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Research Article

Optimal Inequalities for Generalized Logarithmic, Arithmetic, and Geometric Means

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For $p \in \mathbb{R}$, the generalized logarithmic mean $L_p(a,b)$, arithmetic mean $A(a,b)$, and geometric mean $G(a,b)$ of two positive numbers a and b are defined by $L_p(a,b) = a$, for $a = b$, $L_p(a,b) = [(b^{p+1} - a^{p+1})/((p+1)(b-a))]^{1/p}$, for $p \neq 0$, $p \neq -1$, and $a \neq b$, $L_p(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}$, for $p = 0$, and $a \neq b$, $L_p(a,b) = (b-a)/(\log b - \log a)$, for $p = -1$, and $a \neq b$, $A(a,b) = (a+b)/2$, and $G(a,b) = \sqrt{ab}$, respectively. In this paper, we find the greatest value p (or least value q, resp.) such that the inequality $L_p(a,b) < \alpha A(a,b) + (1-\alpha)G(a,b)$ (or $\alpha A(a,b) + (1-\alpha)G(a,b) < L_q(a,b)$, resp.) holds for $\alpha \in (0,1/2)$(or $\alpha \in (1/2,1)$, resp.) and all $a,b > 0$ with $a \neq b$.

1. Introduction

For $p \in \mathbb{R}$, the generalized logarithmic mean $L_p(a,b)$ and power mean $M_p(a,b)$ with parameter p of two positive numbers a and b are defined by

$$L_p(a,b) = \begin{cases} a, & a = b, \\ \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq 0,\ p \neq -1,\ a \neq b, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0,\ a \neq b, \\ \dfrac{b-a}{\log b - \log a}, & p = -1,\ a \neq b, \end{cases} \tag{1.1}$$

and

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \tag{1.2}$$

respectively. It is well known that both means are continuous and increasing with respect to $p \in \mathbb{R}$ for fixed a and b. Recently, both means have been the subject of intensive research. In particular, many remarkable inequalities involving $L_p(a,b)$ and $M_p(a,b)$ can be found in the literature [1–9]. Let

$$A(a,b) = \frac{a+b}{2}, \qquad I(a,b) = \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & b \neq a, \\ a, & b = a, \end{cases} \qquad L(a,b) = \begin{cases} \dfrac{b-a}{\log b - \log a}, & b \neq a, \\ a, & b = a, \end{cases} \tag{1.3}$$

$G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ be the arithmetic, identric, logarithmic, geometric, and harmonic means of two positive numbers a and b, respectively. Then

$$\begin{aligned} \min\{a,b\} &< H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) = L_{-2}(a,b) \\ &< L(a,b) = L_{-1}(a,b) < I(a,b) = L_0(a,b) < A(a,b) \\ &= M_1(a,b) = L_1(a,b) < \max\{a,b\} \end{aligned} \tag{1.4}$$

for all $a \neq b$.

In [10], Carlson proved that

$$L(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b) \tag{1.5}$$

for all $a, b > 0$ with $a \neq b$.

The following inequality is due to Sándor [11, 12]:

$$I(a,b) > \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b). \tag{1.6}$$

In [13], Lin established the following results: (1) $p \geq 1/3$ implies that $L(a,b) < M_p(a,b)$ for all $a,b > 0$ with $a \neq b$; (2) $p \leq 0$ implies that $L(a,b) > M_p(a,b)$ for all $a,b > 0$ with $a \neq b$; (3) $p < 1/3$ implies that there exist $a,b > 0$ such that $L(a,b) > M_p(a,b)$; (4) $p > 0$ implies that there exist $a,b > 0$ such that $L(a,b) < M_p(a,b)$. Hence the question was answered: what are the least value q and the greatest value p such that the inequality $M_p(a,b) < L(a,b) < M_q(a,b)$ holds for all $a,b > 0$ with $a \neq b$.

Pittenger [14] established that

$$M_{p_1}(a,b) \leq L_p(a,b) \leq M_{p_2}(a,b) \tag{1.7}$$

for all $a, b > 0$, where

$$p_1 = \begin{cases} \min\left\{\dfrac{p+2}{3}, \dfrac{p\log 2}{\log(p+1)}\right\}, & p > -1,\ p \neq 0, \\ \dfrac{2}{3}, & p = 0, \\ \min\left\{\dfrac{p+2}{3}, 0\right\}, & p \leq -1, \end{cases}$$
$$p_2 = \begin{cases} \max\left\{\dfrac{p+2}{3}, \dfrac{p\log 2}{\log(p+1)}\right\}, & p > -1,\ p \neq 0, \\ \log 2, & p = 0, \\ \max\left\{\dfrac{p+2}{3}, 0\right\}, & p \leq -1. \end{cases} \tag{1.8}$$

Here, p_1 and p_2 are sharp and inequality (1.7) becomes equality if and only if $a = b$ or $p = 1, -2$ or $-1/2$. The case $p = -1$ reduces to Lin's results [13]. Other generalizations of Lin's results were given by Imoru [15].

Recently, some monotonicity results of the ratio between generalized logarithmic means were established in [16–18].

The aim of this paper is to prove the following Theorem 1.1.

Theorem 1.1. *Let $\alpha \in (0, 1)$ and $a, b > 0$ with $a \neq b$, then*

(1) $L_{3\alpha-2}(a, b) = \alpha A(a, b) + (1-\alpha)G(a, b)$ *for* $\alpha = 1/2$;

(2) $L_{3\alpha-2}(a, b) < \alpha A(a, b) + (1-\alpha)G(a, b)$ *for* $0 < \alpha < 1/2$, *and* $L_{3\alpha-2}(a, b) > \alpha A(a, b) + (1-\alpha)G(a, b)$ *for* $1/2 < \alpha < 1$, *moreover, in each case, the bound* $L_{3\alpha-2}(a, b)$ *for the sum* $\alpha A(a, b) + (1-\alpha)G(a, b)$ *is optimal.*

2. Proof of Theorem 1.1

In order to prove our Theorem 1.1 we need a lemma, which we present in this section.

Lemma 2.1. *For $\alpha \in (0, 1)$ and $h(t) = (6\alpha-1)t^{6\alpha-4} - (6\alpha-1)t^{6\alpha-5} - (6\alpha-5)t^{6\alpha-6} + (6\alpha-5)t^{6\alpha-7}$ one has*

(1) *If* $\alpha \in [1/6, 1)$, *then* $h(t) > 0$ *for* $t > 1$;

(2) *If* $\alpha \in (0, 1/6)$, *then* $h(t) < 0$ *for* $t > \sqrt{(5-6\alpha)/(1-6\alpha)}$, $h(t) > 0$ *for* $1 < t < \sqrt{(5-6\alpha)/(1-6\alpha)}$, *and* $h(t) = 0$ *for* $t = \sqrt{(5-6\alpha)/(1-6\alpha)}$.

Proof. (1) If $\alpha = 1/6$, then we clearly see that

$$h(t) = 4t^{-6}(t-1) > 0 \tag{2.1}$$

for $t > 1$.

If $\alpha \in (1/6, 1)$, then

$$h(t) = (6\alpha - 1)(t-1)\left(t^2 - 1 + \frac{4}{6\alpha - 1}\right)t^{6\alpha-7} > 0 \tag{2.2}$$

for $t > 1$.

Therefore, Lemma 2.1(1) follows from (2.1) and (2.2).

(2) If $\alpha \in (0, 1/6)$, then

$$h(t) = (6\alpha - 1)(t-1)\left(t + \sqrt{\frac{5-6\alpha}{1-6\alpha}}\right)\left(t - \sqrt{\frac{5-6\alpha}{1-6\alpha}}\right)t^{6\alpha-7}. \tag{2.3}$$

Therefore, Lemma 2.1(2) follows from (2.3). □

Proof of Theorem 1.1.

Proof. (1) If $\alpha = 1/2$, then (1.1) leads to

$$L_{3\alpha-2}(a,b) = L_{-1/2}(a,b) = \frac{a+b}{4} + \frac{\sqrt{ab}}{2} = \frac{1}{2}A(a,b) + \frac{1}{2}G(a,b) = \alpha A(a,b) + (1-\alpha)G(a,b). \tag{2.4}$$

(2) We divide the proof into two cases. □

Case 1. $\alpha = 1/3$ or $\alpha = 2/3$. From inequalities (1.5) and (1.6) we clearly see that

$$L_{3\alpha-2}(a,b) < \alpha A(a,b) + (1-\alpha)G(a,b) \tag{2.5}$$

for $\alpha = 1/3$, and

$$L_{3\alpha-2}(a,b) > \alpha A(a,b) + (1-\alpha)G(a,b) \tag{2.6}$$

for $\alpha = 2/3$.

Case 2. $\alpha \in (0,1) \setminus \{1/3, 1/2, 2/3\}$. Without loss of generality, we assume that $a > b$. Let $t = \sqrt{a/b} > 1$, then (1.1) leads to

$$\begin{aligned} &\log L_{3\alpha-2}(a,b) - \log[\alpha A(a,b) + (1-\alpha)G(a,b)] \\ &\quad = \frac{1}{3\alpha-2}\log\frac{t^{6\alpha-2}-1}{(3\alpha-1)(t^2-1)} - \log\left[\frac{\alpha}{2}\left(1+t^2\right) + (1-\alpha)t\right]. \end{aligned} \tag{2.7}$$

Let $f(t) = (1/(3\alpha-2))\log[(t^{6\alpha-2}-1)/((3\alpha-1)(t^2-1))] - \log[(\alpha/2)(1+t^2) + (1-\alpha)t]$, then simple computations yield

$$\lim_{t\to 1} f(t) = 0, \tag{2.8}$$

$$f'(t) = \frac{g(t)}{(3\alpha-2)(t^{6\alpha-2}-1)(t^2-1)[(\alpha/2)(1+t^2)+(1-\alpha)t]}, \tag{2.9}$$

where

$$\begin{aligned} g(t) = {} & (1-\alpha)(3\alpha-2)t^{6\alpha} + 3\alpha(\alpha-1)t^{6\alpha-1} - 3\alpha(1-\alpha)t^{6\alpha-2} \\ & - \alpha(3\alpha-1)t^{6\alpha-3} + \alpha(3\alpha-1)t^3 + 3\alpha(1-\alpha)t^2 \\ & + 3\alpha(1-\alpha)t - (1-\alpha)(3\alpha-2). \end{aligned} \tag{2.10}$$

Note that

$$g(1) = 0, \tag{2.11}$$

$$\begin{aligned} g'(t) = {} & 6\alpha(1-\alpha)(3\alpha-2)t^{6\alpha-1} + 3\alpha(\alpha-1)(6\alpha-1)t^{6\alpha-2} \\ & - 6\alpha(1-\alpha)(3\alpha-1)t^{6\alpha-3} - 3\alpha(3\alpha-1)(2\alpha-1)t^{6\alpha-4} \\ & + 3\alpha(3\alpha-1)t^2 + 6\alpha(1-\alpha)t + 3\alpha(1-\alpha), \end{aligned} \tag{2.12}$$

$$g'(1) = 0, \tag{2.13}$$

$$\begin{aligned} g''(t) = {} & 6\alpha(1-\alpha)(3\alpha-2)(6\alpha-1)t^{6\alpha-2} + 6\alpha(\alpha-1)(6\alpha-1) \\ & \times (3\alpha-1)t^{6\alpha-3} - 18\alpha(1-\alpha)(3\alpha-1)(2\alpha-1)t^{6\alpha-4} \\ & - 6\alpha(3\alpha-1)(2\alpha-1)(3\alpha-2)t^{6\alpha-5} + 6\alpha(3\alpha-1)t \\ & + 6\alpha(1-\alpha), \end{aligned} \tag{2.14}$$

$$g''(1) = 0, \tag{2.15}$$

$$\begin{aligned} g'''(t) = {} & 12\alpha(1-\alpha)(3\alpha-2)(6\alpha-1)(3\alpha-1)t^{6\alpha-3} \\ & + 18\alpha(\alpha-1)(6\alpha-1)(3\alpha-1)(2\alpha-1)t^{6\alpha-4} \\ & - 36\alpha(1-\alpha)(3\alpha-1)(2\alpha-1)(3\alpha-2)t^{6\alpha-5} \\ & - 6\alpha(3\alpha-1)(2\alpha-1)(3\alpha-2)(6\alpha-5)t^{6\alpha-6} \\ & + 6\alpha(3\alpha-1), \end{aligned} \tag{2.16}$$

$$g'''(1) = 0, \tag{2.17}$$

$$g^{(4)}(t) = 36\alpha(3\alpha-1)(3\alpha-2)(2\alpha-1)(1-\alpha)h(t), \tag{2.18}$$

where $h(t)$ is defined as in Lemma 2.1.

We divide the proof into five subcases.

Subcase A. $\alpha \in (0, 1/6)$. From (2.18) and Lemma 2.1(2) we clearly see that $g^{(4)}(t) < 0$ for $t \in (1, \sqrt{(5-6\alpha)/(1-6\alpha)})$ and $g^{(4)}(t) > 0$ for $t \in (\sqrt{(5-6\alpha)/(1-6\alpha)}, +\infty)$, then we know that $g'''(t)$ is strictly decreasing in $(1, \sqrt{(5-6\alpha)/(1-6\alpha)})$ and strictly increasing in $(\sqrt{(5-6\alpha)/(1-6\alpha)}, +\infty)$. Now from the monotonicity of $g'''(t)$ and (2.17) together with the fact that $\lim_{t\to+\infty} g'''(t) = 6\alpha(3\alpha-1) < 0$ we clearly see that $g'''(t) < 0$ for $t > 1$, then from (2.7)–(2.15) and $(3\alpha-2)(t^{6\alpha-2}-1) > 0$ for $t > 1$ we get $L_{3\alpha-2}(a,b) < \alpha A(a,b) + (1-\alpha)G(a,b)$ for $\alpha \in (0, 1/6)$.

Subcase B. $\alpha \in [1/6, 1/3)$. Then (2.18) and Lemma 2.1(1) lead to

$$g^{(4)}(t) < 0 \tag{2.19}$$

for $t > 1$.

From (2.7)–(2.17) and (2.19) together with the fact that $(3\alpha-2)(t^{6\alpha-2}-1) > 0$ for $t > 1$ we know that $L_{3\alpha-2}(a,b) < \alpha A(a,b) + (1-\alpha)G(a,b)$ for $\alpha \in [1/6, 1/3)$.

Subcase C. $\alpha \in (1/3, 1/2)$. Then (2.18) and Lemma 2.1(1) imply that

$$g^{(4)}(t) > 0 \tag{2.20}$$

for $t > 1$.

From (2.7)–(2.17), (2.20) and $(3\alpha-2)(t^{6\alpha-2}-1) < 0$ for $t > 1$ we know that $L_{3\alpha-2}(a,b) < \alpha A(a,b) + (1-\alpha)G(a,b)$ for $\alpha \in (1/3, 1/2)$.

Subcase D. $\alpha \in (1/2, 2/3)$. Then (2.19) again yields, and $L_{3\alpha-2}(a,b) > \alpha A(a,b) + (1-\alpha)G(a,b)$ for $\alpha \in (1/2, 2/3)$ follows from (2.7)–(2.17) and (2.19) together with $(3\alpha-2)(t^{6\alpha-2}-1) < 0$.

Subcase E. $\alpha \in (2/3, 1)$. Then (2.20) is also true, and $L_{3\alpha-2}(a,b) > \alpha A(a,b) + (1-\alpha)G(a,b)$ for $\alpha \in (2/3, 1)$ follows from (2.7)–(2.17), (2.20) and the fact that $(3\alpha-2)(t^{6\alpha-2}-1) > 0$.

Next, we prove that the bound $L_{3\alpha-2}(a,b)$ for the sum $\alpha A(a,b)+(1-\alpha)G(a,b)$ is optimal in each case. The proof is divided into six cases.

Case 1. $\alpha = 1/3$. For any $\epsilon \in (0,1)$ and $x \in (0,1)$, then (1.1) leads to

$$\begin{aligned}
&[L_{3\alpha-2+\epsilon}(1,1+x)]^{1-\epsilon} - [\alpha A(1,1+x) + (1-\alpha)G(1,1+x)]^{1-\epsilon} \\
&\quad = [L_{\epsilon-1}(1,1+x)]^{1-\epsilon} - \left[\frac{1}{3}A(1,1+x) + \frac{2}{3}G(1,1+x)\right]^{1-\epsilon} \\
&\quad = \frac{\epsilon x}{(1+x)^{\epsilon}-1} - \left[\frac{1}{3} + \frac{x}{6} + \frac{2}{3}(1+x)^{1/2}\right]^{1-\epsilon} \\
&\quad = \frac{f_1(x)}{(1+x)^{\epsilon}-1},
\end{aligned} \tag{2.21}$$

where $f_1(x) = \epsilon x - [(1+x)^{\epsilon}-1][1/3 + x/6 + (2/3)(1+x)^{1/2}]^{1-\epsilon}$.

Let $x \to 0$; making use of Taylor expansion, one has

$$f_1(x) = \frac{1}{24}\epsilon^2(1-\epsilon)x^3 + o\left(x^3\right). \tag{2.22}$$

Equations (2.21) and (2.22) imply that for any $\epsilon \in (0,1)$, there exists $0 < \delta_1 = \delta_1(\epsilon) < 1$, such that $L_{3\alpha-2+\epsilon}(1,1+x) > \alpha A(1,1+x) + (1-\alpha)G(1,1+x)$ for any $x \in (0,\delta_1)$ and $\alpha = 1/3$.

Case 2. $\alpha = 2/3$. For any $\epsilon \in (0,1)$ and $x \in (0,1)$, from (1.1) we have

$$\begin{aligned}
&[\alpha A(1,1+x) + (1-\alpha)G(1,1+x)]^{\epsilon} - [L_{3\alpha-2-\epsilon}(1,1+x)]^{\epsilon} \\
&\quad = \left[\frac{2}{3}A(1,1+x) + \frac{1}{3}G(1,1+x)\right]^{\epsilon} - [L_{-\epsilon}(1,1+x)]^{\epsilon} \\
&\quad = \left[\frac{2}{3} + \frac{x}{3} + \frac{1}{3}(1+x)^{1/2}\right]^{\epsilon} - \frac{(1-\epsilon)x}{(1+x)^{1-\epsilon}-1} \\
&\quad = \frac{f_2(x)}{(1+x)^{1-\epsilon}-1},
\end{aligned} \tag{2.23}$$

where $f_2(x) = [(1+x)^{1-\epsilon}-1][2/3 + x/3 + (1/3)(1+x)^{1/2}]^{\epsilon} - (1-\epsilon)x$.

Let $x \to 0$; making use of Taylor expansion, one has

$$f_2(x) = \frac{1}{24}\epsilon^2(1-\epsilon)x^3 + o\left(x^3\right). \tag{2.24}$$

Equations (2.23) and (2.24) imply that for any $\epsilon \in (0,1)$, there exists $0 < \delta_2 = \delta_2(\epsilon) < 1$, such that $L_{3\alpha-2-\epsilon}(1,1+x) < \alpha A(1,1+x) + (1-\alpha)G(1,1+x)$ for $x \in (0,\delta_2)$ and $\alpha = 2/3$.

Case 3. $\alpha \in (0,1/3)$. For $\epsilon \in (0,1-3\alpha)$ and $x \in (0,1)$, we get

$$\begin{aligned}
&[L_{3\alpha-2+\epsilon}(1,1+x)]^{2-3\alpha-\epsilon} - [\alpha A(1,1+x) + (1-\alpha)G(1,1+x)]^{2-3\alpha-\epsilon} \\
&\quad = \frac{(1-3\alpha-\epsilon)x(1+x)^{1-3\alpha-\epsilon}}{(1+x)^{1-3\alpha-\epsilon}-1} - \left[\alpha + \frac{\alpha}{2}x + (1-\alpha)(1+x)^{1/2}\right]^{2-3\alpha-\epsilon} \\
&\quad = \frac{f_3(x)}{(1+x)^{1-3\alpha-\epsilon}-1},
\end{aligned} \tag{2.25}$$

where $f_3(x) = (1-3\alpha-\epsilon)x(1+x)^{1-3\alpha-\epsilon} - [(1+x)^{1-3\alpha-\epsilon}-1][\alpha + (\alpha/2)x + (1-\alpha)(1+x)^{1/2}]^{2-3\alpha-\epsilon}$.

Let $x \to 0$; making use of Taylor expansion, one has

$$f_3(x) = \frac{1}{24}\epsilon(1-3\alpha-\epsilon)(2-3\alpha-\epsilon)x^3 + o\left(x^3\right). \tag{2.26}$$

Equations (2.25) and (2.26) imply that for any $\alpha \in (0, 1/3)$ and any $\epsilon \in (0, 1-3\alpha)$, there exists $0 < \delta_3 = \delta_3(\epsilon, \alpha) < 1$, such that $L_{3\alpha-2+\epsilon}(1, 1+x) > \alpha A(1, 1+x) + (1-\alpha)G(1, 1+x)$ for $x \in (0, \delta_3)$.

Case 4. $\alpha \in (1/3, 1/2)$. For any $\epsilon \in (0, 2-3\alpha)$ and $x \in (0, 1)$, we get

$$\begin{aligned}
&[L_{3\alpha-2+\epsilon}(1, 1+x)]^{2-3\alpha-\epsilon} - [\alpha A(1, 1+x) + (1-\alpha)G(1, 1+x)]^{2-3\alpha-\epsilon} \\
&\quad = \frac{(3\alpha - 1 + \epsilon)x}{(1+x)^{3\alpha+\epsilon-1} - 1} - \left[\alpha + \frac{\alpha}{2}x + (1-\alpha)(1+x)^{1/2}\right]^{2-3\alpha-\epsilon} \\
&\quad = \frac{f_4(x)}{(1+x)^{3\alpha-1+\epsilon} - 1},
\end{aligned} \tag{2.27}$$

where $f_4(x) = (3\alpha - 1 + \epsilon)x - [(1+x)^{3\alpha-1+\epsilon} - 1][\alpha + (\alpha/2)x + (1-\alpha)(1+x)^{1/2}]^{2-3\alpha-\epsilon}$.

Let $x \to 0$; using Taylor expansion we have

$$f_4(x) = \frac{1}{24}\epsilon(3\alpha - 1 + \epsilon)(2 - 3\alpha - \epsilon)x^3 + o\left(x^3\right). \tag{2.28}$$

Equations (2.27) and (2.28) show that for any $\alpha \in (1/3, 1/2)$ and any $\epsilon \in (0, 2-3\alpha)$, there exists $0 < \delta_4 = \delta_4(\epsilon, \alpha) < 1$, such that $L_{3\alpha-2+\epsilon}(1, 1+x) > \alpha A(1, 1+x) + (1-\alpha)G(1, 1+x)$ for $x \in (0, \delta_4)$.

Case 5. $\alpha \in (1/2, 2/3)$. For any $\epsilon \in (0, 3\alpha - 1)$ and $x \in (0, 1)$, we have

$$\begin{aligned}
&[\alpha A(1, 1+x) + (1-\alpha)G(1, 1+x)]^{2-3\alpha+\epsilon} - [L_{3\alpha-2-\epsilon}(1, 1+x)]^{2-3\alpha+\epsilon} \\
&\quad = \left[\alpha + \frac{\alpha}{2}x + (1-\alpha)(1+x)^{1/2}\right]^{2-3\alpha+\epsilon} - \frac{(3\alpha - 1 - \epsilon)x}{(1+x)^{3\alpha-\epsilon-1} - 1} \\
&\quad = \frac{f_5(x)}{(1+x)^{3\alpha-1-\epsilon} - 1},
\end{aligned} \tag{2.29}$$

where $f_5(x) = [(1+x)^{3\alpha-1-\epsilon} - 1][\alpha + (\alpha/2)x + (1-\alpha)(1+x)^{1/2}]^{2-3\alpha+\epsilon} - (3\alpha - 1 - \epsilon)x$.

Let $x \to 0$; making use of Taylor expansion we get

$$f_5(x) = \frac{1}{24}\epsilon(3\alpha - 1 - \epsilon)(2 - 3\alpha + \epsilon)x^3 + o\left(x^3\right). \tag{2.30}$$

Equations (2.29) and (2.30) imply that for any $\alpha \in (1/2, 2/3)$ and any $\epsilon \in (0, 3\alpha - 1)$, there exists $0 < \delta_5 = \delta_5(\epsilon, \alpha) < 1$, such that $L_{3\alpha-2-\epsilon}(1, 1+x) < \alpha A(1, 1+x) + (1-\alpha)G(1, 1+x)$ for $x \in (0, \delta_5)$.

Case 6. $\alpha \in (2/3, 1)$. For any $\epsilon \in (0, 3\alpha - 2)$ and $x \in (0,1)$, we get

$$\begin{aligned} &[\alpha A(1,1+x) + (1-\alpha)G(1,1+x)]^{3\alpha-2-\epsilon} - [L_{3\alpha-2-\epsilon}(1,1+x)]^{3\alpha-2-\epsilon} \\ &\quad = \left[\alpha + \frac{\alpha}{2}x + (1-\alpha)(1+x)^{1/2}\right]^{3\alpha-2-\epsilon} - \frac{(1+x)^{3\alpha-\epsilon-1} - 1}{(3\alpha-1-\epsilon)x} \\ &\quad = \frac{f_6(x)}{(3\alpha-1-\epsilon)x}, \end{aligned} \tag{2.31}$$

where $f_6(x) = (3\alpha - 1 - \epsilon)x[\alpha + (\alpha/2)x + (1-\alpha)(1+x)^{1/2}]^{3\alpha-2-\epsilon} - [(1+x)^{3\alpha-1-\epsilon} - 1]$.

Let $x \to 0$, using Taylor expansion we have

$$f_6(x) = \frac{1}{24}\epsilon(3\alpha - 2 - \epsilon)(3\alpha - 1 - \epsilon)x^3 + o\left(x^3\right). \tag{2.32}$$

From (2.31) and (2.32) we know that for any $\alpha \in (2/3, 1)$ and any $\epsilon \in (0, 3\alpha - 2)$, there exists $0 < \delta_6 = \delta_6(\epsilon, \alpha) < 1$, such that $L_{3\alpha-2-\epsilon}(1, 1+x) < \alpha A(1, 1+x) + (1-\alpha)G(1, 1+x)$ for $x \in (0, \delta_6)$. □

At last, we propose two open problems as follows.

Open Problem 1

What is the least value p such that the inequality

$$\alpha A(a,b) + (1-\alpha)G(a,b) < L_p(a,b) \tag{2.33}$$

holds for $\alpha \in (0, 1/2)$ and all $a, b > 0$ with $a \neq b$?

Open Problem 2

What is the greatest value q such that the inequality

$$\alpha A(a,b) + (1-\alpha)G(a,b) > L_q(a,b) \tag{2.34}$$

holds for $\alpha \in (1/2, 1)$ and all $a, b > 0$ with $a \neq b$?

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