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Research Article

Implicit Iteration Methods with Errors for Discrete Non-Lipschitzian Families with Perturbed Mappings

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Motivated and inspired by the recent works on implicit iteration methods, we prove necessary and sufficient conditions for strong convergence of the eventually implicit iteration methods with errors to a common fixed point of a discrete family which is continuous total asymptotically nonexpansive (in brief, TAN) on q-uniformly smooth Banach spaces with a perturbed mapping F, $1 < q \le 2$, under some suitable control conditions of parameters. Some applications to viscosity approximation methods or to the eventually implicit algorithms with errors for a finite family of TAN self-mappings in real Banach spaces are also added.

1. Introduction

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X. Denote by $\langle \cdot, \cdot \rangle$ the duality product. When $\{x_n\}$ is a sequence in X, we denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. Let K be a nonempty closed convex subset of X and let $T : K \to K$ be a mapping. Now let $\mathrm{Fix}(T)$ be the fixed point set of T; namely,

$$\mathrm{Fix}(T) := \{x \in K : Tx = x\}. \tag{1.1}$$

Also, for viscosity approximation methods in Section 4, we use Π_K to denote the collection of all contractions on K, that is, $f \in \Pi_K$ means that there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \le \alpha \|x - y\|$ for all $x, y \in K$.

Recently, Alber et al. [1] introduced the wider class of total asymptotically nonexpansive mappings to unify various definitions of classes of nonlinear mappings associated with the class of asymptotically nonexpansive mappings; see also Definition 1 of [2]. They say that a mapping $T : K \to K$ is said to be *total asymptotically nonexpansive* (in brief, TAN) [1] (or [2]) if there exist two nonnegative real sequences $\{c_n\}$ and $\{d_n\}$ with $c_n,\ d_n \to 0$, and $\phi \in \Gamma(\mathbb{R}^+)$ such that

$$\|T^n x - T^n y\| \le \|x - y\| + c_n \phi(\|x - y\|) + d_n, \tag{1.2}$$

for all $x, y \in K$ and $n \ge 1$, where $\mathbb{R}^+ := [0, \infty)$ and

$$\phi \in \Gamma(\mathbb{R}^+) \Longleftrightarrow \phi \text{ is strictly increasing, continuous on } \mathbb{R}^+ \text{and } \phi(0) = 0. \tag{1.3}$$

In this case, T is often said to be TAN on K with respect to (in short, w.r.t.) $\{c_n\}$, $\{d_n\}$, and ϕ. Sometimes we also write $c_n(T)$, $d_n(T)$, and ϕ_T instead of c_n, d_n, and ϕ whenever the distinction of the mapping T is needed.

Remark 1.1. Note firstly that if T is continuous, the property (1.2) with $c_n = 0$ for all $n \ge 1$ is equivalent to the following one which is said to be *asymptotically nonexpansive in the intermediate sense* [3]:

$$\limsup_{n \to \infty} \sup_{x,y \in K} \{\|T^n x - T^n y\| - \|x - y\|\} \le 0. \tag{1.4}$$

Indeed, taking $c_n \equiv 0$ in (1.2) firstly, we have

$$\sup_{x,y \in K} \{\|T^n x - T^n y\| - \|x - y\|\} \le d_n \tag{1.5}$$

for each $n \ge 1$, and next taking the lim sup on both sides as $n \to \infty$ immediately gives the property (1.4) because $d_n \to 0$ as $n \to \infty$. Conversely, taking

$$d_n = \max\left\{0, \sup_{x,y \in K} \{\|T^n x - T^n y\| - \|x - y\|\}\right\} \tag{1.6}$$

for each $n \ge 1$, (1.4) immediately implies $d_n \to 0$ as $n \to \infty$; see also [2]. Note also that a mapping satisfying the property (1.4) is non-Lipschitzian; see [4]. Also, if we take $\phi(t) = t$ for all $t \ge 0$ and $d_n = 0$ for all $n \ge 1$ in (1.2), it can be reduced to the well-known concept of *asymptotically nonexpansive* mapping [5] as

$$\|T^n x - T^n y\| \le (1 + c_n)\|x - y\| \tag{1.7}$$

for all $x, y \in K$ and $n \geq 1$. Furthermore, in addition, taking $c_n = 0$ for all $n \geq 1$ in (1.2), it can be reduced to the concept of *nonexpansive* mapping as

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.8}$$

for all $x, y \in K$.

Recently, motivated and stimulated by (1.2), Kim and Park [6] introduced a discrete family $\mathfrak{I} = \{T_n : K \to K\}$ of non-Lipschitzian mappings, called TAN on K, namely, $\mathfrak{I} = \{T_n : K \to K\}$ is said to be TAN on K w.r.t. $\{c_n\}$, $\{d_n\}$, and ϕ if there exist nonnegative real sequences $\{c_n\}$ and $\{d_n\}$, $n \geq 1$ with c_n, $d_n \to 0$, and $\phi \in \Gamma(\mathbb{R}^+)$ such that

$$\|T_n x - T_n y\| \leq \|x - y\| + c_n \phi(\|x - y\|) + d_n, \tag{1.9}$$

for all $x, y \in K$ and $n \geq 1$. Furthermore, we say that $\mathfrak{I}$ is *continuous* on K provided each $T_n \in \mathfrak{I}$ is continuous on K; see [6] for examples of *continuous* TAN families. In particular, we say that $\mathfrak{I} = \{T_n : K \to K\}$ is simply TAN when $K = X$. Then they established necessary and sufficient conditions for strong convergence of the sequence $\{x_n\}$ defined recursively by the following explicit algorithm:

$$x_{n+1} = T_n x_n, \quad n \geq 1, \tag{1.10}$$

starting from an initial guess $x_1 \in K$, to a common fixed point of $\mathfrak{I}$ in Banach spaces as follows.

Theorem 1.2 (see [6]). *Let X be a real Banach space K a nonempty closed convex subset of X. Let a discrete family $\mathfrak{I} = \{T_n : K \to K\}$ be continuous TAN on K w.r.t. $\{c_n\}$, $\{d_n\}$, and ϕ with $C := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$. Assume that $\{c_n\}$, $\{d_n\}$, and ϕ satisfy the following two conditions:*

(C1) *there exist $\alpha, \beta > 0$ such that $\phi(t) \leq \alpha t$ for all $t \geq \beta$;*

(C2) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$.

Then the sequence $\{x_n\}$ defined by the explicit iteration method (1.10) *converges strongly to a common fixed point of $\mathfrak{I}$ if and only if* $\liminf_{n\to\infty} d(x_n, C) = 0$, *where* $d(x_n, C) = \inf_{z \in C} \|x_n - z\|$.

Now let us briefly investigate the recent history concerning the implicit iterative algorithms (with errors) for nonexpansive mappings and asymptotically nonexpansive mappings. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive self-mappings of a nonempty closed convex subset K of a Hilbert space H with $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \neq \emptyset$. In 2001, Xu and Ori [7] introduced the following iterative algorithm defined implicitly in H:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{[n]} x_n, \quad n \geq 1, \tag{1.11}$$

where $\{\alpha_n\} \subset (0, 1)$, $x_0 \in K$ is arbitrarily chosen, and $T_{[n]} := T_{n \bmod N}$, namely, the mod function takes values in the set $\{1, 2, \ldots, N\}$ as $T_{[n]} = T_N$ for $r = 0$; $T_{[n]} = T_r$ for $0 < r < N$ whenever $n = kN + r$ for some integers $k \geq 0$ and $0 \leq r < N$. As $C := \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \neq \emptyset$ and

$\alpha_n \to 0$, they proved the weak convergence of the sequence $\{x_n\}$ defined by the implicit iteration process (1.11) to a common fixed point of $\{T_i\}_{i=1}^N$.

In 2002, Zhou and Chang [8] introduced the following implicit iterative process with errors for a finite family $\{T_i\}_{i=1}^N$ of N asymptotically nonexpansive self-mappings of K:

$$x_n = \alpha_n x_{n-1} + \beta_n T_{[n]}^n x_n + \gamma_n u_n, \quad n \geq 1, \tag{1.12}$$

where $x_0 \in K$ is arbitrarily chosen, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, and $\{u_n\}$ is a bounded sequence in K. As $C := \bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$ and $\sum_{n=1}^\infty c_n < \infty$, assuming, in addition, that there exists a constant $L > 0$ such that for any $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$,

$$\left\| T_i^n x - T_j^n y \right\| \leq L \|x - y\|, \quad n \geq 1, \tag{1.13}$$

for all $x, y \in K$, they established the weak and strong convergence of the implicit iterative process (1.12) with errors in uniformly convex Banach spaces, under some suitable conditions of parameters and the assumption (1.13). Subsequently, Sun [9] introduced the following modified implicit process for a finite family $\{T_i\}_{i=1}^N$ of N asymptotically nonexpansive self-mappings of K:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{[n]}^{k(n)} x_n, \quad n \geq 1, \tag{1.14}$$

where $k(n)$ is given as $k(n) = k$ for $r = 0$; $k(n) = k + 1$ for $0 < r < N$ whenever $n = kN + r$ for some integers $k \geq 0$ and $0 \leq r < N$ (in this case, note that $k(n) \to \infty$ as $n \to \infty$, $k(n - N) = k(n) - 1$ and $T_{[n-N]} = T_{[n]}$ for $n \geq N$), and he studied the necessary and sufficient conditions for the strong convergence of the implicit iteration scheme (1.14) for such a finite family $\{T_i\}_{i=1}^N$ as the conclusion in Theorem 1.2, under assumption of the existence of $\{x_n\}$ generated implicitly by (1.14). Recently, for removing the condition (1.13), Chang et al. [10] also introduced the following modified implicit iteration scheme with errors for a finite family $\{T_i\}_{i=1}^N$ of N asymptotically nonexpansive self-mappings of K satisfying $K + K \subset K$:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{[n]}^{k(n)} x_n + u_n, \quad n \geq 1, \tag{1.15}$$

where $\{u_n\}$ is a bounded sequence in K. In case the control sequence $\{\alpha_n\}$ is bounded away from 0 and 1, they studied the weak and strong convergence of the sequence $\{x_n\}$ generated implicitly by (1.15).

On the other hand, Zeng and Yao [11] recently consider the following iterative process defined implicitly for a finite family $\{T_i\}_{i=1}^N$ of nonexpansive mappings from a whole Hilbert space H into itself with a perturbed mapping F:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) \left[T_{[n]} x_n - \lambda_n \mu F\left(T_{[n]} x_n \right) \right], \quad n \geq 1, \tag{1.16}$$

where $x_0 \in H$ is arbitrarily chosen, $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset [0,1)$, $\mu \in (0,\rho)$ for some $\rho > 0$, and $F : H \to H$ is κ-Lipschitzian and η-strongly monotone, that is,

$$\|Fx - Fy\| \leq \kappa\|x - y\|, \qquad \langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2 \tag{1.17}$$

for all $x, y \in X$ and for some $\kappa > 0$ and $\eta > 0$. They established necessary and sufficient conditions for the strong convergence of the implicit iteration scheme (1.16) for such a finite family $\{T_i\}_{i=1}^N$ with the perturbed mapping F.

Remark 1.3. Notice that, in the implicit iterative algorithm (1.16), taking all the $\lambda_n \equiv 0$ reduces to (1.11) equipped with $K = H$.

Let a discrete family $\mathfrak{I} = \{T_n : X \to X\}$ be continuous TAN with a perturbed mapping F, namely, $F : X \to X$ is κ-Lipschitzian and η-strongly accretive on X, that is,

$$\|Fx - Fy\| \leq \kappa\|x - y\|, \qquad \langle Fx - Fy, J(x - y)\rangle \geq \eta\|x - y\|^2 \tag{1.18}$$

for all $x, y \in X$ and for some $\kappa > 0$ and $\eta > 0$, where J denotes the normalized duality mapping on X. Then we consider the following *eventually* implicit iterative algorithm with errors for such a family $\mathfrak{I} = \{T_n : X \to X\}$ with the perturbed mapping F:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n - \beta_n)\left[T_n x_n - \lambda_n \mu_n F(T_n x_n)\right] + \beta_n w_n \tag{1.19}$$

for all sufficiently large $n \geq n_0$, where $x_{n_0-1} := u \in X$ is arbitrarily chosen, $\alpha_n \in (0,1]$, $\beta_n \in [0, 1 - \alpha_n]$, $\lambda_n \in [0,1)$, $\mu_n \in (0,\rho)$ for some $\rho > 0$, and $\{w_n\}$ is bounded in X. We next exhibit that the sequence $\{x_n\}$ defined implicitly and eventually for the family $\mathfrak{I} = \{T_n : X \to X\}$ with the perturbed mapping F as in (1.19) is well defined for all sufficiently large n, using the well-known Meir-Keeler's theorem due to Meir and Keeler [12].

Remark 1.4. Note that taking all the $\lambda_n \equiv 0$ in (1.19) reduces to the following eventually implicit algorithms:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n - \beta_n)T_n x_n + \beta_n w_n \tag{1.20}$$

for all sufficiently large n, where $\alpha_n \in (0,1]$ and $\beta_n \in [0, 1 - \alpha_n]$. In this case, note also that the domain X of all the self-mappings T_n can be restricted within a nonempty closed convex subset K of X.

Finally, inspired and motivated by recent works of Chidume and Ofoedu [2], Zhou and Chang [8], Sun [9], Chang et al. [10], and Zeng and Yao [11], we shall give necessary and sufficient conditions for strong convergence of the eventually implicit iteration processes (1.19) with errors to a common fixed point of such a discrete family $\mathfrak{I} = \{T_n : X \to X\}$ with a perturbed mapping F on reflexive, strictly convex and q-uniformly smooth Banach spaces, $1 < q \leq 2$, under some suitable conditions of parameters and $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$. Some applications to viscosity approximation methods or to the eventually implicit algorithm (1.20) with errors for a finite family of TAN self-mappings in real Banach spaces are also added.

2. Preliminaries

Let X be a real Banach space and let X^* be its dual. For $1 < p < \infty$, the mapping $J_p : X \to 2^{X^*}$ defined by

$$J_p(x) = \left\{ x^* \in X^* : \langle x, x^* \rangle = \|x^*\| \|x\|, \|x^*\| = \|x\|^{p-1} \right\} \tag{2.1}$$

for each $x \in X$ is called the (generalized) *duality mapping* on X. In particular, $J := J_2$ is called the *normalized duality mapping* on X. It is well known that $J_p(x) = \|x\|^{p-2} J(x)$ for $x \neq 0$; see [13] or [14] for more properties of duality mappings.

The moduli of convexity and smoothness of X are functions $\delta_X : [0,2] \to [0,1]$ and $\rho_X : [0,\infty) \to [0,\infty)$ defined respectively by

$$\begin{aligned} \delta_X(\epsilon) &= \inf\left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \epsilon \right\}, \\ \rho_X(t) &= \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2-1} : \|x\| \leq 1,\ \|y\| \leq t \right\}. \end{aligned} \tag{2.2}$$

X is said to be *uniformly convex* if $\delta_X(\epsilon) > 0$ for all $\epsilon > 0$ and *uniformly smooth* if $\lim_{t\downarrow 0} \rho_X(t)/t = 0$. Let $q > 1$ be a given real number. Then X is said to be *q-uniformly smooth* if there is a constant $c > 0$ such that $\rho_X(t) \leq ct^q$; see also [14–16] for more details. It is well known [17] that no Banach space is q-uniformly smooth for $q > 2$, and also that if $1 < r \leq q \leq 2$, then q-uniformly smooth space is r-uniformly smooth. Hilbert spaces, L^p (or ℓ^p) spaces, $1 < p < \infty$, and the Sobolev spaces, $W_m^p, 1 < p < \infty$, are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$L^p(\text{or } \ell^p) \text{ or } W_m^p \text{ is } \begin{cases} p\text{-uniformly smooth} & \text{if } 1 < p \leq 2; \\ 2\text{-uniformly smooth} & \text{if } p \geq 2. \end{cases} \tag{2.3}$$

The following result due to Xu [18] is very useful for our argument.

Lemma 2.1 (see [18]). *Let $1 < q \leq 2$ be a given number. X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + c_q \|y\|^q \tag{2.4}$$

for all $x, y \in X$.

Remark 2.2. Note that $c_2 = 1$ in a Hilbert space H because $\|x+y\|^2 = \|x\|^2 + 2\langle y, x \rangle + \|y\|^2$ for $x, y \in H$.

Let X be a q-uniformly smooth Banach space with $1 < q \le 2$ and let the mapping $F : X \to X$ be κ-Lipschitzian and η-strongly accretive on X. Then using (2.4), $J_q(x) = \|x\|^{q-2}J(x)$ and (1.18), we compute for all $x, y \in X$,

$$\begin{aligned}
\|(I-\mu F)x-(I-\mu F)y\|^q &= \|(x-y)-\mu(Fx-\mathrm{F}y)\|^q \\
&\le \|x-y\|^q - q\mu\langle Fx-Fy, J_q(x-y)\rangle + c_q\|\mu(Fx-Fy)\|^q \\
&\le \|x-y\|^q - q\mu\eta\,\|x-y\|^q + c_q\mu^q\kappa^q\|x-y\|^q \\
&= (1-q\mu\eta + c_q\mu^q\kappa^q)\|x-y\|^q \\
&= \left[1-\mu\left(q\eta - c_q\mu^{q-1}\kappa^q\right)\right]\|x-y\|^q,
\end{aligned} \tag{2.5}$$

where I denotes the identity operator on X. Let $\mu \in (0, \rho)$, where

$$\rho := \min\left\{\frac{1}{q\eta}, \left(\frac{q\eta}{c_q\kappa^q}\right)^{1/(q-1)}\right\} \tag{2.6}$$

(the choice of $\rho < 1/q\eta$ is just a way to ensure that $1-\mu(q\eta - c_q\mu^{q-1}\kappa^q) > 0$ to take the q-root.). Then $I-\mu F$ is a contraction because

$$\|(I-\mu F)x-(I-\mu F)y\| \le \sqrt[q]{1-\mu(q\eta - c_q\mu^{q-1}\kappa^q)}\ \|x-y\| \tag{2.7}$$

for all $x, y \in X$ and $0 < \sqrt[q]{1-\mu(q\eta - c_q\mu^{q-1}\kappa^q)} < 1$ for $\mu \in (0, \rho)$.

Let a discrete family $\Im = \{T_n : X \to X\}$ be TAN. Given $\lambda \in [0, 1)$, $\mu > 0$, and $k \ge 1$, let the mapping $\Phi^{(\lambda,\mu,k)} : X \to X$ be defined by

$$\Phi^{(\lambda,\mu,k)}x := T_kx - \lambda\mu F(T_kx) \tag{2.8}$$

for all $x \in X$. Then the eventually implicit iteration algorithm (1.19) is simply expressed as

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n-\beta_n)\Phi^{(\lambda_n,\mu_n n)}x_n + \beta_n w_n \tag{2.9}$$

for all sufficiently large n.

The following lemmas will be used frequently throughout this paper.

Lemma 2.3. *If $0 \le \lambda < 1$, $0 < \mu < \rho$, and $k \ge 1$, then there holds for $\Phi^{(\lambda,\mu,k)} : X \to X$,*

$$\left\|\Phi^{(\lambda,\mu,k)}x - \Phi^{(\lambda,\mu,k)}y\right\| \le (1-\lambda\tau)\|T_kx - T_ky\|, \quad x, y \in X, \tag{2.10}$$

where $\tau := 1 - \sqrt[q]{1-\mu(q\eta - c_q\mu^{q-1}\kappa^q)} \in (0, 1)$.

Proof. Using (2.7), we get for all $x, y \in X$,

$$\begin{aligned}
\left\|\Phi^{(\lambda,\mu,k)} x - \Phi^{(\lambda,\mu,k)} y\right\| &= \left\|T_k x - \lambda\mu F(T_k x) - (T_k y - \lambda\mu F(T_k y))\right\| \\
&= \left\|\lambda(I-\mu F)T_k x + (1-\lambda)T_k x - \left[\lambda(I-\mu F)T_k y + (1-\lambda)T_k y\right]\right\| \\
&\le \lambda\left\|(I-\mu F)T_k x - (I-\mu F)T_k y\right\| + (1-\lambda)\left\|T_k x - T_k y\right\| \\
&\le \lambda\sqrt[q]{1-\mu(q\eta - c_q\mu^{q-1}\kappa^q)}\ \left\|T_k x - T_k y\right\| + (1-\lambda)\left\|T_k x - T_k y\right\| \\
&= (1-\lambda\tau)\left\|T_k x - T_k y\right\|.
\end{aligned} \tag{2.11}$$

This completes the proof. □

In 1969, Meir and Kleeler [12] established the following fixed point theorem which is a remarkable generalization of the Banach contraction principle.

Theorem 2.4 (see [12]). *Let (X, d) be a complete metric space and let Φ be a Meir-Keeler contraction (MKC, for short) on X, namely, for each $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d(x, y) < \epsilon + \delta \quad \textit{implies } d(\Phi x, \Phi y) < \epsilon \tag{2.12}$$

for all $x, y \in X$. Then Φ has a unique fixed point $z \in X$ and also $\{\Phi^n x\}$ converges strongly to z for all $x \in X$.

Then the following easy observation is crucial for the construction of the eventually implicit iteration algorithm (2.9).

Proposition 2.5. *Let a discrete family $\Im = \{T_n : X \to X\}$ be TAN, $t \in (0,1)$, and $k \ge 1$. Then $(1-t)T_k$ is an MKC on X for all sufficiently large k.*

Proof. Given $\epsilon > 0$, choose $\delta = t\epsilon/2(1-t)$. On setting

$$A_n := (1-t)\left[\epsilon + \delta + c_n\phi(\epsilon+\delta) + d_n\right], \tag{2.13}$$

$A_n \to (1-t)(\epsilon+\delta)$ as $n \to \infty$ because $c_n, d_n \to 0$, and so we can see

$$0 \le A_n \le (1-t)(\epsilon+\delta) + \frac{t\epsilon}{2} \tag{2.14}$$

for all sufficiently large k. Let $x, y \in X$ with $\|x - y\| < \epsilon + \delta$. Then use (1.9), the strictly increasing property of ϕ, and (2.14), in turn, to derive

$$\begin{aligned}\left\|(1-t)T_k x-(1-t)T_k y\right\| &= (1-t)\left\|T_k x-T_k y\right\| \\ &\leq (1-t)\left[\left\|x-y\right\|+c_k\phi\left(\left\|x-y\right\|\right)+d_k\right] \\ &< (1-t)\left[\epsilon+\delta+c_k\phi(\epsilon+\delta)+d_k\right]=A_k \\ &\leq (1-t)(\epsilon+\delta)+\frac{t\epsilon}{2}=\epsilon.\end{aligned} \tag{2.15}$$

Hence $(1-t)T_k$ is an MKC on X for all sufficiently large k. □

Lemma 2.6 (see [19, 20]). *Let $\{a_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1+\alpha_n)a_n + \beta_n \tag{2.16}$$

for all $n \geq 1$. Suppose that $\sum_{n=1}^{\infty}\alpha_n < \infty$ and $\sum_{n=1}^{\infty}\beta_n < \infty$. Then $\lim_{n\to\infty}a_n$ exists. Moreover, if in addition, $\liminf_{n\to\infty}a_n = 0$, then $\lim_{n\to\infty}a_n = 0$.

3. Necessary and Sufficient Conditions for Convergence

Let X be a q-uniformly smooth Banach space with $1 < q \leq 2$ and let the mapping $F : X \to X$ be κ-Lipschitzian and η-strongly accretive on X. Let a discrete family $\Im = \{T_n : X \to X\}$ be TAN. Let $t \in (0,1]$, $s \in [0,1-t]$. Let $k \geq 1$, $\lambda \in [0,1)$, and $\mu \in (0,\rho)$, where ρ is the constant in (2.6). Fix $u, w \in X$ and let the mapping $\Gamma^{(t,s,k)} : X \to X$ be defined by

$$\Gamma^{(t,s,k)}x = tu + (1-t-s)\Phi^{(\lambda,\mu,k)}x + sw \tag{3.1}$$

for all $x \in X$. Using Lemma 2.3, we have

$$\begin{aligned}\left\|\Gamma^{(t,s,k)}x-\Gamma^{(t,s,k)}y\right\| &= \left\|(1-t-s)\Phi^{(\lambda,\mu,k)}x-(1-t-s)\Phi^{(\lambda,\mu,k)}y\right\| \\ &= (1-t-s)\left\|\Phi^{(\lambda,\mu,k)}x-\Phi^{(\lambda,\mu,k)}y\right\| \\ &\leq (1-t-s)\left\|(1-\lambda\tau)T_k x-(1-\lambda\tau)T_k y\right\|\end{aligned} \tag{3.2}$$

for all $x, y \in X$. First, in case of $\lambda \neq 0$, by Proposition 2.5, $(1-\lambda\tau)T_k$ in (3.2) is an MKC on X for all sufficiently large k and hence so is $\Gamma^{(t,s,k)}$. In the other case of $\lambda = 0$, note that $\Phi^{(\lambda,\mu,k)} = T_k$ and $(1-t-s)\Phi^{(\lambda,\mu,k)} = (1-t-s)T_k$. Applying Proposition 2.5 again, $(1-t-s)T_k$ (hence, $\Gamma^{(t,s,k)}$) is an MKC on X for all sufficiently large k. Therefore, in any case, $\Gamma^{(t,s,k)}$ is an MKC on X for all sufficiently large k. Applying Theorem 2.4 (Meir-Keeler), there exists a unique fixed point $x_{(t,s,k)}$ of $\Gamma^{(t,s,k)}$ in X, that is,

$$x_{(t,s,k)} = tu + (1-t-s)\left[T_k x_{(t,s,k)} - \lambda\mu F\left(T_k x_{(t,s,k)}\right)\right] + sw \tag{3.3}$$

for all sufficiently large k, where $t \in (0,1]$, $s \in [0,1-t]$, $\lambda \in [0,1)$, and $\mu \in (0,\rho)$. This equation (3.3) exhibits that the eventually implicit iteration schemes (1.19) with perturbed mapping F are well defined and so we can present necessary and sufficient conditions for strong convergence of the sequence $\{x_n\}$ defined implicitly and eventually by (1.19) on q-uniformly smooth Banach spaces.

Theorem 3.1. *Let $1 < q \le 2$ be a real number and $N \ge 1$. Let X be a q-uniformly smooth Banach space. Let $F : X \to X$ be κ-Lipschitzian and η-strongly accretive for some constants $\kappa > 0$, $\eta > 0$. Let also a discrete family $\Im = \{T_n : X \to X\}$ be continuous TAN with $C := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by the implicit iteration method* (1.19) *with bounded errors $\{w_n\}$ in X. Assume that $\{c_n\}$ and $\{d_n\}$ satisfy the condition (C2) in Theorem 1.2, and that ϕ satisfies the following property:*

(C1)$'$ *there exist $\alpha > 0$, $\beta \ge 0$ such that $\phi(t) \le \alpha t$ for all $t \ge \beta$.*

Assume also that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, and $\{\mu_n\}$ are sequences of nonnegative real numbers satisfying the following control conditions:

(C3) $\{\lambda_n\} \subset [0,1)$ *for all sufficiently large n and $\sum_{n=1}^{\infty} \lambda_n < \infty$;*

(C4) $0 < \mu_n < \rho$ *for for all sufficiently large n, where ρ and c_p are constants in* (2.6) *and Lemma 2.1, respectively;*

(C5) $0 < a := \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$*;*

(C6) $\beta_n \in [0, 1-\alpha_n]$ *for all sufficiently large n and $\sum_{n=1}^{\infty} \beta_n < \infty$.*

Then $\{x_n\}$ converges strongly to a common fixed point of $\Im$ if and only if $\liminf_{n\to\infty} d(x_n, C) = 0$.

Lemma 3.2. *Under the same hypotheses as Theorem 3.1, there hold the following properties:*

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all $p \in C$, and hence $\{x_n\}$ and $\{\Phi^{(\lambda_n,\mu_n,n)} x_n\}$ are bounded;*

(ii) $\lim_{n\to\infty} d(x_n, C)$ *exists.*

Proof. First, note that it follows from (C1)$'$ and the strictly increasing property of ϕ that

$$\phi(t) \le \phi(\beta) + \alpha t, \quad t \ge 0. \tag{3.4}$$

In fact, if $t \le \beta$, since ϕ is nondecreasing, we have $\phi(t) \le \phi(\beta)$. For any $t \ge \beta$, by (C1)$'$, we get $\phi(t) \le \alpha t$. Hence (3.4) is required. Now to prove (i), let $p \in C$ and let $n \ge 1$ be arbitrarily given. Use (2.8), (2.10), (1.9), and (3.4), in turn, to derive

$$\begin{aligned}
\left\|\Phi^{(\lambda_n,\mu_n,n)} x_n - p\right\| &\le \left\|\Phi^{(\lambda_n,\mu_n,n)} x_n - \Phi^{(\lambda_n,\mu_n,n)} p\right\| + \left\|\Phi^{(\lambda_n,\mu_n,n)} p - p\right\| \\
&\le (1-\lambda_n\tau_n)\|T_n x_n - T_n p\| + \lambda_n\mu_n\|Fp\| \\
&\le (1-\lambda_n\tau_n)\left[\|x_n - p\| + c_n\phi(\|x_n - p\|) + d_n\right] + \lambda_n\mu_n\|Fp\| \\
&\le (1-\lambda_n\tau_n)\left[(1+\alpha c_n)\|x_n - p\| + \phi(\beta)c_n + d_n\right] + \lambda_n\mu_n\|Fp\|,
\end{aligned} \tag{3.5}$$

where $\tau_n := 1-\sqrt[q]{1-\mu_n(q\eta - c_q\mu_n{}^{q-1}\kappa^q)} \in (0,1)$ as in Lemma 2.3. This inequality (3.5) together with (C4) yields

$$\begin{aligned}
\|x_n - p\| &\le \alpha_n\|x_{n-1}-p\| + (1-\alpha_n-\beta_n)\left\|\Phi^{(\lambda_n,\mu_n,n)}x_n - p\right\| + \beta_n\|w_n - p\| \\
&\le \alpha_n\|x_{n-1}-p\| + (1-\alpha_n-\beta_n) \\
&\quad \times \left[(1-\lambda_n\tau_n)(1+\alpha c_n)\|x_n-p\| + \phi(\beta)c_n + d_n\right] + \lambda_n\mu_n\|Fp\| \\
&\quad + \beta_n(\|w_n\| + \|p\|) \\
&\le \alpha_n\|x_{n-1}-p\| + (1-\alpha_n+\alpha_n\lambda_n\tau_n)(1+\alpha c_n)\|x_n-p\| \\
&\quad + \phi(\beta)c_n + d_n + \lambda_n\rho\|Fp\| + \beta_n M_1 \\
&\le \alpha_n\|x_{n-1}-p\| + (1-\alpha_n+\alpha_n\lambda_n)(1+\alpha c_n)\|x_n-p\| \\
&\quad + \phi(\beta)c_n + d_n + \lambda_n\rho\|Fp\| + \beta_n M_1 \\
&\le \alpha_n\|x_{n-1}-p\| + [1-\alpha_n+\alpha_n\lambda_n(1+\alpha c_n)+\alpha c_n]\|x_n-p\| \\
&\quad + \phi(\beta)c_n + d_n + \lambda_n\rho\|Fp\| + \beta_n M_1
\end{aligned} \tag{3.6}$$

for all sufficiently large n, where $M_1 = \sup\{\|w_n\| : n \ge 1\} + \|p\| < \infty$. Since $a = \liminf_{n\to\infty}\alpha_n \in (0,1)$, $\lambda_n \to 0$, and $c_n \to 0$, we can choose

$$0 \le \eta_n := \lambda_n(1+\alpha c_n) + \frac{\alpha c_n}{\alpha_n} < 1 \tag{3.7}$$

for all sufficiently large n and $\eta_n \to 0$ as $n \to \infty$. Now using (3.7), we get

$$\begin{aligned}
\|x_n - p\| &\le \frac{1}{1-\eta_n}\ \|x_{n-1}-p\| + \frac{1}{\alpha_n(1-\eta_n)}(\phi(\beta)c_n + d_n + \lambda_n\rho\|Fp\| + \beta_n M_1) \\
&\le \left(1+\frac{\eta_n}{1-\eta_n}\right)\|x_{n-1}-p\| + \frac{1}{a(1-\eta_n)}(\phi(\beta)c_n + d_n + \lambda_n\rho\|Fp\| + \beta_n M_1) \\
&\le (1+M_2\eta_n)\|x_{n-1}-p\| + \frac{M_2}{a}(\phi(\beta)c_n + d_n + \lambda_n\rho\|Fp\| + \beta_n M_1),
\end{aligned} \tag{3.8}$$

for all sufficiently large n, where $M_2 := \sup\{1/(1-\eta_n) : n \ge 1\} < \infty$ and hence there exists a suitable constant $M > 0$ such that

$$\|x_n - p\| \le (1+M\eta_n)\|x_{n-1}-p\| + M\sigma_n \tag{3.9}$$

for all sufficiently large n, where $\sigma_n := c_n + d_n + \lambda_n + \beta_n$. Since $0 \le \eta_n < \lambda_n + \alpha c_n(1+1/a)$, it follows from (C2), (C3), and (C6) that $\sum\eta_n < \infty$ and $\sum\sigma_n < \infty$. Hence the limit $\lim_{n\to\infty}\|x_n - p\|$ exists from Lemma 2.6. Since $\{x_n\}$ is bounded, so is $\{T_n x_n\}$ because

$$\|T_n x - p\| \le \|x_n - p\| + c_n\phi(\|x_n - p\|) + d_n \tag{3.10}$$

for a fixed $p \in C$ by (1.9). Then it follows from Lemma 2.3 that $\{\Phi^{(\lambda_n,\mu_n,n)}x_n\}$ is also bounded. Hence (i) is obtained.

Now to show (ii), taking the infimum over all $p \in C$ on the both sides of inequality (3.9), we obtain

$$d(x_n, C) \leq (1 + M\eta_n)d(x_{n-1}, C) + M\sigma_n \tag{3.11}$$

for all sufficiently large n. Applying Lemma 2.6 again, (ii) is quickly obtained. □

Proof of Theorem 3.1. It suffices to show the sufficiency. Assume that

$$\liminf_{n\to\infty} d(x_n, C) = 0. \tag{3.12}$$

Then it follows from (ii) of Lemma 3.2 that $\lim_{n\to\infty} d(x_n, C) = 0$. Since $0 \leq \eta_n < 1$ for all sufficiently large n in (3.7) and $\sum \eta_n < \infty$ in the proving process of Lemma 3.2, we see that

$$1 \leq K := \prod (1 + M\eta_n) \leq e^{M\sum \eta_n} < \infty. \tag{3.13}$$

Given $\epsilon > 0$, since $\lim_{n\to\infty} d(x_n, C) = 0$ and $\sum M\sigma_n < \infty$, we can choose a positive integer n_0 sufficiently large so that (3.9) holds for all $n \geq n_0$, and

$$d(x_n, C) < \frac{\epsilon}{4K}, \quad \sum_{i=n}^{\infty} M\sigma_i < \frac{\epsilon}{4K}, \quad n \geq n_0. \tag{3.14}$$

Let $n, m \geq n_0$ and $p \in C$. First, use the inequality (3.9) repeatedly together with (3.13) to derive

$$\begin{aligned}
\|x_n - p\| &\leq \prod_{i=n_0+1}^{n} (1 + M\eta_i)\|x_{n_0} - p\| + \sum_{i=n_0+1}^{n-1} M\sigma_i \prod_{k=i+1}^{n} (1 + M\eta_k) + M\sigma_n \\
&\leq K\left[\|x_{n_0} - p\| + \sum_{i=n_0+1}^{n} M\sigma_i\right],
\end{aligned} \tag{3.15}$$

which implies that

$$\begin{aligned}
\|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\
&\leq K\left[\|x_{n_0} - p\| + \sum_{i=n_0+1}^{n} M\sigma_i\right] + K\left[\|x_{n_0} - p\| + \sum_{i=n_0+1}^{m} M\sigma_i\right] \\
&\leq 2K\left[\|x_{n_0} - p\| + \sum_{i=n_0+1}^{\infty} M\sigma_i\right].
\end{aligned} \tag{3.16}$$

Taking the infimum over all $p \in C$ firstly on both sides and next using (3.14), we have

$$\begin{aligned} \|x_n - x_m\| &\le 2K\left[d(x_{n_0}, C) + \sum_{i=n_0+1}^{\infty} M\sigma_i\right] \\ &\le 2K\left(\frac{\epsilon}{4K} + \frac{\epsilon}{4K}\right) = \epsilon, \quad n, m \ge n_0. \end{aligned} \tag{3.17}$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. Say $x_n \to x^* \in X$. Finally, we claim that $x^* \in C$. In fact, note first that

$$\|x^* - p\| \le \|x^* - x_n\| + \|x_n - p\| \tag{3.18}$$

for all $p \in C$ and $n \ge 1$. Taking the infimum again over all $p \in C$ on both sides ensures that

$$d(x^*, C) \le \|x^* - x_n\| + d(x_n, C) \longrightarrow 0 \tag{3.19}$$

as $n \to \infty$. Since C is closed by continuity of $\Im$, it follows that $x^* \in C$ and the proof is complete. □

Corollary 3.3. *Under the same hypotheses as Theorem 3.1, the sequence $\{x_n\}$ converges strongly to a common fixed point $p \in \Im$ if and only if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to p.*

As taking all the $\lambda_n = 0$ in (1.19), in view of Remark 1.4, we have the following direct consequence of Theorem 3.1 in real Banach spaces.

Theorem 3.4. *Let $N \ge 1$, let K be a nonempty closed convex subset of a real Banach space X, and let a discrete family $\Im = \{T_n : K \to K\}$ be continuous TAN on K w.r.t. $\{c_n\}$, $\{d_n\}$, and ϕ with $C := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \ne \emptyset$. Let $\{x_n\}$ be the sequence defined implicitly and eventually by* (1.20) *with bounded errors $\{w_n\}$ in K. Assume that $\{c_n\}$ and $\{d_n\}$ satisfy the condition (C2) and ϕ satisfies the property (C1)$'$, and also that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying (C5) and (C6) in Theorem 3.1. Then $\{x_n\}$ converges strongly to a common fixed point of $\Im$ in K if and only if* $\liminf_{n\to\infty} d(x_n, C) = 0$.

Remark 3.5. Theorem 3.4 is just an implicit iterative version with error terms of Theorem 1.2.

4. Viscosity Approximation Methods

Recall firstly that a mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an *L-function* if $\psi(0) = 0$, $\phi(t) > 0$ for each $t > 0$, and for every $s > 0$ there exists $u > s$ such that $\psi(t) \le s$ $(t \in [s, u])$. As a consequence, every L-function ψ satisfies $\psi(t) < t$ for each $t > 0$. Also, for a metric space (X, d), a mapping $f : X \to X$ is said to be *(ψ, L)-contraction* if $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an L-function and $d(f(x), f(y)) < \psi(d(x, y))$ for all $x, y \in X$ with $x \ne y$; see [21] for more details.

In 2001, Lim [22] established the following characterization of Meir-Keeler type mappings in terms of (ψ, L)-functions.

Theorem 4.1 (see [22]). *Let (X, d) be a metric space and let $f : X \to X$ be a mapping. Then f is an MKC if and only if there exists an L-function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that f is a (φ, L)-contraction.*

Let K be a nonempty closed convex subset of a real Banach space X and let $T : K \to K$ be a nonexpansive mapping. Given a real number $t \in (0, 1)$ and an $f \in \Pi_K$, we define $T_t^f : K \to K$ by

$$T_t^f x = tf(x) + (1-t)Tx \tag{4.1}$$

for all $x \in K$. For simplicity, we will write T_t for T_t^f provided no confusion occurs. Obviously, T_t is a contraction on K and the well-known Banach Contraction Principle guarantees the unique fixed point of T_t, say x_t. Then x_t is the unique solution of the following fixed point equation

$$x_t = tf(x_t) + (1-t)Tx_t, \tag{4.2}$$

which is later called the (continuous) implicit method, while the (discrete) implicit algorithm is specially expressed as

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \tag{4.3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ tending to zero. On the other hand, the first explicit viscosity algorithm is given in 2000 year by Moudafi [23] as

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 1, \tag{4.4}$$

where $x_1 \in C$ is arbitrarily chosen and $\{\alpha_n\}$ is a sequence in $(0, 1)$. Recently, Petruşel and Yao [21] introduced the following implicit viscosity approximation scheme for a uniformly asymptotically regular sequence $\{T_n\}$ of nonexpansive mappings from K into itself in a reflexive Banach space X:

$$x_n = t_n f(x_n) + (1-t_n)T_n x_n, \quad n \geq 1, \tag{4.5}$$

where $\{t_n\}$ is a sequence in $(0, 1)$ with $t_n \to 0$, and $f : K \to K$ is a generalized contraction (recall that it is called a *generalized contraction* in [21] whenever f is either an MKC or a (φ, L)-contraction in the sense of Theorem 4.1). As $C := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$, they studied strong convergence of the sequence $\{x_n\}$ generated by (4.5) to a common fixed point p of $\{T_n\}$ such that $p \in C$ is the unique solution to the variational inequality

$$\langle f(p) - p, J(y-p) \rangle \leq 0, \quad y \in C; \tag{4.6}$$

see Theorem 3.6 of [21] for more details. In a similar way, Lin [24] very recently established strong convergence of an explicit viscosity approximation scheme generated by a generalized contraction and a nonexpansive semigroup in reflexive Banach spaces; see also [25] for approximating fixed points of nonexpansive mappings.

From now on, unless other specified, assume that K is a nonempty closed convex subset of a real Banach space X and a discrete family $\mathfrak{I} = \{T_n : X \to X\}$ is TAN. In this section, as a special case of (1.20), we shall consider the following eventually implicit iteration method with no errors and a fixed anchor $x_0 \in K$ for such a family $\mathfrak{I}$:

$$x_n = \alpha_n x_0 + (1-\alpha_n)T_n x_n \tag{4.7}$$

for all sufficiently large n, where $\alpha_n \in [a,1]$ for some $a \in (0,1)$.

Here we need the following properties for Meir-Keeler contractions, studied recently by Suzuki [26].

Lemma 4.2 (see [26]). *Let K be a convex subset of a Banach space X. Let Φ be an MKC on K. Then there hold the following properties:*

(i) *given $\epsilon > 0$, there exists $r \in (0,1)$ such that, for all $x, y \in K$,*

$$\|x-y\| \geq \epsilon \quad \textit{implies } \|\Phi x - \Phi y\| \leq r\|x-y\|; \tag{4.8}$$

(ii) *if $T : K \to K$ is nonexpansive, then $T \circ \Phi$ is an MKC on K.*

Proposition 4.3. *Fix $a \in (0,1)$, $\alpha_n \in [a,1]$ for all n. Let Φ be an MKC on K and let a discrete family $\mathfrak{I} = \{T_n : X \to X\}$ be TAN on K. Then a mapping $x \mapsto \alpha_n \Phi x + (1-\alpha_n)T_n x$ is an MKC on K for all sufficiently large n.*

Proof. We employ the ideas of Proposition 3(ii) in [26] and Proposition 2.5. Define

$$\Gamma_n x := \alpha_n \Phi x + (1-\alpha_n)T_n x \tag{4.9}$$

for all $x \in K$. Given $\epsilon > 0$, by (i) of Lemma 4.2, there exists $r \in (0,1)$ satisfying (4.8). Choose

$$\delta := \frac{a\epsilon(1-r)}{2[1-a(1-r)]}. \tag{4.10}$$

Fix $x, y \in K$ with $\|x-y\| < \epsilon + \delta$. Then we must claim that $\|\Gamma_n x - \Gamma_n y\| < \epsilon$ for all sufficiently large n. Indeed, in case of $\|x-y\| \geq \epsilon$, we first use (4.8), (1.9), and the strictly increasing property of ϕ to derive

$$\begin{aligned}
\|\Gamma_n x - \Gamma_n y\| &\leq \alpha_n \|\Phi x - \Phi y\| + (1-\alpha_n)\|T_n x - T_n y\| \\
&\leq \alpha_n r\|x-y\| + (1-\alpha_n)\left[\|x-y\| + c_n\phi(\|x-y\|) + d_n\right] \\
&\leq [1-\alpha_n(1-r)]\|x-y\| + c_n\phi(\|x-y\|) + d_n \\
&\leq [1-a(1-r)](\epsilon+\delta) + c_n\phi(\epsilon+\delta) + d_n \\
&:= A_n,
\end{aligned} \tag{4.11}$$

where $A_n \to [1-a(1-r)](\epsilon+\delta)$ as $n \to \infty$ because $c_n, d_n \to 0$. Then we can see

$$
\begin{aligned}
0 \le A_n &\le [1-a(1-r)](\epsilon+\delta) + \frac{a\epsilon(1-r)}{2} \\
&= [1-a(1-r)]\left(\epsilon + \frac{a\epsilon(1-r)}{2[1-a(1-r)]}\right) + \frac{a\epsilon(1-r)}{2} = \epsilon
\end{aligned}
\tag{4.12}
$$

for all sufficiently large n. In the other case of $0 < \|x-y\| < \epsilon$ (because $x = y$ is trivial), we can take ϵ' such that $\|x-y\| < \epsilon' < \epsilon$. For this $\epsilon' > 0$, it follows from the definition of MKC in Theorem 2.4 that there exists $\delta' > 0$ such that

$$
\|x-y\| < \epsilon' + \delta' \quad \text{implies } \|\Phi x - \Phi y\| < \epsilon'
\tag{4.13}
$$

for all $x, y \in K$. Now repeating the previous techniques, we have

$$
\begin{aligned}
\|\Gamma_n x - \Gamma_n y\| &\le \alpha_n \|\Phi x - \Phi y\| + (1-\alpha_n)\|T_n x - T_n y\| \\
&< \alpha_n \epsilon' + (1-\alpha_n)\left[\|x-y\| + c_n\phi(\|x-y\|) + d_n\right] \\
&< \alpha_n \epsilon' + (1-a_n)\left[\epsilon' + c_n\phi(\epsilon') + d_n\right] \\
&< \epsilon' + c_n\phi(\epsilon') + d_n \\
&:= B_n,
\end{aligned}
\tag{4.14}
$$

where $B_n \to \epsilon'$ as $n \to \infty$. Then we can also see

$$
0 \le B_n < \epsilon' + (\epsilon - \epsilon') = \epsilon
\tag{4.15}
$$

for all sufficiently large n. This completes the proof. □

Remark 4.4. Let $T : K \to K$ be nonexpansive. If we take $T_n = T$, $\alpha_n = a \in (0,1)$ for all $n \ge 1$, then a mapping $x \mapsto a\Phi x + (1-a)Tx$ is an MKC on K; see (ii) of Proposition 3 in [26].

Proposition 4.5. *Let ϕ satisfy the condition* (C1)$'$ *in Theorem 3.1. Assume that* $\lim_{n\to\infty} x_n$ *exists for any anchor $x_0 \in K$, where the sequence $\{x_n\}$ equipped with the anchor x_0 is defined implicitly and eventually as $x_n = \alpha_n x_0 + (1-\alpha_n)T_n x_n$ for all sufficiently large n. Define*

$$
P(x_0) := \lim_{n\to\infty} x_n.
\tag{4.16}
$$

Then P is nonexpansive on K.

Proof. Let $x_0, y_0 \in K$ be any different anchors. Let $\{x_n\}$ and $\{y_n\}$ be defined as

$$
\begin{aligned}
x_n &= \alpha_n x_0 + (1-\alpha_n)T_n x_n, \\
y_n &= \alpha_n y_0 + (1-\alpha_n)T_n y_n
\end{aligned}
\tag{4.17}
$$

for all sufficiently large n. By virtue of (1.9) and (C1)$'$, we can compute

$$\begin{aligned}\|x_n - y_n\| &\le \alpha_n\|x_0 - y_0\| + (1-\alpha_n)\|T_n x_n - T_n y_n\| \\ &\le \alpha_n\|x_0 - y_0\| + (1-\alpha_n)\left[\|x_n - y_n\| + c_n\phi(\|x_n - y_n\|) + d_n\right] \\ &\le \alpha_n\|x_0 - y_0\| + (1-\alpha_n)(1+\alpha c_n)\|x_n - y_n\| + c_n\phi(\beta) + d_n \\ &\le \alpha_n\|x_0 - y_0\| + (1-\alpha_n+\alpha c_n)\|x_n - y_n\| + c_n\phi(\beta) + d_n\end{aligned} \tag{4.18}$$

for all sufficiently large n. A simple calculation yields

$$\|x_n - y_n\| \le \left(1 + \frac{\alpha c_n}{a - \alpha c_n}\right)\|x_0 - y_0\| + \frac{1}{a - \alpha c_n}(c_n\phi(b) + d_n) \tag{4.19}$$

because $\alpha_n \in [a,1]$ for some $a \in (0,1)$. Since $c_n, d_n \to 0$ as $n \to \infty$, we have $\|Px_0 - Py_0\| \le \|x_0 - y_0\|$ and the proof is complete. □

Now, in case all the $\beta_n \equiv 0$ in (1.20), we prove strong convergence of the following viscosity approximation methods for a discrete family $\mathfrak{I} = \{T_n : K \to K\}$ which is continuous TAN on a nonempty closed convex subset K of a real Banach space X.

Theorem 4.6. *Under the same hypotheses of K, X, $\mathfrak{I}$, C, $\{c_n\}$, $\{d_n\}$, ϕ, $\{\alpha_n\}$, and $\{\beta_n\}$ as Theorem 3.4, assume that $Px_0 = \lim_{n\to\infty} x_n$ exists for any anchor $x_0 \in K$. Let Φ be an MKC on K, and let $\{y_n\}$ be the sequence defined implicitly and eventually as*

$$y_n = \alpha_n\Phi(y_n) + (1-\alpha_n)T_n y_n \tag{4.20}$$

for all sufficiently large n. Then $\{y_n\}$ converges strongly to the unique point $z \in C$ satisfying $P\circ\Phi z = z$.

Proof. Our proving method employs the idea used for proving Theorem 7 in [26]. Note first that $P \circ \Phi$ and $\alpha_n\Phi + (1-\alpha_n)T_n$ are MKCs on K from (ii) of Lemma 4.2 and Proposition 4.3, respectively. Hence, Theorem 2.4 (Meir-Keeler) ensures the existence and uniqueness of y_n and z. By our assumption, $P\circ\Phi z = P(\Phi z) = \lim_{n\to\infty} x_n = z \in C$ exists for the anchor $\Phi z \in K$, where $\{x_n\}$ is a sequence defined implicitly and eventually by

$$x_n = \alpha_n\Phi z + (1-\alpha_n)T_n x_n \tag{4.21}$$

for all sufficiently large n. We must claim that $y_n \to z$ as $n \to \infty$, too. Indeed, repeat the proving technique used for arriving at (4.19) to get

$$\|y_n - x_n\| \le \left(1 + \frac{\alpha c_n}{a - \alpha c_n}\right)\|\Phi y_n - \Phi z\| + \frac{1}{a - \alpha c_n}(c_n\phi(\beta) + d_n) \tag{4.22}$$

for all sufficiently large n, where $a = \liminf_{n\to\infty}\alpha_n \in (0,1)$. Now to prove the claim by contradiction, assume that $\{y_n\}$ does not converge to z. Then there exist $\epsilon > 0$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\|y_{n_k} - z\| \ge \epsilon$ for all $k \ge 1$. By (i) of Lemma 4.2, for this $\epsilon > 0$, there exists $r \in (0,1)$ satisfying (4.8). Then using (4.22) combined with (4.8), we can compute

$$\begin{aligned}\|y_{n_k} - z\| &\le \|y_{n_k} - x_{n_k}\| + \|x_{n_k} - z\| \\ &\le \left(1 + \frac{\alpha c_{n_k}}{a - \alpha c_{n_k}}\right)\|\Phi y_{n_k} - \Phi z\| + \frac{1}{a - \alpha c_{n_k}}\left(c_{n_k}\phi(\beta) + d_{n_k}\right) + \|x_{n_k} - z\| \\ &\le \left(1 + \frac{\alpha c_{n_k}}{a - \alpha c_{n_k}}\right) r\|y_{n_k} - z\| + \frac{1}{a - \alpha c_{n_k}}\left(c_{n_k}\phi(\beta) + d_{n_k}\right) + \|x_{n_k} - z\|\end{aligned} \tag{4.23}$$

for all sufficiently large k. Since $c_{n_k}, d_{n_k} \to 0$, and $x_{n_k} \to z$ as $k \to \infty$, taking the lim sup as $k \to \infty$ on both sides yields

$$\limsup_{k\to\infty}\|y_{n_k} - z\| \le r \limsup_{k\to\infty}\|y_{n_k} - z\|, \tag{4.24}$$

which immediately shows that $\lim_{k\to\infty} y_{n_k} = z$. This contradicts to the construction of $\{y_{n_k}\}$. Therefore, it must be $y_n \to z$ as $n \to \infty$. This completes the proof. □

Remark 4.7. Note that if all the $T_n : K \to K$ are nonexpansive, our Theorem 4.6 can be reduced to Theorem 7 of [26]. However, if all the $T_n : K \to K$ are asymptotically nonexpansive, it still seems new.

5. Applications to a Finite Family of TAN Self-Mappings

Let X be a smooth Banach space and let $N \ge 1$ be fixed. Let $\{T_i\}_{i=1}^N$ be a finite family of N continuous TAN mappings defined on X; more precisely, for each $1 \le i \le N$, T_i is continuous TAN w.r.t. $\{c_n(T_i)\}$, $\{d_n(T_i)\}$, and ϕ_{T_i}. In this section, as special cases, we consider the following eventually implicit iteration algorithm with errors for such a finite family $\{T_i\}_{i=1}^N$ with a perturbed mapping F:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n - \beta_n)\left[A_n x_n - \lambda_n \mu_n F(A_n x_n)\right] + \beta_n w_n \tag{5.1}$$

for all sufficient large n, where A_n is any one among $T_{[n]}^n$, $T_{[n]}^{k(n)}$, and $\sum_{i=1}^N \lambda_i^{(n)} T_i^{(n)}$, all the $\lambda_i^{(n)} \in [0,1]$ with $\sum_{i=1}^n \lambda_i^{(n)} = 1$ and $\overline{\lambda}_i := \inf\{\lambda_i^{(n)} : n \ge 1\} > 0$ for $1 \le i \le N$; see Proposition 1.2 in [6] for more details.

Remark 5.1. Note that the discrete family $\mathfrak{I} := \{A_n : X \to X\}$ is obviously continuous TAN w.r.t. $\{c_n\}$, $\{d_n\}$, and ϕ, where

$$c_n := \max_{1\le i\le N} c_n(T_i), \qquad d_n := \max_{1\le i\le N} d_n(T_i), \qquad \phi := \max_{1\le i\le N} \phi_{T_i}, \tag{5.2}$$

and also that $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \subset \bigcap_{n=1}^{\infty} \mathrm{Fix}(A_n)$. As in Remark 1.4, taking all the $\lambda_n \equiv 0$ in (5.1) reduces to the following eventually implicit iteration algorithm for a finite family $\{T_i\}_{i=1}^{N}$ of N continuous TAN self-mappings of K, a nonempty closed convex subset of a real Banach space X:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n - \beta_n) A_n x_n + \beta_n w_n \tag{5.3}$$

for all sufficiently large n.

As direct consequences of Theorems 3.1 and 3.4, we have the following necessary and sufficient conditions for strong convergence of the eventually implicit iteration methods (5.1) and (5.3), respectively, for such a finite family $\{T_i\}_{i=1}^{N}$ of N continuous TAN self-mappings.

Theorem 5.2. *Let $1 < q \le 2$ be a real number and $N \ge 1$. Let X be a q-uniformly smooth Banach space. Let $F : X \to X$ be κ-Lipschitzian and η-strongly accretive for some constants $\kappa > 0$, $\eta > 0$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of N continuous TAN mappings defined on X with $C := \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \ne \emptyset$. Let $\{x_n\}$ be the sequence defined by the eventually implicit iteration method* (5.1) *with bounded errors $\{w_n\}$ in X. Assume that $\{c_n\}$, $\{d_n\}$, and ϕ given as in* (5.2) *satisfy the conditions (C2) and* (C1)$'$, *and also that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and $\{\mu_n\}$ are sequences satisfying the control conditions (C2)–(C6) in Theorem 3.1. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{N}$ if and only if* $\liminf_{n\to\infty} d(x_n, C) = 0$.

As taking all the $\lambda_n \equiv 0$ in (5.1), in view of Remarks 5.1 and 1.4, we have the following direct consequence of Theorem 5.2.

Theorem 5.3. *Let $N \ge 1$, let K be a nonempty closed convex subset of a real Banach space X, and let $\{T_i\}_{i=1}^{N}$ be a finite family of N continuous TAN self-mappings of K with $C := \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \ne \emptyset$. Let $\{x_n\}$ be the sequence defined implicitly by* (5.3) *with bounded errors $\{w_n\}$ in K. Assume that $\{c_n\}$, $\{d_n\}$, and ϕ given as in* (5.2) *satisfy the conditions (C2) and* (C1)$'$, *and also that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in* $[0,1]$ *satisfying (C5) and (C6) in Theorem 3.1. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{N}$ in K if and only if* $\liminf_{n\to\infty} d(x_n, C) = 0$.

Remark 5.4. (i) Theorem 5.2 improves and extends the corresponding Theorem 2.2 due to Zeng and Yao [11] for a finite family of nonexpansive self-mappings in Hilbert space settings in case when $\mu_n := \mu$ is fixed and $\beta_n = 0$ for all $n \ge 1$.

(ii) Our results are still new when all the $\lambda_n = 0$ and $\beta_n = 0$; compare with the corresponding results of Xu and Ori [7] in Hilbert spaces.

(iii) Theorem 5.3 is just an implicit iterative version of Theorem 8 of Chidume and Ofoedu [2] for a finite family of N continuous TAN self-mappings in real Banach spaces, and it still seems new.

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