \\
&= C_{15}(n,\rho,t,\alpha,\lambda,N,\Omega)\int_\Omega |u|^t g(x)dx.
\end{aligned}
\tag{3.9}
$$

Substituting (3.4) and (3.9) in (3.3), we have completed the proof of Theorem 3.1. □

Using the proof method for Theorem 3.1 and Lemma 2.8, we get the following theorem.

Theorem 3.2. *Let $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, be a smooth differential form satisfying the A-harmonic equation* (1.10)*, G Green's operator, and $\mathcal{M}_s^\#$ the sharp maximal operator defined in* (1.18)*. Then, there exists a constant $C(n,\rho,s,\alpha,\lambda,N,Q_0,\Omega)$, independent of u, such that*

$$
\begin{aligned}
&\left(\int_\Omega \left|\mathcal{M}_s^\#(G(u)) - (\mathcal{M}_s^\#(G(u))_{Q_0}\right|^s \frac{1}{d(x,\partial\Omega)^\alpha}dx\right)^{1/s} \\
&\quad \le C(n,\rho,s,\alpha,\lambda,N,Q_0,\Omega)\left(\int_\Omega |u|^s g(x)dx\right)^{1/s}
\end{aligned}
\tag{3.10}
$$

for any bounded and convex δ-John domain $\Omega \subset \mathbb{R}^n$, where

$$
g(x) = \sum_i \chi_{\rho Q_i}\frac{1}{\left|x - x_{Q_i}\right|^\lambda},
\tag{3.11}
$$

$\rho > 1$ and $\alpha > \lambda \ge 0$ are constants, the fixed cube $Q_0 \subset \Omega$, the cubes $Q_i \subset \Omega$, the constant $N > 1$ appeared in Lemma 2.3, and x_{Q_i} is the center of Q_i.

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