

## Research Article

# Global Caccioppoli-Type and Poincaré Inequalities with Orlicz Norms

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We obtain global weighted Caccioppoli-type and Poincaré inequalities in terms of Orlicz norms for solutions to the nonhomogeneous  $A$ -harmonic equation  $d^*A(x, d\omega) = B(x, d\omega)$ .

## 1. Introduction

The  $L^p$ -theory of solutions of the homogeneous  $A$ -harmonic equation  $d^*A(x, d\omega) = 0$  for differential forms has been very well developed in recent years. Many  $L^p$ -norm estimates and inequalities, including the Hardy-Littlewood inequalities, Poincaré inequalities, Caccioppoli-type estimates, and Sobolev imbedding inequalities, for solutions of the homogeneous  $A$ -harmonic equation have been established; see [1–11]. Among these results, the Caccioppoli-type inequalities and the Poincaré inequalities for differential forms have become more and more important tools in analysis and related fields, including partial differential equations and potential theory. However, the study of the nonhomogeneous  $A$ -harmonic equation  $d^*A(x, d\omega) = B(x, d\omega)$  just began [4, 6]. Roughly, the Caccioppoli-type inequalities or estimates provide upper bounds for the norms of  $\nabla u$  or  $du$  in terms of the corresponding norm of  $u$  or  $u - c$ , where  $u$  is a differential form or a function satisfying certain conditions. For example,  $u$  may be a solution of an  $A$ -harmonic equation or a minimizer of a functional, and  $c$  is some constant if  $u$  is a function or a closed form if  $u$  is a differential form. Different versions of the Caccioppoli-type inequalities and the Poincaré inequalities have been established during the past several decades. For instance, Sbordone proved in [12] the following version of the Caccioppoli-type inequality:

$$\int_{B_{R/2}} A(|du|) dx \leq C \int_{B_R} A\left(\frac{|u - u_R|}{R}\right) dx \quad (1.1)$$

for a quasiminimizer  $u$  of the functional  $F(\Omega; v) = \int_{\Omega} A(|dv|)dx$ , where  $A$  is a continuous, convex, and strictly increasing function satisfying the so-called  $\Delta_2$ -condition,  $B_R$  is a ball with radius  $R > 0$ , and  $u_R = \int_{B_R} u dx$ ; see [12]. Using the above Caccioppoli-type inequality, Fusco and Sbordone obtained in [13] the higher integrability result

$$\int_{B_{R/2}} A^r(|du|)dx \leq C \left( \int_{B_R} A(|du|)dx \right)^r \quad (1.2)$$

for the gradient of minimizers of the functional  $I(\Omega; v)$ , where  $r > 1$  is some constant. In [14], Greco et al. studied the variational integrals whose integrand grows almost linearly with respect to the gradient and the related equation  $\operatorname{div} A(x, f + \nabla u) = 0$ , where  $A$  is slowly increasing to  $\infty$ . For instance,  $A(t) = \log^{\alpha}(1 + t)$ ,  $\alpha > 0$ , or  $A(t) = \log \log(e + t)$ . They proved that the minimizer  $u$  subject to the Dirichlet data  $v$  satisfies the estimate

$$\int_{\Omega} |\nabla u| A^{1\pm\epsilon}(|\nabla u|)dx \leq C \int_{\Omega} |\nabla v| A^{1\pm\epsilon}(|\nabla v|)dx \quad (1.3)$$

at least for some small  $\epsilon > 0$ . In [15], Cianchi and Fusco investigated the higher integrability properties of the gradient of local minimizers of an integral functional of the form  $J(u, \Omega) = \int_{\Omega} f(x, u, du)dx$ , where  $\Omega$  is an open subset of  $\mathbb{R}^n$ ,  $n \geq 2$ , and  $f$  is a Carathodory function defined in  $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$  satisfying some growth conditions. Using a new form of the Caccioppoli inequality and some other tools, such as the Sobolev inequality and a generalized version of the Gehring lemma, they proved that if  $u$  is a local minimizer of  $J(u, \Omega)$ , for  $\Omega_0 \subset\subset \Omega$  there exists  $\delta > 0$  such that

$$\int_{\Omega_0} A(|du|) \left( \frac{A(|du|)}{|du|} \right)^{\delta} dx < \infty, \quad (1.4)$$

where  $A$  satisfies the so-called  $\Delta_2$ -condition. However, all versions of the Caccioppoli-type inequality developed or used in [12–15] are about the minimizer  $u$  of some functional. In this paper, we will prove the Caccioppoli-type inequalities and the Poincaré inequalities with the  $L^s(\log L)^{\alpha}$ -norm for differential forms satisfying the nonhomogeneous  $A$ -harmonic equation. The method developed in this paper could be used to establish other  $L^s(\log L)^{\alpha}$ -norm inequalities for solutions of the homogeneous  $A$ -harmonic equation or the nonhomogeneous  $A$ -harmonic equation.

Throughout this paper, we always assume that  $\Omega$  is an open subset of  $\mathbb{R}^n$ ,  $n \geq 2$ . The  $n$ -dimensional Lebesgue measure of a set  $E \subseteq \mathbb{R}^n$  is denoted by  $|E|$ . We say that  $w$  is a weight if  $w \in L^1_{\text{loc}}(\mathbb{R}^n)$  and  $w > 0$  a.e. For  $0 < p < \infty$ , we denote the weighted  $L^p$ -norm of a measurable function  $f$  over  $E$  by  $\|f\|_{p,E,w^{\alpha}} = \left( \int_E |f(x)|^p w^{\alpha}(x) dx \right)^{1/p}$ , where  $\alpha$  is a real number. We write  $\|f\|_{p,E} = \|f\|_{p,E,w^{\alpha}}$  if  $w = 1$ . A continuously increasing function  $\varphi : [0, \infty) \rightarrow [0, \infty)$  with  $\varphi(0) = 0$  and  $\varphi(\infty) = \infty$  is called an Orlicz function. The Orlicz space  $L^{\varphi}(\Omega)$  consists of all measurable functions  $f$  on  $\Omega$  such that

$$\int_{\Omega} \varphi \left( \frac{|f|}{k} \right) dx < \infty \quad (1.5)$$

for some  $k = k(f) > 0$ .  $L^\varphi(\Omega)$  is equipped with the nonlinear Luxemburg functional

$$\|f\|_\varphi = \inf \left\{ k > 0 : \frac{1}{|\Omega|} \int_\Omega \varphi \left( \frac{|f|}{k} \right) dx \leq 1 \right\}. \quad (1.6)$$

A convex Orlicz function  $\varphi$  is often called a Young function. If  $\varphi$  is a Young function, then  $\|\cdot\|_\varphi$  defines a norm in  $L^\varphi(\Omega)$ , which is called the Luxemburg norm or Orlicz norm. The Orlicz space  $L^\varphi(\Omega)$  with  $\varphi(t) = t^p \log^\alpha(e + t/c)$  will be denoted by  $L^p(\log L)^\alpha(\Omega)$  and the corresponding norm will be denoted by  $\|f\|_{L^p(\log L)^\alpha(\Omega)}$ , where  $1 \leq p < \infty$ ,  $\alpha \geq 0$ , and  $c > 0$  are constants. The spaces  $L^p(\log L)^0(\Omega)$  and  $L^1(\log L)^1(\Omega)$  are usually referred as  $L^p(\Omega)$  and  $L \log L(\Omega)$ , respectively. From [16], we have the equivalence

$$\|f\|_{L^p(\log L)^\alpha(\Omega)} \approx \left( \int_\Omega |f|^p \log^\alpha \left( e + \frac{|f|}{\|f\|_{p,\Omega}} \right) dx \right)^{1/p}. \quad (1.7)$$

Similarly, we have

$$\|f\|_{L^p(\log L)^\alpha(\Omega, \mu)} \approx \left( \int_\Omega |f|^p \log^\alpha \left( e + \frac{|f|}{\|f\|_{p,\Omega}} \right) d\mu \right)^{1/p}, \quad (1.8)$$

where  $\mu$  is a measure defined by  $d\mu = w(x)dx$  and  $w(x)$  is a weight. In this paper, we simply write

$$\|f\|_{L^p(\log L)^\alpha(E, w^\alpha)} = \left( \int_E |f|^p \log^\alpha \left( e + \frac{|f|}{\|f\|_{p,E}} \right) w^\alpha dx \right)^{1/p}, \quad (1.9)$$

and  $\|f\|_{L^p(\log L)^\alpha(E)} = \|f\|_{L^p(\log L)^\alpha(E,1)}$ , where  $w$  is a weight.

We keep using the traditional notations related to differential forms in this paper. Let  $\Lambda^\ell = \Lambda^\ell(\mathbb{R}^n)$  be the linear space of the  $\ell$ -covectors on  $\mathbb{R}^n$ ,  $\ell = 1, 2, \dots, n$ . It is a normed space of dimension  $\binom{n}{\ell}$ . A differential  $\ell$ -form  $\omega$  on  $\Omega$  is a Schwartz distribution on  $\Omega$  with values in  $\Lambda^\ell(\mathbb{R}^n)$ . We write  $D'(\Omega, \Lambda^\ell)$  for the space of all differential  $\ell$ -forms and  $L^p(\Omega, \Lambda^\ell)$  for all  $\ell$ -forms  $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \dots i_\ell}(x) dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_\ell}$  with  $\omega_I \in L^p(\Omega, \mathbb{R})$  for all ordered  $\ell$ -tuples  $I$ . Thus,  $L^p(\Omega, \Lambda^\ell)$  is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left( \int_\Omega |\omega(x)|^p dx \right)^{1/p} = \left( \int_\Omega \left( \sum_I |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}. \quad (1.10)$$

We use  $L^p(\log L)^\alpha(\Omega, \Lambda^\ell)$  to denote the space of all differential  $\ell$ -forms  $u$  on  $\Omega$  with

$$\|u\|_{L^p(\log L)^\alpha(\Omega)} = \left( \int_\Omega |u|^p \log^\alpha \left( e + \frac{|u|}{\|u\|_{p,\Omega}} \right) dx \right)^{1/p} < \infty. \quad (1.11)$$

We use  $d : D'(\Omega, \Lambda^\ell) \rightarrow D'(\Omega, \Lambda^{\ell+1})$  to denote the differential operator and  $d^* : D'(\Omega, \Lambda^{\ell+1}) \rightarrow D'(\Omega, \Lambda^\ell)$  to denote the Hodge codifferential operator given by  $d^* = (-1)^{n\ell+1} \star d \star$  on  $D'(\Omega, \Lambda^{\ell+1})$ ,  $\ell = 0, 1, \dots, n$ . Here  $\star$  is the well-known Hodge star operator. We use  $B$  to denote a ball and  $\sigma B$ ,  $\sigma > 0$ , is the ball with the same center as  $B$  and with diameter  $\sigma \text{diam}(B)$ . A differential form  $u$  is called closed if  $du = 0$  and a differential form  $v$  is called coclosed if  $d^*v = 0$ .

*Definition 1.1.* Let  $A : \Omega \times \Lambda^\ell(\mathbb{R}^n) \rightarrow \Lambda^\ell(\mathbb{R}^n)$  and  $B : \Omega \times \Lambda^\ell(\mathbb{R}^n) \rightarrow \Lambda^{\ell-1}(\mathbb{R}^n)$  be two operators satisfying the conditions:

$$|A(x, \xi)| \leq a|\xi|^{p-1}, \quad A(x, \xi) \cdot \xi \geq |\xi|^p, \quad |B(x, \xi)| \leq b|\xi|^{p-1} \quad (1.12)$$

for almost every  $x \in \Omega$  and all  $\xi \in \Lambda^\ell(\mathbb{R}^n)$ . Then the nonlinear elliptic equation

$$d^*A(x, d\omega) = B(x, d\omega) \quad (1.13)$$

is called the nonhomogeneous  $A$ -harmonic equation for differential forms. Here  $a, b > 0$  are constants and  $1 < p < \infty$  is a fixed exponent associated with (1.13).

We should notice that if the operator  $B$  equals 0 in (1.13), then (1.13) reduces to the following homogeneous  $A$ -harmonic equation, or the  $A$ -harmonic equation:

$$d^*A(x, d\omega) = 0, \quad (1.14)$$

which has received much investigation during the recent years; see [3, 5, 7–11]. A solution to (1.13) is an element of the Sobolev space  $W_{\text{loc}}^{1,p}(\Omega, \Lambda^{\ell-1})$  such that

$$\int_{\Omega} A(x, d\omega) \cdot d\varphi + B(x, d\omega) \cdot \varphi = 0 \quad (1.15)$$

for all  $\varphi \in W_{\text{loc}}^{1,p}(\Omega, \Lambda^{\ell-1})$  with compact support. The solutions of the  $A$ -harmonic equation are called  $A$ -harmonic tensors. For any differential form  $\omega$  defined in a bounded and convex domain  $D$ , there is a decomposition

$$\omega = d(T\omega) + T(d\omega). \quad (1.16)$$

Using the operator  $T$ , we can define the  $l$ -form  $\omega_D \in D'(D, \Lambda^\ell)$  by

$$\omega_D = |D|^{-1} \int_D \omega(y) dy, \quad \ell = 0, \quad \text{and } \omega_D = d(T\omega), \quad \ell = 1, 2, \dots, n, \quad (1.17)$$

for all  $\omega \in L^p(D, \Lambda^\ell)$ ,  $1 \leq p < \infty$ . It is known that  $u_D$  is a closed form. Hence,  $u - u_D$  is still a solution of (1.13) whenever  $u$  is a solution of (1.13).

## 2. Preliminaries

The purpose of this section is to establish some preliminary results that will be used in the proof of our main theorems. In [6], the weighted Poincaré inequality for solutions of the nonhomogeneous  $A$ -harmonic equation was established. From [7], we have the following local Poincaré inequality.

**Lemma 2.1.** *Let  $u \in D'(\Omega, \wedge^\ell)$  be a differential form in a domain  $\Omega \subset \mathbb{R}^n$  and  $du \in L^s(\Omega, \wedge^{\ell+1})$ ,  $\ell = 0, 1, \dots, n$ . Assume that  $1 < s < \infty$ . Then*

$$\|u - u_B\|_{s,B} \leq C|B|^{1/n} \|du\|_{s,\sigma B} \quad (2.1)$$

for all balls  $B$  with  $\sigma B \subset \Omega$ . Here  $C$  is a constant independent of  $u$  and  $\sigma > 1$  is some constant.

From [7], we have the following local Caccioppoli-type inequality.

**Lemma 2.2.** *Let  $u \in D'(\Omega, \wedge^l)$ ,  $l = 0, 1, \dots, n$ , be a solution of the nonhomogeneous  $A$ -harmonic equation (1.13) in a domain  $\Omega \subset \mathbb{R}^n$  and let  $\rho > 1$  be some constant. Assume that  $1 < s < \infty$  is a fixed exponent associated with the  $A$ -harmonic equation (1.13). Then there exists a constant  $C$ , independent of  $u$ , such that*

$$\|du\|_{s,B} \leq C|B|^{-1/n} \|u - c\|_{s,\rho B} \quad (2.2)$$

for all balls  $B$  with  $\rho B \subset \Omega$  and all closed forms  $c$ .

The following weak reverse Hölder inequality appears in [7].

**Lemma 2.3.** *Let  $u$  be a solution of (1.13) in  $\Omega$  and  $0 < s, t < \infty$ . Then there exists a constant  $C$ , independent of  $u$ , such that*

$$\|u\|_{s,B} \leq C|B|^{(t-s)/st} \|u\|_{t,\sigma B} \quad (2.3)$$

for all balls or cubes  $B$  with  $\sigma B \subset \Omega$  for some  $\sigma > 1$ .

Now, we prove the following local Orlicz norm estimates.

**Proposition 2.4.** *Let  $u$  be a solution of (1.13) in  $\Omega$ ,  $\alpha > 0$ ,  $\sigma > 1$ , and  $1 < p < \infty$ . Then there exists a constant  $C$ , independent of  $u$ , such that*

$$\|u_B\|_{L^p(\log L)^\alpha(B)} \leq C \|u\|_{L^p(\log L)^\alpha(\sigma B)}, \quad (2.4)$$

$$\|u - u_B\|_{L^p(\log L)^\alpha(B)} \leq C \|u - c\|_{L^p(\log L)^\alpha(\sigma B)} \quad (2.5)$$

for all balls  $B$  with  $\sigma B \subset \Omega$  and  $\text{diam}(B) \geq d_0$ . Here  $d_0 > 0$  is a constant and  $c$  is any closed form.

*Proof.* Let  $B \subset \Omega$  be a ball with  $\text{diam}(B) \geq d_0 > 0$ . Choose  $\varepsilon > 0$  small enough and a constant  $M$  large enough such that  $|B|^{-\varepsilon/p^2} \leq M$ . From Lemma 2.3, we have

$$\|u_B\|_{p+\varepsilon, B} \leq C_1 |B|^{(p-(p+\varepsilon))/p(p+\varepsilon)} \|u_B\|_{p, \sigma B} \quad (2.6)$$

for some  $\sigma > 1$ . Similar to (3.4) in the proof of Theorem 3.1, we may assume that  $|u_B|/\|u_B\|_{p, B} \geq 1$  on  $B$ . For above  $\varepsilon > 0$ , there exists  $C_2 > 0$  such that

$$\log^\alpha \left( e + \frac{|u_B|}{\|u_B\|_{p, B}} \right) \leq C_2 \left( \frac{|u_B|}{\|u_B\|_{p, \sigma B}} \right)^\varepsilon. \quad (2.7)$$

From (2.6) and (2.7), it follows that

$$\begin{aligned} \|u_B\|_{L^p(\log L)^\alpha(B)} &= \left( \int_B |u_B|^p \log^\alpha \left( e + \frac{|u_B|}{\|u_B\|_{p, B}} \right) dx \right)^{1/p} \\ &\leq C_3 \left( \frac{1}{\|u_B\|_{p, \sigma B}^\varepsilon} \int_B |u_B|^{p+\varepsilon} dx \right)^{1/p} \\ &\leq \frac{C_3}{\|u_B\|_{p, \sigma B}^{\varepsilon/p}} \left( \left( \int_B |u_B|^{p+\varepsilon} dx \right)^{1/(p+\varepsilon)} \right)^{(p+\varepsilon)/p} \\ &\leq \frac{C_4}{\|u_B\|_{p, \sigma B}^{\varepsilon/p}} \left( |B|^{(p-(p+\varepsilon))/p(p+\varepsilon)} \|u_B\|_{p, \sigma B} \right)^{(p+\varepsilon)/p} \\ &\leq C_5 |B|^{-\varepsilon/p^2} \|u_B\|_{p, \sigma B}. \end{aligned} \quad (2.8)$$

From [17], we know that

$$\|u_B\|_{p, \sigma B} \leq C_6 \|u\|_{p, \sigma B}. \quad (2.9)$$

Putting (2.9) into (2.8) and noting that

$$\log^\alpha \left( e + \frac{|u|}{\|u\|_{p, \sigma B}} \right) \geq 1 \quad (2.10)$$

for  $\alpha > 0$ , we obtain

$$\|u_B\|_{L^p(\log L)^\alpha(B)} \leq C_7 \|u\|_{p, \sigma B} \leq C_8 \|u\|_{L^p(\log L)^\alpha(\sigma B)}. \quad (2.11)$$

This ends the proof of inequality (2.4). If  $c$  is a closed differential form, from (1.16) and (1.17), we find that

$$c = dT(c) + T(dc) = dT(c) = c_B. \quad (2.12)$$

Applying triangle inequality and (2.4), we conclude that

$$\begin{aligned}
 \|u - u_B\|_{L^p(\log L)^\alpha(B)} &= \|(u - c) - (u_B - c_B)\|_{L^p(\log L)^\alpha(B)} \\
 &= \|(u - c) - (u - c)_B\|_{L^p(\log L)^\alpha(B)} \\
 &\leq \|u - c\|_{L^p(\log L)^\alpha(B)} + \|(u - c)_B\|_{L^p(\log L)^\alpha(B)} \quad (2.13) \\
 &\leq \|u - c\|_{L^p(\log L)^\alpha(B)} + C_9 \|u - c\|_{L^p(\log L)^\alpha(\sigma B)} \\
 &\leq C_{10} \|u - c\|_{L^p(\log L)^\alpha(\sigma B)}
 \end{aligned}$$

for any closed form  $c$ . The proof of Proposition 2.4 has been completed.  $\square$

Next, extend the weak reverse Hölder inequality above to the case of Orlicz norms.

**Lemma 2.5.** *Let  $u$  be a solution of (1.13) in  $\Omega$ ,  $\sigma > 1$ , and  $0 < s, t < \infty$ . Then there exists a constant  $C$ , independent of  $u$ , such that*

$$\|u\|_{L^s(\log L)^\alpha(B)} \leq C |B|^{(t-s)/st} \|u\|_{L^t(\log L)^\beta(\sigma B)} \quad (2.14)$$

for any constants  $\alpha > 0$  and  $\beta > 0$ , and all balls  $B$  with  $\sigma B \subset \Omega$  and  $\text{diam}(B) \geq d_0 > 0$ , where  $d_0$  is a fixed constant.

The proof of Lemma 2.5 is similar to that of Proposition 2.4. For completeness, we prove Lemma 2.5 as follows.

*Proof.* For any ball  $B \subset \Omega$  with  $\text{diam}(B) \geq d_0 > 0$ , we may choose  $\varepsilon > 0$  small enough and a constant  $C_1$  such that

$$|B|^{-\varepsilon/st} \leq C_1. \quad (2.15)$$

From Lemma 2.3, we have

$$\|u\|_{s+\varepsilon, B} \leq C_2 |B|^{(t-(s+\varepsilon))/t(s+\varepsilon)} \|u\|_{t, \sigma B} \quad (2.16)$$

for some  $\sigma > 1$ . Similar to (3.5) in the proof of Theorem 3.1, we may assume that  $|u|/\|u\|_{t, B} \geq 1$  on  $B$ . For above  $\varepsilon > 0$ , there exists  $C_3 > 0$  such that

$$\log^\alpha \left( e + \frac{|u|}{\|u\|_{s, B}} \right) \leq C_3 \left( \frac{|u|}{\|u\|_{t, \sigma B}} \right)^\varepsilon. \quad (2.17)$$

From (2.16) and (2.17), we have

$$\begin{aligned}
\|u\|_{L^s(\log L)^\alpha(B)} &= \left( \int_B |u|^s \log^\alpha \left( e + \frac{|u|}{\|u\|_{s,B}} \right) dx \right)^{1/s} \\
&\leq C_4 \left( \frac{1}{\|u\|_{t,\sigma B}^\varepsilon} \int_B |u|^{(s+\varepsilon)} dx \right)^{1/s} \\
&\leq \frac{C_4}{\|u\|_{t,\sigma B}^{\varepsilon/s}} \left( \left( \int_B |u|^{s+\varepsilon} dx \right)^{1/(s+\varepsilon)} \right)^{(s+\varepsilon)/s} \\
&\leq \frac{C_5}{\|u\|_{t,\sigma B}^{\varepsilon/s}} \left( |B|^{(t-(s+\varepsilon))/t(s+\varepsilon)} \|u\|_{t,\sigma B} \right)^{(s+\varepsilon)/s} \\
&\leq C_6 |B|^{(t-s-\varepsilon)/st} \|u\|_{t,\sigma B}.
\end{aligned} \tag{2.18}$$

From (2.15) and (2.18) and using  $\log^\beta(e + |u|/\|u\|_{t,\sigma B}) \geq 1$ ,  $\beta > 0$ , we obtain

$$\begin{aligned}
\|u\|_{L^s(\log L)^\alpha(B)} &\leq C_6 |B|^{(t-s-\varepsilon)/st} \|u\|_{t,\sigma B} \\
&\leq C_7 |B|^{(t-s-\varepsilon)/st} \left( \int_{\sigma B} |u|^t \log^\beta \left( e + \frac{|u|}{\|u\|_{t,\sigma B}} \right) dx \right)^{1/t} \\
&\leq C_7 |B|^{(t-s-\varepsilon)/st} \|u\|_{L^t(\log L)^\beta(\sigma B)} \\
&\leq C_8 |B|^{(t-s)/st} \|u\|_{L^t(\log L)^\beta(\sigma B)}.
\end{aligned} \tag{2.19}$$

This ends the proof of Lemma 2.5. □

Using a similar method developed in the proof of Lemma 2.5 and from Lemma 2.9 in [6], we can prove the following version of the weak reverse Hölder inequality with Orlicz norms. Note that the following version of the weak reverse Hölder inequality cannot be obtained by replacing  $u$  by  $du$  in Lemma 2.5 since  $du$  may not be a solution of (1.13).

**Lemma 2.6.** *Let  $u$  be a solution of (1.13) in  $\Omega$ ,  $\sigma > 1$ , and  $0 < s, t < \infty$ . Then there exists a constant  $C$ , independent of  $u$ , such that*

$$\|du\|_{L^s(\log L)^\alpha(B)} \leq C |B|^{(t-s)/st} \|du\|_{L^t(\log L)^\beta(\sigma B)} \tag{2.20}$$

for all balls  $B$  with  $\sigma B \subset \Omega$  and  $\text{diam}(B) \geq d_0 > 0$ . Here  $d_0$  is a fixed constant, and  $\alpha > 0$  and  $\beta > 0$  are any constants.



It is easy to see that for any constant  $k$ , there exist constants  $m > 0$  and  $M > 0$ , such that

$$m \log(e+t) \leq \log\left(e + \frac{t}{k}\right) \leq M \log(e+t), \quad t > 0. \quad (2.21)$$

From the weak reverse Hölder inequality (Lemma 2.3), we know that the norms  $\|u\|_{s,B}$  and  $\|u\|_{t,B}$  are comparable when  $0 < d_1 \leq \text{diam}(B) \leq d_2 < \infty$ . Hence, we may assume that  $0 < m_1 \leq \|u\|_{s,B} \leq M_1 < \infty$  and  $0 < m_2 \leq \|u\|_{t,B} \leq M_2 < \infty$  for some constants  $m_i$  and  $M_i$ ,  $i = 1, 2$ . Thus, we have

$$\begin{aligned} C_1 \log(e+|u|) &\leq \log\left(e + \frac{|u|}{\|u\|_{s,B}}\right) \leq C_2 \log(e+|u|), \\ C_3 \log(e+|u|) &\leq \log\left(e + \frac{|u|}{\|u\|_{t,B}}\right) \leq C_4 \log(e+|u|) \end{aligned} \quad (2.22)$$

for any  $s > 0$  and  $t > 0$ , where  $C_i$  is a constant,  $i = 1, 2, 3, 4$ . Using (2.22), we obtain

$$C_5 \left( \int_B |u|^s \log^\alpha \left( e + \frac{|u|}{\|u\|_{t,B}} \right) dx \right)^{1/s} \leq \|u\|_{L^s(\log L)^\alpha(B)} \leq C_6 \left( \int_B |u|^s \log^\alpha \left( e + \frac{|u|}{\|u\|_{t,B}} \right) dx \right)^{1/s}, \quad (2.23)$$

$$C_7 \|u\|_{L^t(\log L)^\alpha(B)} \leq \left( \int_B |u|^t \log^\alpha \left( e + \frac{|u|}{\|u\|_{s,B}} \right) dx \right)^{1/t} \leq C_8 \|u\|_{L^t(\log L)^\alpha(B)} \quad (2.24)$$

for any ball  $B$  and any  $s > 0$ ,  $t > 0$ , and  $\alpha > 0$ . Consequently, we see that  $\|u\|_{L^s(\log L)^\alpha(B)} < \infty$  if and only if

$$\left( \int_B |u|^s \log^\alpha \left( e + \frac{|u|}{\|u\|_{t,B}} \right) dx \right)^{1/s} < \infty. \quad (2.25)$$

We recall the Muckenhoupt weights as follows. More properties and applications of Muckenhoupt weights can be found in [1].

*Definition 2.7.* A weight  $w(x)$  is called an  $A_r(E)$  weight in a set  $E \subset \mathbb{R}^n$  for  $r > 1$ , write  $w \in A_r(E)$ , if

$$\sup_B \left( \frac{1}{|B|} \int_B w dx \right) \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{1/(r-1)} dx \right)^{(r-1)} < \infty \quad (2.26)$$

for any ball  $B \subset E$ .

We will need the following reverse Hölder inequality for  $A_r(E)$ -weights.

**Lemma 2.8.** *If  $\omega \in A_r(E)$ ,  $r > 1$ , then there exist constants  $k > 1$  and  $C$ , independent of  $\omega$ , such that*

$$\|\omega\|_{k,Q} \leq C|q|^{(1-k)/k} \|\omega\|_{1,Q} \quad (2.27)$$

for all balls or cubes  $Q \subset E$ .

### 3. Caccioppoli-Type Estimates

In recent years different versions of Caccioppoli-type estimates have been established; see [1, 2, 4, 12–15, 17–19]. The Caccioppoli-type estimates have become powerful tools in analysis and related fields. The purpose of this section is to prove the following Caccioppoli-type estimates with  $L^p(\log L)^\alpha$ -norms for solutions to the nonhomogeneous  $A$ -harmonic equation.

**Theorem 3.1.** *Let  $u \in L^p(\log L)^\alpha(\Omega, \Lambda^\ell)$ ,  $\ell = 0, 1, \dots, n-1$ , be a solution to the nonhomogeneous  $A$ -harmonic equation (1.13) in  $\Omega \subset \mathbb{R}^n$ . Then, there exists a constant  $C$ , independent of  $u$ , such that*

$$\|du\|_{L^p(\log L)^\alpha(B)} \leq C|B|^{-1/n} \|u - c\|_{L^p(\log L)^\alpha(\sigma B)} \quad (3.1)$$

for some constant  $\sigma > 1$  and all balls  $B$  with  $\sigma B \subset \Omega$  and  $\text{diam}(B) \geq d_0 > 0$ . Here  $d_0$ ,  $1 < p < \infty$  and  $\alpha > 0$  are constants, and  $c \in L^p(\log L)^\alpha(\Omega, \Lambda^\ell)$  is any closed form.

*Proof.* Let  $B \subset \Omega$  be a ball with  $\text{diam}(B) \geq d_0 > 0$ . Let  $\varepsilon > 0$  be small enough and a constant  $C_1$  large enough such that

$$|B|^{-\varepsilon/p^2} \leq C_1. \quad (3.2)$$

Applying Lemma 2.9 in [6], we have

$$\|du\|_{p+\varepsilon, B} \leq C_2|B|^{(p-(p+\varepsilon))/p(p+\varepsilon)} \|du\|_{p, \sigma B} \quad (3.3)$$

for some  $\sigma > 1$ . We may assume that  $|du|/\|du\|_{p,B} \geq 1$  on  $B$ . Otherwise, setting  $B_1 = \{x \in B : |du|/\|du\|_{p,B} \geq 1\}$ ,  $B_2 = \{x \in B : |du|/\|du\|_{p,B} < 1\}$ , and using the elementary inequality

$|a + b|^s \leq 2^s(|a|^s + |b|^s)$ , where  $s > 0$  is any constant, we have

$$\begin{aligned} & \|du\|_{L^p(\log L)^\alpha(B)} \\ &= \left( \int_B |du|^p \log^\alpha \left( e + \frac{|du|}{\|du\|_{p,B}} \right) dx \right)^{1/p} \\ &= \left( \int_{B_1} |du|^p \log^\alpha \left( e + \frac{|du|}{\|du\|_{p,B}} \right) dx + \int_{B_2} |du|^p \log^\alpha \left( e + \frac{|du|}{\|du\|_{p,B}} \right) dx \right)^{1/p} \\ &\leq 2^{1/p} \left( \left( \int_{B_1} |du|^p \log^\alpha \left( e + \frac{|du|}{\|du\|_{p,B}} \right) dx \right)^{1/p} + \left( \int_{B_2} |du|^p \log^\alpha \left( e + \frac{|du|}{\|du\|_{p,B}} \right) dx \right)^{1/p} \right). \end{aligned} \tag{3.4}$$

First, we estimate the first term on the right. Since  $|du|/\|du\|_{p,B} > 1$  on  $B_1$ , then for  $\varepsilon > 0$  appeared in (3.2), there exists  $C_3 > 0$  such that

$$\log^\alpha \left( e + \frac{|du|}{\|du\|_{p,B}} \right) \leq C_3 \left( \frac{|du|}{\|du\|_{p,\sigma_1 B}} \right)^\varepsilon. \tag{3.5}$$

Combining (3.2), (3.3), and (3.5), we obtain

$$\begin{aligned} \left( \int_{B_1} |du|^p \log^\alpha \left( e + \frac{|du|}{\|du\|_{p,B}} \right) dx \right)^{1/p} &\leq C_4 \left( \frac{1}{\|du\|_{p,\sigma_1 B}^\varepsilon} \int_{B_1} |du|^{p+\varepsilon} dx \right)^{1/p} \\ &\leq C_4 \left( \frac{1}{\|du\|_{p,\sigma_1 B}^\varepsilon} \int_B |du|^{p+\varepsilon} dx \right)^{1/p} \\ &= \frac{C_4}{\|du\|_{p,\sigma_1 B}^{\varepsilon/p}} \left( \left( \int_B |du|^{p+\varepsilon} dx \right)^{1/(p+\varepsilon)} \right)^{(p+\varepsilon)/p} \\ &\leq \frac{C_5}{\|du\|_{p,\sigma_1 B}^{\varepsilon/p}} \left( |B|^{(p-(p+\varepsilon))/p(p+\varepsilon)} \|du\|_{p,\sigma_1 B} \right)^{(p+\varepsilon)/p} \\ &\leq C_6 \|du\|_{p,\sigma_1 B}, \end{aligned} \tag{3.6}$$

where  $\sigma_1 > 1$  is a constant. Since

$$\log^\alpha \left( e + \frac{|du|}{\|du\|_{p,B}} \right) \leq M_1 \log^\alpha(e + 1) \leq M_2, \quad x \in B_2, \tag{3.7}$$

we can estimate the second term similarly

$$\left( \int_{B_2} |du|^p \log^\alpha \left( e + \frac{|du|}{\|du\|_{p,B}} \right) dx \right)^{1/p} \leq C_7 \|du\|_{p,\sigma_2 B}, \quad (3.6)'$$

where  $\sigma_2 > 1$  is a constant. From (3.4), (3.6), and (3.6)', we have

$$\|du\|_{L^p(\log L)^\alpha(B)} \leq C_8 \|du\|_{p,\sigma_3 B}, \quad (3.8)$$

where  $\sigma_3 = \max\{\sigma_1, \sigma_2\}$ . By Lemma 2.2, we obtain

$$\|du\|_{p,\sigma_3 B} \leq C_9 |B|^{-1/n} \|u - c\|_{p,\sigma_4 B} \quad (3.9)$$

for some  $\sigma_4 > \sigma_3$  and all closed forms  $c$ . Note that

$$\log^\alpha \left( e + \frac{|u - c|}{\|u - c\|_{p,\sigma_2 B}} \right) \geq 1, \quad \alpha > 0. \quad (3.5)'$$

Combining last three inequalities, we obtain

$$\|du\|_{L^p(\log L)^\alpha(B)} \leq C_{10} |B|^{-1/n} \|u - c\|_{p,\sigma_4 B} \leq C_{10} |B|^{-1/n} \|u - c\|_{L^p(\log L)^\alpha(\sigma_4 B)}. \quad (3.10)$$

The proof of Theorem 3.1 has been completed.  $\square$

If we revise (3.5) and (3.5)' in the proof of Theorem 3.1, we obtain the following version of Caccioppoli-type estimate.

**Corollary 3.2.** *Let  $u \in L^p(\log L)^\alpha(\Omega, \Lambda^\ell)$ ,  $\ell = 0, 1, \dots, n-1$ , be a solution to the nonhomogeneous A-harmonic equation (1.13) in  $\Omega \subset \mathbb{R}^n$ . Then, there exists a constant  $C$ , independent of  $u$ , such that*

$$\left( \int_B |du|^p \log^\alpha \left( e + \frac{|du|}{\|du\|_{p,\Omega}} \right) dx \right)^{1/p} \leq \frac{C}{\text{diam}(B)} \left( \int_{\sigma B} |u - c|^p \log^\alpha \left( e + \frac{|u - c|}{\|u - c\|_{p,\Omega}} \right) dx \right)^{1/p} \quad (3.11)$$

for some constant  $\sigma > 1$  and all balls  $B$  with  $\sigma B \subset \Omega$  and  $\text{diam}(B) \geq d_0 > 0$ . Here  $d_0, 1 < p < \infty$  and  $\alpha > 0$  are constants, and  $c \in L^p(\log L)^\alpha(\Omega, \Lambda^\ell)$  is any closed form.

**Theorem 3.3.** *Let  $u \in L^p(\log L)^\alpha(\Omega, \Lambda^\ell)$ ,  $\ell = 0, 1, \dots, n-1$ , be a solution to the nonhomogeneous A-harmonic equation (1.13) in a bounded domain  $\Omega \subset \mathbb{R}^n$  and  $w(x) \in A_r(\Omega)$  for some  $r > 1$ . Then, there exists a constant  $C$ , independent of  $u$ , such that*

$$\|du\|_{L^p(\log L)^\alpha(B,w)} \leq C |B|^{-1/n} \|u - c\|_{L^p(\log L)^\alpha(\sigma B,w)} \quad (3.12)$$

for any closed form  $c$ , some constant  $\sigma > 1$  and all balls  $B$  with  $\sigma B \subset \Omega$  and  $\text{diam}(B) \geq d_0 > 0$ . Here  $d_0, 1 < p < \infty$  and  $\alpha > 0$  are constants, and  $c \in L^p(\log L)^\alpha(\Omega, \Lambda^\ell)$  is any closed form.

*Proof.* Let  $B$  be a ball with  $\sigma B \subset \Omega$  and  $\text{diam}(B) \geq d_0 > 0$ . Since  $\Omega$  is bounded, then  $d_0 \leq \text{diam}(B) \leq \text{diam}(\Omega) < \infty$ . Thus,  $0 < v_1 \leq |B| \leq v_2 < \infty$  for some constants  $v_1$  and  $v_2$ . By Lemma 2.3, we have

$$m_1 \|u\|_{s, \rho_1 B} \leq \|u\|_{t, B} \leq m_2 \|u\|_{s, \rho_2 B} \quad (3.13)$$

for any solution  $u$  of (1.13) and any constants  $s, t > 0$ , where  $0 < \rho_1 < 1$ ,  $\rho_2 > 1$ ,  $0 < m_1 < 1$ , and  $m_2 > 1$  are some constants. By Lemma 2.8, there exist constants  $k > 1$  and  $C_0 > 0$ , such that

$$\|w\|_{k, B} \leq C_0 |B|^{(1-k)/k} \|w\|_{1, B}. \quad (3.14)$$

Choose  $s = pk/(k-1)$ , then  $1 < p < s$  and  $k = s/(s-p)$ . We know that  $u \in L^p(\log L)^\alpha(\Omega, \Lambda^\ell)$  implies  $u \in L^p(\Omega, \Lambda^\ell)$ . Then, for any closed form  $c \in L^p(\log L)^\alpha(\Omega, \Lambda^\ell)$ , it follows that  $u - c \in L^p(\log L)^\alpha(\Omega, \Lambda^\ell)$ . Thus,  $u - c \in L^p(\Omega, \Lambda^\ell)$ . By Caccioppoli inequality with  $L^p$ -norms, we know that  $du \in L^p(\Omega, \Lambda^\ell)$  which gives  $\|du\|_{p, \Omega} = N < \infty$ . If  $\|du\|_{p, B} = 0$ , then  $du = 0$  a.e. on  $B$  and Theorem 3.3 follows. Thus, we may assume that  $0 < m_1 \leq \|du\|_{s, B} < M_1$  and  $0 < m_2 \leq \|du\|_{p, B} < M_2$  by (3.13). Since  $1/p = 1/s + (s-p)/ps$ , by the Hölder inequality, (3.14) and (2.23), we have

$$\begin{aligned} \|du\|_{L^p(\log L)^\alpha(B, w)} &= \left( \int_B |du|^p \log^\alpha \left( e + \frac{|du|}{\|du\|_{p, B}} \right) w \, dx \right)^{1/p} \\ &= \left( \int_B \left( |du| \log^{\alpha/p} \left( e + \frac{|du|}{\|du\|_{p, B}} \right) w^{1/p} \right)^p dx \right)^{1/p} \\ &\leq \left( \int_B |du|^s \log^{\alpha s/p} \left( e + \frac{|du|}{\|du\|_{p, B}} \right) dx \right)^{1/s} \left( \int_B w^{s/(s-p)} dx \right)^{(s-p)/sp} \\ &\leq C_1 \left( \int_B |du|^s \log^{\alpha s/p} \left( e + \frac{|du|}{\|du\|_{s, B}} \right) dx \right)^{1/s} \left( \left( \int_B w^k dx \right)^{1/k} \right)^{1/p} \\ &\leq C_2 |B|^{(1-k)/kp} \|w\|_{1, B}^{1/p} \left( \int_B |du|^s \log^{\alpha s/p} \left( e + \frac{|du|}{\|du\|_{s, B}} \right) dx \right)^{1/s}. \end{aligned} \quad (3.15)$$

Applying Theorem 3.1 yields

$$\left( \int_B |du|^s \log^{\alpha s/p} \left( e + \frac{|du|}{\|du\|_{s, B}} \right) dx \right)^{1/s} \leq C_3 |B|^{-1/n} \|u - c\|_{L^s(\log L)^{\alpha s/p}(\sigma_1 B)}. \quad (3.16)$$

Here  $c$  is any closed form. Next, choose  $t = p/r$ . Using (3.16) and Lemma 2.5 with  $\beta = \alpha/r$ , we obtain

$$\left( \int_B |du|^s \log^{\alpha s/p} \left( e + \frac{|du|}{\|du\|_{p,B}} \right) dx \right)^{1/s} \leq C_4 |B|^{-1/n} |B|^{(t-s)/st} \|u - c\|_{L^t(\log L)^\beta(\sigma_2 B)} \quad (3.17)$$

for some  $\sigma_2 > \sigma_1$ . Using the Hölder inequality again with  $1/t = 1/p + (p-t)/pt$ , we obtain

$$\begin{aligned} & \|u - c\|_{L^t(\log L)^\beta(\sigma_2 B)} \\ &= \left( \int_{\sigma_2 B} |u - c|^t \log^\beta \left( e + \frac{|u - c|}{\|u - c\|_{t, \sigma_2 B}} \right) dx \right)^{1/t} \\ &= \left( \int_{\sigma_2 B} \left( |u - c| \log^{\beta/t} \left( e + \frac{|u - c|}{\|u - c\|_{t, \sigma_2 B}} \right) w^{1/p} w^{-1/p} \right)^t dx \right)^{1/t} \\ &\leq \left( \int_{\sigma_2 B} |u - c|^p \log^{\beta p/t} \left( e + \frac{|u - c|}{\|u - c\|_{t, \sigma_2 B}} \right) w dx \right)^{1/p} \left( \int_{\sigma_2 B} \left( \frac{1}{w} \right)^{t/(p-t)} dx \right)^{(p-t)/pt} \\ &\leq \|u - c\|_{L^p(\log L)^\alpha(\sigma_2 B, w)} \left( \int_{\sigma_2 B} \left( \frac{1}{w} \right)^{1/(r-1)} dx \right)^{(r-1)/p}. \end{aligned} \quad (3.18)$$

Combining (3.15), (3.17), and (3.18), we conclude that

$$\begin{aligned} & \|du\|_{L^p(\log L)^\alpha(B, w)} \\ &\leq C_5 |B|^{-r/p-1/n} \|u - c\|_{L^p(\log L)^\alpha(\sigma_2 B, w)} \left( \int_B w dx \left( \int_{\sigma_2 B} \left( \frac{1}{w} \right)^{1/(r-1)} dx \right)^{(r-1)} \right)^{1/p} \\ &\leq C_5 |B|^{-r/p-1/n} \|u - c\|_{L^p(\log L)^\alpha(\sigma_2 B, w)} \left( \|w\|_{1, \sigma_2 B} \cdot \left\| \frac{1}{w} \right\|_{1/(r-1), \sigma_2 B} \right)^{1/p}. \end{aligned} \quad (3.19)$$

Since  $w \in A_r(\Omega)$ , then

$$\begin{aligned} & \left( \|w\|_{1,\sigma_2 B} \cdot \left\| \frac{1}{w} \right\|_{1/(r-1),\sigma_2 B} \right)^{1/p} \\ &= \left( \left( \int_{\sigma_2 B} w \, dx \right) \left( \int_{\sigma_2 B} \left( \frac{1}{w} \right)^{1/(r-1)} dx \right)^{r-1} \right)^{1/p} \\ &= \left( |\sigma_2 B|^r \left( \frac{1}{|\sigma_2 B|} \int_{\sigma_2 B} w \, dx \right) \left( \frac{1}{|\sigma_2 B|} \int_{\sigma_2 B} \left( \frac{1}{w} \right)^{1/(r-1)} dx \right)^{r-1} \right)^{1/p} \\ &\leq C_6 |B|^{r/p}. \end{aligned} \tag{3.20}$$

Substituting the last inequality into (3.19) it follows obviously that

$$\|du\|_{L^p(\log L)^\alpha(B,w)} \leq C_7 |B|^{-1/n} \|u - c\|_{L^p(\log L)^\alpha(\sigma_2 B,w)}. \tag{3.21}$$

This ends the proof of Theorem 3.3. □

Let  $\alpha = 1$  in Theorem 3.3; we obtain the following corollary.

**Corollary 3.4.** *Let  $u \in L^p(\log L)(\Omega, \Lambda^\ell)$ ,  $\ell = 0, 1, \dots, n - 1$ , be a solution to the nonhomogeneous  $A$ -harmonic equation (1.13) in a bounded domain  $\Omega \subset \mathbb{R}^n$  and  $w(x) \in A_r(\Omega)$  for some  $r > 1$ . Then, there exists a constant  $C$ , independent of  $u$ , such that*

$$\|du\|_{L^p(\log L)(B,w)} \leq C |B|^{-1/n} \|u - c\|_{L^p(\log L)(\sigma B,w)} \tag{3.22}$$

for any closed form  $c$ , some constant  $\sigma > 1$  and all balls  $B$  with  $\sigma B \subset \Omega$  and  $\text{diam}(B) \geq d_0 > 0$ . Here  $d_0$  and  $1 < p < \infty$  are constants, and  $c \in L^p(\log L)(\Omega, \Lambda^\ell)$  is any closed form.

We know that if  $w \in A_r(E)$  and  $0 < \lambda \leq 1$ , then  $w^\lambda \in A_r(E)$ . Thus, under the same conditions of Theorem 3.3, we also have the following estimate:

$$\|du\|_{L^p(\log L)^\alpha(B,w^\lambda)} \leq C |B|^{-1/n} \|u - c\|_{L^p(\log L)^\alpha(\sigma B,w^\lambda)}, \tag{3.23}$$

where  $c$  is any closed form, and  $0 < \lambda \leq 1$  and  $\alpha > 0$  are any constants. Choose  $\lambda = 1/p$ ,  $1 < p < \infty$ , in (3.23). Then, for closed form  $c$  and any constant  $\alpha > 0$ , we have

$$\|du\|_{L^p(\log L)^\alpha(B,w^{1/p})} \leq C |M|^{-1/n} \|u - c\|_{L^p(\log L)^\alpha(\sigma B,w^{1/p})}. \tag{3.24}$$

We have proved Caccioppoli-type inequalities with  $L^p(\log L)^\alpha$ -norms for solutions to the nonhomogeneous  $A$ -harmonic equation. Using the same method developed in [12], we can obtain the more general version of the Caccioppoli-type inequality for differential forms satisfying certain conditions. A special useful Young function  $\psi : [0, \infty) \rightarrow [0, \infty)$ , termed

an  $N$ -function, is a continuous Young function such that  $\varphi(x) = 0$  if and only if  $x = 0$  and  $\lim_{x \rightarrow 0} \varphi(x)/x = 0$ ,  $\lim_{x \rightarrow \infty} \varphi(x)/x = +\infty$ . We say that a differential form  $u \in W_{\text{loc}}^{1,1}(\Omega, \Lambda^\ell)$  is a  $k$ -quasiminimizer for the functional

$$I(\Omega; v) = \int_{\Omega} \varphi(|dv|) dx \quad (3.25)$$

if and only if, for every  $\phi \in W_{\text{loc}}^{1,1}(\Omega, \Lambda^\ell)$  with compact support,

$$I(\text{supp } \phi; u) \leq k \cdot I(\text{supp } \phi; u + \phi), \quad (3.26)$$

where  $k > 1$  is a constant. We say that  $\varphi$  satisfies the so-called  $\Delta_2$ -condition if there exists a constant  $p > 1$  such that  $\varphi(2t) \leq p\varphi(t)$  for all  $t > 0$ , from which it follows that

$$\varphi(\lambda t) \leq \lambda^p \varphi(t) \quad (3.27)$$

for any  $t > 0$  and  $\lambda \geq 1$ ; see [12].

We will need the following lemma which can be found in [19] or [12].

**Lemma 3.5.** *Let  $f(t)$  be a nonnegative function defined on the interval  $[a, b]$  with  $a \geq 0$ . Suppose that for  $s, t \in [a, b]$  with  $t < s$ ,*

$$f(t) \leq \frac{A}{(s-t)^\alpha} + B + \theta f(s) \quad (3.28)$$

*holds, where  $A, B, \alpha$ , and  $\theta$  are nonnegative constants with  $\theta < 1$ . Then, there exists a constant  $C = C(\alpha, \theta)$  such that*

$$f(\rho) \leq C \left( \frac{A}{(R-\rho)^\alpha} + B \right) \quad (3.29)$$

*for any  $\rho, R \in [a, b]$  with  $\rho < R$ .*

**Theorem 3.6.** *Let  $u$  be a  $k$ -quasiminimizer for the functional (3.25) and let  $\varphi$  be a Young function satisfying the  $\Delta_2$ -condition. Then, for any ball  $B_R \subset \Omega$  with radius  $R$ , there exists a constant  $C$ , independent of  $u$ , such that*

$$\int_{B_{R/2}} \varphi(|du|) dx \leq C \int_{B_R} \varphi\left(\frac{|u-c|}{R}\right) dx, \quad (3.30)$$

*where  $c$  is any closed form.*

The proof of Theorem 3.6 is the same as that of Theorem 6.1 developed in [12]. For the complete purpose, we include the proof of Theorem 3.6 as follows.



*Proof.* Let  $B_R = B(x_0, R) \subset \mathbb{R}^n$  be a ball with radius  $R$  and center  $x_0$ ,  $R/2 < t < s < R$ . Set  $\eta(x) = g(|x - x_0|)$ , where

$$g(\tau) = \begin{cases} 1, & 0 \leq \tau \leq t, \\ \text{affine}, & t < \tau < s, \\ 0, & \tau \geq s. \end{cases} \quad (3.31)$$

Then,  $\eta \in W_0^{1,\infty}(B_s)$ ,  $\eta(x) = 1$  on  $B_t$ , and

$$|d\eta(x)| = \begin{cases} (s-t)^{-1}, & t \leq |x - x_0| \leq s, \\ 0, & \text{otherwise.} \end{cases} \quad (3.32)$$

Let  $v(x) = u(x) + (\eta(x))^p(c - u(x))$ . We find that

$$dv = (1 - \eta^p)du + \eta^p p \frac{d\eta}{\eta} (c - u(x)). \quad (3.33)$$

Since  $\psi$  is an increasing convex function satisfying the  $\Delta_2$ -condition, we obtain

$$\psi(|dv|) \leq (1 - \eta^p)\psi(|du|) + \eta^p \psi\left(p \frac{|d\eta|}{\eta} |c - u(x)|\right). \quad (3.34)$$

Using the definition of the  $k$ -quasiminimizer and (3.27), it follows that

$$\begin{aligned} \int_{B_s} \psi(|du|) dx &\leq k \int_{B_s} \psi(|dv|) dx \\ &\leq k \left( \int_{B_s \setminus B_t} (1 - \eta^p) \psi(|du|) dx + \int_{B_s} \eta^p \psi\left(p \frac{|d\eta|}{\eta} |c - u(x)|\right) dx \right) \\ &\leq k \left( \int_{B_s \setminus B_t} \psi(|du|) dx + p^p \int_{B_s} \psi(|d\eta| |u - c|) dx \right). \end{aligned} \quad (3.35)$$

Applying (3.35), (3.32), and (3.27), we have

$$\begin{aligned} \int_{B_t} \psi(|du|) dx &\leq \int_{B_s} \psi(|du|) dx \\ &\leq k \left( \int_{B_s \setminus B_t} \psi(|du|) dx + p^p \int_{B_s} \psi\left(2R \frac{|u - c|}{s - tR}\right) dx \right) \\ &\leq k \left( \int_{B_s \setminus B_t} \psi(|du|) dx + \frac{(2pR)^p}{(s-t)^p} \int_{B_s} \psi\left(\frac{|u - c|}{R}\right) dx \right). \end{aligned} \quad (3.36)$$

Adding  $k \int_{B_t} \psi(|du|)dx$  to both sides of inequality (3.36) yields

$$\int_{B_t} \psi(|du|)dx \leq \frac{k}{k+1} \left( \int_{B_s} \psi(|du|)dx + \frac{(2pR)^p}{(s-t)^p} \int_{B_s} \psi\left(\frac{|u-c|}{R}\right)dx \right). \quad (3.37)$$

Next, write  $f(t) = \int_{B_t} \psi(|du|)dx$ ,  $f(s) = \int_{B_s} \psi(|du|)dx$ ,  $A = (2pR)^p \int_{B_s} \psi(|u-c|/R)dx$ , and  $B = 0$ . From (3.37), we find that the conditions of Lemma 3.5 are satisfied. Hence, using Lemma 3.5 with  $\rho = R/2$  and  $\alpha = p$ , we obtain (3.30) immediately. The proof of Theorem 3.6 has been completed.  $\square$

It should be noticed that  $c \in L^p(\log L)^\alpha(\Omega, \Lambda^\ell)$  is any closed form on the right side of each version of the Caccioppoli-type inequality. Hence, we may choose  $c = 0$  in each of the above Caccioppoli-type inequalities. For example, if we choose  $c = 0$  in Theorem 3.1 and Theorem 3.6, we obtain the following Corollaries 3.7 and 3.8, respectively, which can be considered as the special version of the Caccioppoli-type inequality.

**Corollary 3.7.** *Let  $u \in L^p(\log L)^\alpha(\Omega, \Lambda^\ell)$ ,  $\ell = 0, 1, \dots, n-1$ , be a solution to the nonhomogeneous  $A$ -harmonic equation (1.13) in  $\Omega \subset \mathbb{R}^n$ . Then, there exists a constant  $C$ , independent of  $u$ , such that*

$$\|du\|_{L^p(\log L)^\alpha(B)} \leq C|B|^{-1/n} \|u\|_{L^p(\log L)^\alpha(\sigma B)} \quad (3.38)$$

for some constant  $\sigma > 1$  and all balls  $B$  with  $\sigma B \subset \Omega$  and  $\text{diam}(B) \geq d_0 > 0$ . Here  $d_0$ ,  $1 < p < \infty$  and  $\alpha > 0$  are constants.

**Corollary 3.8.** *Let  $u$  be a  $k$ -quasiminimizer for the functional (3.25) and  $\psi$  be a Young function satisfying the  $\Delta_2$ -condition. Then, for any ball  $B_R \subset \Omega$  with radius  $R$ , there exists a constant  $C$ , independent of  $u$ , such that*

$$\int_{B_{R/2}} \psi(|du|)dx \leq C \int_{B_R} \psi\left(\frac{|u|}{R}\right)dx. \quad (3.39)$$

## 4. Poincaré Inequalities

In this section, we focus our attention on the local and global Poincaré inequalities with  $L^p(\log L)^\alpha$ -norms. The main result for this section is Theorem 4.2, the global Poincaré inequality for solutions of the nonhomogeneous  $A$ -harmonic equation. The following definition of  $L^\varphi(\mu)$ -domains can be found in [1].

*Definition 4.1.* Let  $\varphi$  be a Young function on  $[0, \infty)$  with  $\varphi(0) = 0$ . We call a proper subdomain  $\Omega \subset \mathbb{R}^n$  an  $L^\varphi(\mu)$ -domain, if there exists a constant  $C$  such that

$$\int_{\Omega} \varphi(\sigma|u - u_\Omega|)d\mu \leq C \sup_{B \subset \Omega} \int_B \varphi(\sigma\tau|u - u_B|)d\mu \quad (4.1)$$

for all  $u$  such that  $\varphi(|u|) \in L^1_{\text{loc}}(\Omega; \mu)$ , where the measure  $\mu$  is defined by  $d\mu = w(x)dx$ ,  $w(x)$  is a weight and  $\tau, \sigma$  are constants with  $0 < \tau \leq 1$ ,  $0 < \sigma \leq 1$ , and the supremum is over all balls  $B \subset \Omega$ .

**Theorem 4.2.** Assume that  $\Omega \subset \mathbb{R}^n$  is a bounded  $L^\varphi(\mu)$ -domain with  $\varphi(t) = t^p \log^\alpha(e + t/k)$ , where  $k = \|u - u_{B_0}\|_{p,\Omega}$ ,  $1 < p < \infty$ , and  $B_0 \subset \Omega$  is a fixed ball. Let  $u \in D'(\Omega, \Lambda^0)$  be a solution of the nonhomogeneous  $A$ -harmonic equation in  $\Omega$  and  $du \in L^p(\Omega, \Lambda^1)$  as well as  $\omega \in A_r(\Omega)$  for some  $r > 1$ . Then, there is a constant  $C$ , independent of  $u$ , such that

$$\|u - u_\Omega\|_{L^p(\log L)^\alpha(\Omega, \omega)} \leq C |\Omega|^{1/n} \|du\|_{L^p(\log L)^\alpha(\Omega, \omega)} \quad (4.2)$$

for any constant  $\alpha > 0$ .

To prove Theorem 4.2, we need the following local Poincaré inequalities, Theorems 4.3 and 4.4, with Orlicz norms.

**Theorem 4.3.** Let  $u \in D'(\Omega, \Lambda^\ell)$  be a solution of the nonhomogeneous  $A$ -harmonic equation in a domain  $\Omega \subset \mathbb{R}^n$  and  $du \in L^p(\Omega, \Lambda^{\ell+1})$ ,  $\ell = 0, 1, \dots, n$ . Assume that  $1 < p < \infty$ . Then, there is a constant  $C$ , independent of  $u$ , such that

$$\|u - u_B\|_{L^p(\log L)^\alpha(B)} \leq C |B|^{1/n} \|du\|_{L^p(\log L)^\alpha(\rho B)} \quad (4.3)$$

for all balls  $B$  with  $\rho B \subset \Omega$  and  $\text{diam}(B) \geq d_0$ . Here  $\alpha > 0$  is any constant and  $\rho > 1$  and  $d_0 > 0$  are some constants.

*Proof.* Let  $B \subset \Omega$  be a ball with  $\text{diam}(B) \geq d_0 > 0$ . Choose  $\varepsilon > 0$  small enough and a constant  $C_1$  such that

$$|B|^{-\varepsilon/p^2} \leq C_1. \quad (4.4)$$

From Lemma 2.3, we have

$$\|u - u_B\|_{p+\varepsilon, B} \leq C_2 |B|^{(p-(p+\varepsilon))/p(p+\varepsilon)} \|u - u_B\|_{p, \rho_1 B} \quad (4.5)$$

for some  $\rho_1 > 1$ . Similar to the proof of Theorem 3.1, we may assume that  $|u - u_B| / \|u - u_B\|_{p, B} \geq 1$ . Hence, for above  $\varepsilon > 0$ , there exists  $C_3 > 0$  such that

$$\log^\alpha \left( e + \frac{|u - u_B|}{\|u - u_B\|_{p, B}} \right) \leq C_3 \left( \frac{|u - u_B|}{\|u - u_B\|_{p, \rho_1 B}} \right)^\varepsilon. \quad (4.6)$$

From (4.5) and (4.6) and Lemma 2.1, we have

$$\begin{aligned}
\|u - u_B\|_{L^p(\log L)^\alpha(B)} &= \left( \int_B |u - u_B|^p \log^\alpha \left( e + \frac{|u - u_B|}{\|u - u_B\|_{p,B}} \right) dx \right)^{1/p} \\
&\leq C_4 \left( \frac{1}{\|u - u_B\|_{p,\rho_1 B}^\varepsilon} \int_B |u - u_B|^{p+\varepsilon} dx \right)^{1/p} \\
&\leq \frac{C_4}{\|u - u_B\|_{p,\rho_1 B}^{\varepsilon/p}} \left( \left( \int_B |u - u_B|^{p+\varepsilon} dx \right)^{1/(p+\varepsilon)} \right)^{(p+\varepsilon)/p} \quad (4.7) \\
&\leq \frac{C_5}{\|u - u_B\|_{p,\rho_1 B}^{\varepsilon/p}} \left( |B|^{(p-(p+\varepsilon))/p(p+\varepsilon)} \|u - u_B\|_{p,\rho_1 B} \right)^{(p+\varepsilon)/p} \\
&\leq C_6 |B|^{-\varepsilon/p^2} \|u - u_B\|_{p,\rho_1 B} \\
&\leq C_6 |B|^{-\varepsilon/p^2} |B|^{1/n} \|du\|_{p,\rho_2 B}
\end{aligned}$$

for some  $\rho_2 > \rho_1$ . For any  $\alpha > 0$ , we have

$$\log^\alpha \left( e + \frac{|du|}{\|du\|_{p,\rho_2 B}} \right) \geq 1. \quad (4.8)$$

Combining (4.8), (4.7), and (4.4), we obtain

$$\begin{aligned}
\|u - u_B\|_{L^p(\log L)^\alpha(B)} &\leq C_6 |B|^{-\varepsilon/p^2} |B|^{1/n} \|du\|_{p,\rho_2 B} \\
&\leq C_6 |B|^{-\varepsilon/p^2} |B|^{1/n} \|du\|_{L^p(\log L)^\alpha(\rho_2 B)} \quad (4.9) \\
&\leq C_7 |B|^{1/n} \|du\|_{L^p(\log L)^\alpha(\rho_2 B)}.
\end{aligned}$$

The proof of Theorem 4.3 has been completed.  $\square$

**Theorem 4.4.** *Let  $u \in D'(\Omega, \Lambda^\ell)$  be a solution of the nonhomogeneous  $A$ -harmonic equation in a domain  $\Omega \subset \mathbb{R}^n$  and  $du \in L^p(\Omega, \Lambda^{\ell+1})$ ,  $\ell = 0, 1, \dots, n$ . Assume that  $1 < p < \infty$  and  $w \in A_r(\Omega)$  for some  $r > 1$ . Then, there is a constant  $C$ , independent of  $u$ , such that*

$$\|u - u_B\|_{L^p(\log L)^\alpha(B,w)} \leq C |B|^{1/n} \|du\|_{L^p(\log L)^\alpha(\sigma B,w)} \quad (4.10)$$

for all balls  $B$  with  $\sigma B \subset \Omega$  and  $\text{diam}(B) \geq d_0$ . Here  $\alpha > 0$  is any constant, and  $\sigma > 1$  and  $d_0 > 0$  are some constants.

The proof of Theorem 4.4 is similar to that of Theorem 3.3. For completeness of the paper, we prove Theorem 4.4 as follows.

*Proof.* Choose  $s = kp/(k - 1)$ , where  $k > 1$  is a constant involved in (3.14). Using the Hölder inequality with  $1/p = 1/s + (s - p)/sp$ , (3.14), and (2.23), we obtain

$$\begin{aligned}
 & \|u - u_B\|_{L^p(\log L)^\alpha(B,w)} \\
 &= \left( \int_B \left( |u - u_B| \log^{\alpha/p} \left( e + \frac{|u - u_B|}{\|u - u_B\|_{p,B}} \right) w^{1/p} \right)^p dx \right)^{1/p} \\
 &\leq \left( \int_B |u - u_B|^s \log^{\alpha s/p} \left( e + \frac{|u - u_B|}{\|u - u_B\|_{p,B}} \right) dx \right)^{1/s} \left( \int_B w^{s/(s-p)} dx \right)^{(s-p)/ps} \\
 &\leq C_1 \left( \int_B |u - u_B|^s \log^{\alpha s/p} \left( e + \frac{|u - u_B|}{\|u - u_B\|_{s,B}} \right) dx \right)^{1/s} \left( \int_B w^k dx \right)^{1/k} \\
 &\leq C_2 |B|^{(1-k)/kp} \|w\|_{1,B}^{1/p} \left( \int_B |u - u_B|^s \log^{\alpha s/p} \left( e + \frac{|u - u_B|}{\|u - u_B\|_{s,B}} \right) dx \right)^{1/s}.
 \end{aligned} \tag{4.11}$$

Applying Theorem 4.3 yields

$$\left( \int_B |u - u_B|^s \log^{\alpha s/p} \left( e + \frac{|u - u_B|}{\|u - u_B\|_{s,B}} \right) dx \right)^{1/s} \leq C_2 |B|^{1/n} \|du\|_{L^s(\log L)^{\alpha s/p}(\sigma_1 B)}, \tag{4.12}$$

where  $\sigma_1 > 1$  is some constant. Let  $t = p/r$ . Using Lemma 2.6 with  $\beta = \alpha/r$ , we have

$$\left( \int_{\sigma_1 B} |du|^s \log^{\alpha s/p} \left( e + \frac{|du|}{\|du\|_{s,B}} \right) dx \right)^{1/s} \leq C_2 |B|^{(t-s)/st} \|du\|_{L^t(\log L)^\beta(\sigma_2 B)} \tag{4.13}$$

for some  $\sigma_2 > \sigma_1$ . Using the Hölder inequality again with  $1/t = 1/p + (p - t)/pt$ , we obtain

$$\begin{aligned}
 \|du\|_{L^t(\log L)^\beta(\sigma_2 B)} &= \left( \int_{\sigma_2 B} |du|^t \log^\beta \left( e + \frac{|du|}{\|du\|_{t,\sigma_2 B}} \right) dx \right)^{1/t} \\
 &= \left( \int_{\sigma_2 B} \left( |du| \log^{\beta/t} \left( e + \frac{|du|}{\|du\|_{t,\sigma_2 B}} \right) w^{1/p} w^{-1/p} \right)^t dx \right)^{1/t} \\
 &\leq \left( \int_{\sigma_2 B} |du|^p \log^{\beta p/t} \left( e + \frac{|du|}{\|du\|_{t,\sigma_2 B}} \right) w dx \right)^{1/p} \left( \int_{\sigma_2 B} \left( \frac{1}{w} \right)^{t/(p-t)} dx \right)^{(p-t)/pt} \\
 &\leq \|du\|_{L^p(\log L)^\alpha(\sigma_2 B,w)} \left( \int_{\sigma_2 B} \left( \frac{1}{w} \right)^{1/(r-1)} dx \right)^{(r-1)/p}.
 \end{aligned} \tag{4.14}$$

Combining (4.11), (4.12), (4.13), and (4.14), we have

$$\begin{aligned} & \|u - u_B\|_{L^p(\log L)^\alpha(B,w)} \\ & \leq C_3 |B|^{1/n-r/p} \|du\|_{L^p(\log L)^\alpha(\sigma_2 B,w)} \left( \int_B w \, dx \left( \int_{\sigma_2 B} \left(\frac{1}{w}\right)^{1/(r-1)} dx \right)^{r-1} \right)^{1/p} \\ & \leq C_4 |B|^{1/n-r/p} \|du\|_{L^p(\log L)^\alpha(\sigma_2 B,w)} \left( \|w\|_{1,\sigma_2 B} \cdot \left\| \frac{1}{w} \right\|_{1/(r-1),\sigma_2 B} \right)^{1/p}. \end{aligned} \quad (4.15)$$

Note that  $w \in A_r(\Omega)$ , then

$$\begin{aligned} & \left( \|w\|_{1,\sigma_2 B} \cdot \left\| \frac{1}{w} \right\|_{1/(r-1),\sigma_2 B} \right)^{1/p} \\ & \leq \left( \left( \int_{\sigma_2 B} w \, dx \right) \left( \int_{\sigma_2 B} \left(\frac{1}{w}\right)^{1/(r-1)} dx \right)^{r-1} \right)^{1/p} \\ & = \left( |\sigma_2 B|^r \left( \frac{1}{|\sigma_2 B|} \int_{\sigma_2 B} w \, dx \right) \left( \frac{1}{|\sigma_2 B|} \int_{\sigma_2 B} \left(\frac{1}{w}\right)^{1/(r-1)} dx \right)^{r-1} \right)^{1/p} \\ & \leq C_5 |B|^{r/p}. \end{aligned} \quad (4.16)$$

Substituting (4.16) into (4.15) it follows obviously that

$$\|u - u_B\|_{L^p(\log L)^\alpha(B,w)} \leq C_6 |B|^{1/n} \|du\|_{L^p(\log L)^\alpha(\sigma_2 B,w)}. \quad (4.17)$$

This ends the proof of Theorem 4.4.  $\square$

*Proof of Theorem 4.2.* For any constants  $k_i > 0$ ,  $i = 1, 2, 3$ , there are constants  $C_1 > 0$  and  $C_2 > 0$  such that

$$C_1 \log\left(e + \frac{t}{k_1}\right) \leq \log\left(e + \frac{t}{k_2}\right) \leq C_2 \log\left(e + \frac{t}{k_3}\right) \quad (4.18)$$

for any  $t > 0$ . Therefore, we have

$$\begin{aligned} C_1 \left( \int_B |u|^t \log^\alpha\left(e + \frac{|u|}{k_1}\right) dx \right)^{1/t} & \leq \left( \int_B |u|^t \log^\alpha\left(e + \frac{|u|}{k_2}\right) dx \right)^{1/t} \\ & \leq C_2 \left( \int_B |u|^t \log^\alpha\left(e + \frac{|u|}{k_3}\right) dx \right)^{1/t}. \end{aligned} \quad (4.19)$$

By properly selecting constants  $k_i$ , we will have different inequalities that we need. Using the definition of  $L^\varphi(\mu)$ -domains with  $\tau = 1, \sigma = 1$  and  $\varphi(t) = t^p \log^\alpha(e+t/k)$ , where  $k = \|u - u_{B_0}\|_{p,\Omega}$ , Theorem 4.4 and (4.19), we obtain

$$\begin{aligned} \|u - u_\Omega\|_{L^p(\log L)^\alpha(\Omega,w)}^p &= \int_\Omega |u - u_\Omega|^p \log^\alpha\left(e + \frac{|u - u_\Omega|}{\|u - u_{B_0}\|_{p,\Omega}}\right) w \, dx, \\ &\leq C_1 \sup_{B \subset \Omega} \int_B |u - u_B|^p \log^\alpha\left(e + \frac{|u - u_B|}{\|u - u_{B_0}\|_{p,\Omega}}\right) w \, dx \\ &\leq C_2 \sup_{B \subset \Omega} \int_B |u - u_B|^p \log^\alpha\left(e + \frac{|u - u_B|}{\|u - u_B\|_{p,B}}\right) w \, dx \tag{4.20} \\ &\leq C_3 \sup_{B \subset \Omega} |B|^{p/n} \|du\|_{L^p(\log L)^\alpha(\sigma B,w)}^p \\ &\leq C_3 \sup_{B \subset \Omega} |\Omega|^{p/n} \|du\|_{L^p(\log L)^\alpha(\Omega,w)}^p \\ &\leq C_3 |\Omega|^{p/n} \|du\|_{L^p(\log L)^\alpha(\Omega,w)}^p \end{aligned}$$

which is equivalent to

$$\|u - u_\Omega\|_{L^p(\log L)^\alpha(\Omega,w)} \leq C |\Omega|^{1/n} \|du\|_{L^p(\log L)^\alpha(\Omega,w)}. \tag{4.21}$$

We have completed the proof of Theorem 4.2. □

*Definition 4.5.* We call  $\Omega$ , a proper subdomain of  $\mathbb{R}^n$ , a  $\delta$ -John domain,  $\delta > 0$ , if there exists a point  $x_0 \in \Omega$  which can be joined with any other point  $x \in \Omega$  by a continuous curve  $\gamma \subset \Omega$  so that

$$d(\xi, \partial\Omega) \geq \delta|x - \xi| \tag{4.22}$$

for each  $\xi \in \gamma$ . Here  $d(\xi, \partial\Omega)$  is the Euclidean distance between  $\xi$  and  $\partial\Omega$ .

We know that any  $\delta$ -John domain is an  $L^\varphi(\mu)$ -domain [1]. Hence, Theorem 4.2 holds if  $\Omega$  is a  $\delta$ -John domain. Specifically, we have the following theorem.

**Theorem 4.6.** *Let  $u \in D'(\Omega, \Lambda^0)$  be a solution of the nonhomogeneous  $A$ -harmonic equation in a  $\delta$ -John domain  $\Omega \subset \mathbb{R}^n$  and  $du \in L^p(\Omega, \Lambda^1)$ . Assume that  $1 < p < \infty$  and  $w \in A_r(\Omega)$  for some  $r > 1$ . Then, there is a constant  $C$ , independent of  $u$ , such that*

$$\|u - u_\Omega\|_{L^p(\log L)^\alpha(\Omega,w)} \leq C |\Omega|^{1/n} \|du\|_{L^p(\log L)^\alpha(\Omega,w)} \tag{4.23}$$

for any constant  $\alpha > 0$ .

## 5. Applications

In this section, we explore some applications of the results obtained in previous sections.

*Example 5.1.* Assume that  $B = 0$  and  $u$  is a function (0-form) in (1.13). Note that  $|du| = |\nabla u|$  if  $u$  is a function. Then, (1.13) reduces to the following  $A$ -harmonic equation:

$$\operatorname{div} A(x, \nabla u) = 0 \quad (5.1)$$

for functions. Let  $A(x, \xi) = \xi|\xi|^{p-2}$  with  $p > 1$  in (5.1). Then, it is easy to see that the operator  $A$  satisfies the required conditions and (5.1) reduces to the usual  $p$ -harmonic equation:

$$\operatorname{div}(\nabla u |\nabla u|^{p-2}) = 0 \quad (5.2)$$

which is equivalent to the following partial differential equation:

$$(p-2) \sum_{k=1}^n \sum_{i=1}^n u_{x_k} u_{x_i} u_{x_k x_i} + |\nabla u|^2 \Delta u = 0. \quad (5.3)$$

Let  $p = 2$  in (5.3); we obtain the Laplace equation  $\Delta u = 0$  for functions in  $\mathbb{R}^n$ . Hence, each version of the Caccioppoli-type inequality developed in Theorems 3.1, and 3.3, and Corollaries 3.2, 3.4, 3.7, and 3.8 holds if  $u$  satisfies one of the equations (5.1), (5.2), (5.3) and the equation  $\Delta u = 0$ .

Each version of the Caccioppoli-type inequality proved in Section 3 can be used to study the properties of the solutions of the different  $A$ -harmonic equations, particularly, the equations (5.1)–(5.3). For example, using Corollary 3.7, we have the following integrability result.

**Corollary 5.2.** *Let  $u$  be a solution to one of the equations (1.13)–(1.14), or (5.1)–(5.3) in  $\Omega \subset \mathbb{R}^n$ . If  $u$  is locally  $L^p(\log L)^\alpha$ -integrable in  $\Omega$ , then  $du$  is also locally  $L^p(\log L)^\alpha$ -integrable in  $\Omega$ .*

From Theorem 3 in [20, page 16], we know that any open subset of  $\mathbb{R}^n$  is the union of a sequence of mutually disjoint Whitney cubes. Also, cubes are convex. Thus, the definition of the homotopy operator  $T$  can be extended into any domain  $\Omega$  in  $\mathbb{R}^n$ . Using the same method developed in the proof of Theorem 4.4, we can extend inequality (2.5) into the weighted case. Then, similar to the proof of Theorem 4.2, we can generalize the local weighted result into the following global estimate.

**Proposition 5.3.** *Let  $u \in D'(\Omega, \Lambda^0)$  be a solution of the nonhomogeneous  $A$ -harmonic equation (1.13) in an  $L^\varphi(\mu)$ -domain. Assume that  $\alpha > 0$ ,  $1 < p < \infty$  and  $w \in A_r(\Omega)$  for some  $r > 1$ . Then there exists a constant  $C$ , independent of  $u$ , such that*

$$\|u - u_\Omega\|_{L^p(\log L)^\alpha(\Omega, w)} \leq C \|u - c\|_{L^p(\log L)^\alpha(\Omega, w)} \quad (5.4)$$

for any closed form  $c$ .



Using (5.4) with  $c = 0$  and the triangle inequality, we have

$$\begin{aligned} \|u_\Omega\|_{L^p(\log L)^\alpha(\Omega, w)} &\leq \|u - u_\Omega\|_{L^p(\log L)^\alpha(\Omega, w)} + \|u\|_{L^p(\log L)^\alpha(\Omega, w)} \\ &\leq C_1 \|u\|_{L^p(\log L)^\alpha(\Omega, w)} + \|u\|_{L^p(\log L)^\alpha(\Omega, w)} \\ &= (C_1 + 1) \|u\|_{L^p(\log L)^\alpha(\Omega, w)} \\ &= C_2 \|u\|_{L^p(\log L)^\alpha(\Omega, w)}. \end{aligned} \quad (5.5)$$

Thus,

$$\|u_\Omega\|_{L^p(\log L)^\alpha(\Omega, w)} \leq C \|u\|_{L^p(\log L)^\alpha(\Omega, w)}. \quad (5.6)$$

**Theorem 5.4.** *Let  $u \in D'(\Omega, \Lambda^0)$  be a solution of the nonhomogeneous  $A$ -harmonic equation (1.13) in a bounded  $L^\varphi(\mu)$ -domain and let  $T$  be the homotopy operator. Assume that  $\alpha > 0$ ,  $1 < p < \infty$ , and  $w \in A_r(\Omega)$  for some  $r > 1$ . Then there exists a constant  $C$ , independent of  $u$ , such that*

$$\begin{aligned} \|T(du)\|_{L^p(\log L)^\alpha(\Omega, w)} &\leq C |\Omega|^{1/n} \|du\|_{L^p(\log L)^\alpha(\Omega, w)}, \\ \|T(du)\|_{L^p(\log L)^\alpha(\Omega, w)} &\leq C \|u - c\|_{L^p(\log L)^\alpha(\Omega, w)} \end{aligned} \quad (5.7)$$

for any closed form  $c$ .

*Proof.* For any differential form  $u$ , from (1.16) and (1.17), we obtain

$$u = d(Tu) + T(du) = u_\Omega + T(du). \quad (5.8)$$

Hence, by (5.8) and Theorem 4.2, it follows that

$$\begin{aligned} \|T(du)\|_{L^p(\log L)^\alpha(\Omega, w)} &= \|u - u_\Omega\|_{L^p(\log L)^\alpha(\Omega, w)} \\ &\leq C |\Omega|^{1/n} \|du\|_{L^p(\log L)^\alpha(\Omega, w)}. \end{aligned} \quad (5.9)$$

Next, combining (5.8) and (5.4) yields

$$\|T(du)\|_{L^p(\log L)^\alpha(\Omega, w)} = \|u - u_\Omega\|_{L^p(\log L)^\alpha(\Omega, w)} \leq C \|u - c\|_{L^p(\log L)^\alpha(\Omega, w)}. \quad (5.10)$$

This ends the proof of Theorem 5.4.  $\square$

The general theory of solutions to above equations is known as potential theory. In the study of heat conduction, the Laplace equation is the steady-state heat equation. Considering the length of the paper, we only discuss applications to the homotopy operator  $T$ ; see [1] for more results about this operator. We leave it to readers to find similar applications to other operators, such as the Laplace-Beltrami operator  $\Delta = dd^* + d^*d$  and Green's operator  $G$  applied to differential forms. Note that there is a parameter  $\alpha$  in our main results. For different

choices of this parameter, we will have different versions of global inequalities. For example, selecting  $\alpha = 1$  in Theorem 4.2, we have

$$\|u - u_{\Omega}\|_{L^p(\log L)(\Omega, w)} \leq C|\Omega|^{1/n}\|du\|_{L^p(\log L)(\Omega, w)}. \quad (5.11)$$

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