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Research Article

Convergence of Iterative Sequences for Generalized Equilibrium Problems Involving Inverse-Strongly Monotone Mappings

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The purpose of this paper is to consider the weak convergence of an iterative sequence for finding a common element in the set of solutions of generalized equilibrium problems, in the set of solutions of classical variational inequalities, and in the set of fixed points of nonexpansive mappings.

1. Introduction and Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ and C is a nonempty closed convex subset of H. Let $S : C \to C$ be a nonlinear mapping. In this paper, we use $F(S)$ to denote the fixed point set of S. Recall that the mapping S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C. \tag{1.2}$$

A is said to be inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.3}$$

A set-valued mapping $R : H \to 2^H$ is said to be monotone if for all $x, y \in H$, $f \in Rx$ and $g \in Ry$ imply $\langle x - y, f - g \rangle > 0$. A monotone mapping $R : H \to 2^H$ is maximal if the graph $G(R)$ of R is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping R is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in G(R)$ implies $f \in Rx$.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers and $A : C \to H$ an inverse-strongly monotone mapping. In this paper, we consider the following generalized equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

In this paper, the set of such an $x \in C$ is denoted by $\mathrm{EP}(F, A)$, that is,

$$\mathrm{EP}(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x \rangle \geq 0, \ \forall y \in C\}. \tag{1.5}$$

Next, we give two special cases of problem (1.4).

(I) If $A \equiv 0$, then the generalized equilibrium problem (1.4) is reduced to the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{1.6}$$

In this paper, the set of such an $x \in C$ is denoted by $\mathrm{EP}(F)$, that is,

$$\mathrm{EP}(F) = \{x \in C : F(x, y) \geq 0, \ \forall y \in C\}. \tag{1.7}$$

Numerous problems in physics, optimization, and economics reduce to finding a solution of the equilibrium problem.

To study problems (1.4) and (1.6), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y); \tag{1.8}$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semicontinuous.

(II) If $F \equiv 0$, then problem (1.4) is reduced to the classical variational inequality. Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.9}$$

It is known that $x \in C$ is a solution to (1.9) if and only if x is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant and I is the identity mapping.

In 2003, Takahashi and Toyoda [1] considered the variational inequality (1.9) and proved the following theorem.

Theorem 1.1. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H, and let S be a nonexpansive mapping of C into itself such that $\mathfrak{F} = F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n), \quad \forall n \geq 0, \tag{1.10}$$

where $\lambda_n \in [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\alpha_n \in [c, d]$ for some $c, d \in (0, 1)$. Then, $\{x_n\}$ converges weakly to $z \in F(S) \cap VI(C, A)$, where $z = \lim_{n\to\infty} P_{\mathfrak{F}} x_n$.

In 2007, Tada and Takahashi [2] considered the equilibrium problem (1.6) and proved the following result.

Theorem 1.2. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let S be a nonexpansive mapping of C into H such that $F(S) \cap \mathrm{EP}(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$ and let*

$$\begin{gathered} u_n \in C \text{ such that } F(u_n, u) + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S u_n, \quad \forall n \geq 1, \end{gathered} \tag{1.11}$$

where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$ converges weakly to $w \in F(S) \cap \mathrm{EP}(F)$, where $w = \lim_{n\to\infty} P_{F(S)\cap\mathrm{EP}(F)} x_n$.

Very recently, Moudafi [3] considered the following iterative process:

$$\begin{gathered} x_0 \in C, \\ F_1(u_n, u) + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0, \quad \forall u \in C, \\ F_2(v_n, v) + \frac{1}{r_n} \langle v - v_n, v_n - x_n \rangle \geq 0, \quad \forall v \in C, \\ x_{n+1} = \frac{u_n + v_n}{2}, \quad \forall n \geq 1, \end{gathered} \tag{1.12}$$

where F_1 and F_2 are bifunctions and $\{r_n\}$ is a control sequence. A weak convergence theorem was established; see [3] for more details.

Weak convergence of iterative sequences has been studied recently for the problems (1.4), (1.6), and (1.9); see [1–14] and the references therein. In this paper, we consider the generalized equilibrium problem (1.4) and a nonexpansive mapping based on an iterative process. We show that the sequence generated in the purposed iterative process converges weakly to a common element in the set of solutions of the variational inequality (1.9), in the fixed point sets of a nonexpansive mapping and in the solution sets of the generalized equilibrium problem (1.4). The results presented in this paper improve and extend the corresponding results announced by Takahashi and Toyoda [1] and Tada and Takahashi [2].

In order to prove our main results, we also need the following lemmas.

Lemma 1.3 (see [1]). *Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $y \in C$,*

$$\|x_{n+1} - y\| \leq \|x_n - y\|, \quad \forall n \geq 1, \tag{1.13}$$

then $\{P_C(x_n)\}$ converges strongly to some $z \in C$.

The following lemma can be found in [15].

Lemma 1.4. *Let T be a monotone mapping of C into H and $N_C v$ the normal cone to C at $v \in C$, that is,*

$$N_C v = \{w \in H : \langle v - u, w\rangle \ \geq 0, \ \forall u \in C\} \tag{1.14}$$

and define a mapping R on C by

$$Rv = \begin{cases} Tv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{1.15}$$

Then R is maximal monotone and $0 \in Rv$ if and only if $\langle Tv, u - v\rangle \geq 0$ for all $u \in C$.

The following lemma can be found in [16, 17].

Lemma 1.5. *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{1.16}$$

Further, define

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\} \tag{1.17}$$

for all $r > 0$ *and* $x \in H$. *Then, the following hold:*

(a) T_r *is single-valued;*

(b) T_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle; \tag{1.18}$$

(c) $F(T_r) = \mathrm{EP}(F)$;

(d) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 1.6 (see [18]). *Let* H *be a Hilbert space and* $0 < p \leq t_n \leq q < 1$ *for all* $n \geq 1$. *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* H *such that*

$$\limsup_{n\to\infty}\|x_n\| \leq r, \qquad \limsup_{n\to\infty}\|y_n\| \leq r,$$
$$\lim_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = r \tag{1.19}$$

hold for some $r \geq 0$, *then* $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 1.7 (see [19]). *Let* H *be a real Hilbert space,* C *a nonempty closed convex subset of* H, *and* $S : C \to C$ *a nonexpansive mapping. Then the mapping* $I - S$ *is demiclosed at zero, that is, if* $\{x_n\}$ *is a sequence in* C *such that* $x_n \rightharpoonup \overline{x}$ *and* $x_n - Sx_n \to 0$, *then* $\overline{x} \in F(S)$.

2. Main Results

Theorem 2.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F_1 *and* F_2 *be two bifunctions from* $C \times C$ *to* $\mathbb{R}$ *which satisfy (A1)–(A4). Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping,* $B : C \to H$ *a* β*-inverse-strongly monotone mapping,* $T : C \to H$ *an* λ*-inverse-strongly monotone mapping, and* $S : C \to C$ *a nonexpansive mapping. Assume that* $\mathcal{F} := \mathrm{EP}(F_1, A) \cap \mathrm{EP}(F_2, B) \cap VI(C,T) \cap F(S) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0,1]$. *Let* $\{a_n\}$ *be a sequence in* $[0, 2\alpha]$, $\{b_n\}$ *a sequence in* $[0, 2\beta]$, *and* $\{t_n\}$ *a sequence in* $[0, 2\lambda]$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{gathered}
x_1 \in C, \\
F_1(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{a_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\
F_2(v_n, v) + \langle Bx_n, v - v_n\rangle + \frac{1}{b_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in C, \\
y_n = \beta_n u_n + (1-\beta_n)v_n, \\
x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(y_n - t_n T y_n), \quad \forall n \geq 1.
\end{gathered} \tag{Δ}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, *and* $\{t_n\}$ *satisfy the following restrictions:*

(a) $0 < a' \le \alpha_n \le a < 1$, $0 < b \le \beta_n \le c < 1$;

(b) $0 < d \le a_n \le e < 2\alpha$, $0 < f \le b_n \le g < 2\beta$, $0 < h \le t_n \le j < 2\lambda$

for some $a', a, b, c, d, e, f, g, h, j \in \mathbb{R}$, *then the sequence* $\{x_n\}$ *generated in* (Δ) *converges weakly to some point* $\overline{x} \in \mathcal{F}$, *where* $\overline{x} = \lim_{n\to\infty} P_{\mathcal{F}} x_n$.

Proof. Fix $p \in \mathcal{F}$. It follows that

$$p = Sp = T_{a_n}(I - a_n A)p = T_{b_n}(I - b_n B)p = P_C(I - t_n T)p, \quad \forall n \ge 1. \tag{2.1}$$

Note that $I - t_n T$ is nonexpansive for each $n \ge 1$. Indeed, for any $x, y \in C$, we see that

$$\begin{aligned}
\left\|(I - t_n T)x - (I - t_n T)y\right\|^2 &= \left\|(x - y) - t_n(Tx - Ty)\right\|^2 \\
&= \left\|x - y\right\|^2 - 2t_n\langle x - y, Tx - Ty\rangle + t_n^2\left\|Tx - Ty\right\|^2 \\
&\le \left\|x - y\right\|^2 - t_n(2\lambda - t_n)\|Tx - Ty\|^2 \\
&\le \left\|x - y\right\|^2.
\end{aligned} \tag{2.2}$$

In a similar way, we can obtain that $I - a_n A$ and $I - b_n B$ are nonexpansive for each $n \ge 1$. Note that

$$\begin{aligned}
\|u_n - p\| &\le \|T_{a_n}(I - a_n A)x_n - p\| \le \|x_n - p\|, \\
\|v_n - p\| &\le \|T_{b_n}(I - b_n B)x_n - p\| \le \|x_n - p\|.
\end{aligned} \tag{2.3}$$

Put $z_n = P_C(y_n - t_n T y_n)$ for each $n \ge 1$. It follows from (2.3) that

$$\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n\|x_n - p\| + (1 - \alpha_n)\|Sz_n - p\| \\
&\le \alpha_n\|x_n - p\| + (1 - \alpha_n)\|z_n - p\| \\
&\le \alpha_n\|x_n - p\| + (1 - \alpha_n)\|y_n - p\| \\
&\le \alpha_n\|x_n - p\| + (1 - \alpha_n)\left(\beta_n\|u_n - p\| + \left(1 - \beta_n\right)\|v_n - p\|\right) \\
&\le \|x_n - p\|.
\end{aligned} \tag{2.4}$$

This implies that $\lim_{n\to\infty}\|x_n - p\|$ exists. This shows that $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$.

On the other hand, we have

$$\left\|u_n - p\right\|^2 = \left\|T_{a_n}(I - a_n A)x_n - p\right\|^2 \le \left\|x_n - p\right\|^2 - a_n(2\alpha - a_n)\left\|Ax_n - Ap\right\|^2, \tag{2.5}$$

$$\left\|v_n - p\right\|^2 = \left\|T_{b_n}(I - b_n B)x_n - p\right\|^2 \le \left\|x_n - p\right\|^2 - b_n(2\beta - b_n)\left\|Bx_n - Bp\right\|^2. \tag{2.6}$$

Combining (2.5) with (2.6) yields that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|Sz_n-p\|^2\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|z_n-p\|^2\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\left(\beta_n\|u_n-p\|^2+(1-\beta_n)\|v_n-p\|^2\right)\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\left(\beta_n\left(\|x_n-p\|^2-a_n(2\alpha-a_n)\|Ax_n-Ap\|^2\right)\right.\\
&\qquad\left.+\ (1-\beta_n)\left(\|x_n-p\|^2-b_n(2\beta-a_n)\|Bx_n-Bp\|^2\right)\right)\\
&\le \|x_n-p\|^2-(1-\alpha_n)\beta_n a_n(2\alpha-a_n)\|Ax_n-Ap\|^2\\
&\quad -(1-\alpha_n)(1-\beta_n)b_n(2\beta-a_n)\|Bx_n-Bp\|^2.
\end{aligned} \tag{2.7}$$

This implies that

$$(1-\alpha_n)\beta_n a_n(2\alpha-a_n)\|Ax_n-Ap\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2. \tag{2.8}$$

In view of the restrictions (a) and (b), we obtain that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{2.9}$$

It also follows from (2.7) that

$$(1-\alpha_n)(1-\beta_n)b_n(2\beta-a_n)\|Bx_n-Bp\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2. \tag{2.10}$$

In view of the restrictions (a) and (b), we obtain that

$$\lim_{n\to\infty}\|Bx_n-Bp\|=0. \tag{2.11}$$

On the other hand, we see from Lemma 1.5 that

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{a_n}(I - a_nA)x_n - T_{a_n}(I - a_nA)p\|^2 \\
&\le \langle (I - a_nA)x_n - (I - a_nA)p, u_n - p\rangle \\
&= \frac{1}{2}\Big(\|(I - a_nA)x_n - (I - a_nA)p\|^2 + \|u_n - p\|^2 \\
&\quad -\|(I - a_nA)x_n - (I - a_nA)p - (u_n - p)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - a_n(Ax_n - Ap)\|^2\Big) \\
&= \frac{1}{2}\Big(\|x_n - p\|^2 + \|u_n - p\|^2 \\
&\quad -\Big(\|x_n - u_n\|^2 - 2a_n\langle x_n - u_n, Ax_n - Ap\rangle + a_n^2\|Ax_n - Ap\|^2\Big)\Big).
\end{aligned} \tag{2.12}$$

This implies that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2a_n\|x_n - u_n\|\|Ax_n - Ap\|. \tag{2.13}$$

In a similar way, we can obtain that

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - v_n\|^2 + 2b_n\|x_n - v_n\|\|Bx_n - Bp\|. \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|Sz_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|z_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|y_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\Big(\beta_n\|u_n - p\|^2 + (1 - \beta_n)\|v_n - p\|^2\Big) \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n) \\
&\quad \times \Big(\beta_n\Big(\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2a_n\|x_n - u_n\|\|Ax_n - Ap\|\Big) \\
&\quad + (1 - \beta_n)\Big(\|x_n - p\|^2 - \|x_n - v_n\|^2 + 2b_n\|x_n - v_n\|\|Bx_n - Bp\|\Big)\Big) \\
&\le \|x_n - p\|^2 - (1 - \alpha_n)\beta_n\|x_n - u_n\|^2 + 2a_n\|x_n - u_n\|\|Ax_n - Ap\| \\
&\quad - (1 - \alpha_n)(1 - \beta_n)\|x_n - v_n\|^2 + 2b_n\|x_n - v_n\|\|Bx_n - Bp\|.
\end{aligned} \tag{2.15}$$

This shows that

$$(1-\alpha_n)\beta_n\|x_n-u_n\|^2 \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2a_n\|x_n-u_n\|\|Ax_n-Ap\| \\ + 2b_n\|x_n-v_n\|\|Bx_n-Bp\|. \tag{2.16}$$

In view of the restriction (a), we obtain from (2.9) and (2.11) that

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{2.17}$$

From (2.15), we also have

$$(1-\alpha_n)(1-\beta_n)\|x_n-v_n\|^2 \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2a_n\|x_n-u_n\|\|Ax_n-Ap\| \\ + 2b_n\|x_n-v_n\|\|Bx_n-Bp\|. \tag{2.18}$$

In view of the restriction (a), we obtain from (2.9) and (2.11) that

$$\lim_{n\to\infty}\|x_n-v_n\| = 0. \tag{2.19}$$

Since $\{x_n\}$ is bounded, we see that there exits a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $\overline{x}$. It follows from (2.17) that u_{n_i} converges weakly to $\overline{x}$. Note that

$$F_1(u_n,u) + \langle Ax_n, u-u_n\rangle + \frac{1}{a_n}\langle u-u_n, u_n-x_n\rangle \ge 0, \quad \forall u\in C. \tag{2.20}$$

From (A2), we see that

$$\langle Ax_n, u-u_n\rangle + \frac{1}{a_n}\langle u-u_n, u_n-x_n\rangle \ge F_1(u,u_n), \quad \forall u\in C. \tag{2.21}$$

Replacing n by n_i, we arrive at

$$\langle Ax_{n_i}, u-u_{n_i}\rangle + \frac{1}{a_{n_i}}\langle u-u_{n_i}, u_{n_i}-x_{n_i}\rangle \ge F_1(u,u_{n_i}), \quad \forall u\in C. \tag{2.22}$$

For t with $0<t\le 1$ and $u\in C$, let $u_t = tu+(1-t)\overline{x}$. Since $u\in C$ and $\overline{x}\in C$, we have $u_t\in C$. It follows from (2.22) that

$$\begin{aligned}\langle u_t-u_{n_i}, Au_t\rangle &\ge \langle u_t-u_{n_i}, Au_t\rangle - \langle Ax_{n_i}, u_t-u_{n_i}\rangle - \left\langle u_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{a_{n_i}}\right\rangle + F_1(u_t,u_{n_i}) \\ &= \langle u_t-u_{n_i}, Au_t-Au_{n_i}\rangle + \langle u_t-u_{n_i}, Au_{n_i}-Ax_{n_i}\rangle \\ &\quad - \left\langle u_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{a_{n_i}}\right\rangle + F_1(u_t,u_{n_i}).\end{aligned} \tag{2.23}$$

From (2.17), we have $Au_{n_i} - Ax_{n_i} \to 0$ as $i \to \infty$. On the other hand, we obtain from the monotonicity of A that $\langle u_t - u_{n_i}, Au_t - Au_{n_i}\rangle \geq 0$. It follows from (A4) that

$$\langle u_t - \overline{x}, Au_t\rangle \geq F_1(u_t, \overline{x}). \tag{2.24}$$

From (A1), (A4), and (2.24), we obtain that

$$\begin{aligned} 0 = F_1(u_t, u_t) &\leq tF_1(u_t, u) + (1-t)F_1(u_t, \overline{x}) \\ &\leq tF_1(u_t, u) + (1-t)\langle u_t - \overline{x}, Au_t\rangle \\ &= tF_1(u_t, u) + (1-t)t\langle u - \overline{x}, Au_t\rangle, \end{aligned} \tag{2.25}$$

which yields that

$$F_1(u_t, u) + (1-t)\langle u - \overline{x}, Au_t\rangle \geq 0. \tag{2.26}$$

Letting $t \to 0$ in the above inequality, we arrive at

$$F_1(\overline{x}, u) + \langle u - \overline{x}, A\overline{x}\rangle \geq 0. \tag{2.27}$$

This shows that $\overline{x} \in \mathrm{EP}(F_1, A)$. In a similar way, we can obtain that $\overline{x} \in \mathrm{EP}(F_2, B)$.

Next, we claim that $\overline{x} \in VI(C, T)$

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|SP_C(y_n - t_nTy_n) - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|(y_n - t_nTy_n) - (p - t_nTp)\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big(\|y_n - p\|^2 - t_n(2\lambda - t_n)\|Ty_n - Tp\|^2\Big) \\ &\leq \|x_n - p\|^2 - (1-\alpha_n)t_n(2\lambda - t_n)\|Ty_n - Tp\|^2. \end{aligned} \tag{2.28}$$

It follows that

$$(1-\alpha_n)t_n(2\lambda - t_n)\|Ty_n - Tp\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{2.29}$$

This implies from conditions (a) and (b) that

$$\lim_{n\to\infty}\|Ty_n - Tp\| = 0. \tag{2.30}$$

Since P_C is firmly nonexpansive, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|P_C(I - t_nT)y_n - P_C(I - t_nT)p\|^2 \\
&\leq \langle (I - t_nT)y_n - (I - t_nT)p, z_n - p\rangle \\
&= \frac{1}{2}\Big(\|(I - t_nT)y_n - (I - t_nT)p\|^2 + \|z_n - p\|^2 \\
&\quad - \|(I - t_nT)y_n - (I - t_nT)p - (z_n - p)\|^2\Big) \\
&\leq \frac{1}{2}\Big(\|y_n - p\|^2 + \|z_n - p\|^2 - \|y_n - z_n - t_n(Ty_n - Tp)\|^2\Big) \\
&= \frac{1}{2}\Big(\|y_n-p\|^2 + \|z_n-p\|^2 - \Big(\|y_n-z_n\|^2 - 2t_n\langle y_n-z_n, Ty_n-Tp\rangle + t_n^2\|Ty_n-Tp\|^2\Big)\Big).
\end{aligned} \tag{2.31}$$

So, we obtain that

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - z_n\|^2 + 2t_n\|y_n - z_n\|\|Ty_n - Tp\|. \tag{2.32}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|Sz_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|z_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\Big(\|x_n - p\|^2 - \|y_n - z_n\|^2 + 2t_n\|y_n - z_n\|\|Ty_n - Tp\|\Big).
\end{aligned} \tag{2.33}$$

Therefore we have

$$(1 - \alpha_n)\|y_n - z_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2t_n\|y_n - z_n\|\|Ty_n - Tp\|. \tag{2.34}$$

From the restriction (a) and (2.30), we get that

$$\lim_{n\to\infty}\|y_n - z_n\| = 0. \tag{2.35}$$

Note that

$$\begin{aligned}
\|x_n - z_n\| &\leq \|x_n - y_n\| + \|y_n - z_n\| \\
&\leq \beta_n\|x_n - u_n\| + (1 - \beta_n)\|x_n - v_n\| + \|y_n - z_n\|.
\end{aligned} \tag{2.36}$$

From (2.17), (2.19), and (2.35), we obtain that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{2.37}$$

Define a mapping R by

$$Rv = \begin{cases} Tv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{2.38}$$

Let $(v, w) \in G(R)$. Since $w - Tv \in N_C v$ and $z_n \in C$, we obtain that

$$\langle v - z_n, w - Tv \rangle \geq 0. \tag{2.39}$$

As $z_n = P_C(y_n - t_n T y_n)$ and $v \in C$, we get that

$$\left\langle v - z_n, \frac{z_n - y_n}{t_n} + T y_n \right\rangle \geq 0. \tag{2.40}$$

From (2.39) and (2.40), we obtain that

$$\begin{aligned} \langle v - z_{n_i}, w \rangle &\geq \langle v - z_{n_i}, Tv \rangle \\ &\geq \langle v - z_{n_i}, Tv \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - y_{n_i}}{t_{n_i}} + T y_{n_i} \right\rangle \\ &= \left\langle v - z_{n_i}, Tv - T y_{n_i} - \frac{z_{n_i} - y_{n_i}}{t_{n_i}} \right\rangle \\ &= \langle v - z_{n_i}, Tv - T z_{n_i} \rangle + \langle v - z_{n_i}, T z_{n_i} - T y_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - y_{n_i}}{t_{n_i}} \right\rangle \\ &\geq \langle v - z_{n_i}, T z_{n_i} - T y_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - y_{n_i}}{t_{n_i}} \right\rangle. \end{aligned} \tag{2.41}$$

Note that T is Lipschitz. On the other hand, we see from (2.37) that $z_{n_i} \rightharpoonup \overline{x}$. Hence, we get that

$$\langle v - \overline{x}, w \rangle \geq 0. \tag{2.42}$$

Since R is maximal monotone, we obtain that $\overline{x} \in R^{-1}(0)$. From Lemma 1.4, we get that $\overline{x} \in VI(C, T)$.

Finally, we show that $\overline{x} \in F(S)$. Note that

$$\begin{gathered} \|x_{n+1} - p\| \leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|S z_n - p\| , \\ \|S z_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \end{gathered} \tag{2.43}$$

Since $\lim_{n\to\infty}\|x_n - p\|$ exists, we may assume that $\lim_{n\to\infty}\|x_n - p\| = r$ for some positive constant r. Then we see that

$$\limsup_{n\to\infty}\|x_{n+1} - p\| \leq r, \qquad \limsup_{n\to\infty}\|x_n - p\| \leq r,$$
$$\limsup_{n\to\infty}\|Sz_n - p\| \leq r. \tag{2.44}$$

In view of Lemma 1.6, we get that

$$\lim_{n\to\infty}\|x_n - Sz_n\| = 0. \tag{2.45}$$

Furthermore, we know that

$$\begin{aligned}\|Sx_n - x_n\| &\leq \|Sx_n - Sz_n\| + \|Sz_n - x_n\| \\ &\leq \|x_n - z_n\| + \|Sz_n - x_n\|.\end{aligned} \tag{2.46}$$

From (2.37) and (2.45), we obtain that

$$\lim_{n\to\infty}\|Sx_n - x_n\| = 0. \tag{2.47}$$

Note that $x_{n_i} \rightharpoonup \overline{x}$ and $Sx_{n_i} - x_{n_i} \to 0$ as $i \to \infty$. From Lemma 1.7, we arrive at $\overline{x} \in F(S)$. Assume that there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$, converges to x', where $x' \neq \overline{x}$. In view of the Opial's condition, we see that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - \overline{x}\| &= \liminf_{i\to\infty}\|x_{n_i} - \overline{x}\| < \liminf_{i\to\infty}\|x_{n_i} - x'\| \\ &= \lim_{n\to\infty}\|x_{\rm n} - x'\| = \liminf_{j\to\infty}\|x_{n_j} - x'\| \\ &< \lim_{j\to\infty}\|x_{n_j} - \overline{x}\| = \lim_{n\to\infty}\|x_n - \overline{x}\|.\end{aligned} \tag{2.48}$$

This is a contradiction. So, we have $x' = \overline{x}$.

Let $h_n = P_{\mathcal{F}}x_n$. Since $\overline{x} \in \mathcal{F}$, we have

$$\langle x_n - h_n, h_n - \overline{x}\rangle \geq 0. \tag{2.49}$$

From (2.4) and Lemma 1.3, we get that $\{h_n\}$ converges strongly to some $v \in \mathcal{F}$. Since $\{x_n\}$ converges weakly to $\overline{x}$, we have

$$\langle \overline{x} - v, v - \overline{x}\rangle \geq 0. \tag{2.50}$$

Hence we obtain that

$$\overline{x} = v = \lim_{n\to\infty} P_{\mathcal{F}} x_n. \tag{2.51}$$

This completes the proof. □

Corollary 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping, $T : C \to H$ an λ-inverse-strongly monotone mapping, and $S : C \to C$ a nonexpansive mapping. Assume that $\mathcal{F} := \mathrm{EP}(F, A) \cap VI(C, T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$. Let $\{a_n\}$ be a sequence in $[0, 2\alpha]$, and $\{t_n\}$ a sequence in $[0, 2\lambda]$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered} x_1 \in C, \\ F(u_n, u) + \langle Ax_n, u - u_n \rangle + \frac{1}{a_n}\langle u - u_n, u_n - x_n \rangle \geq 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(u_n - t_n T u_n), \quad \forall n \geq 1. \end{gathered} \tag{2.52}$$

Assume that the sequences $\{\alpha_n\}$, $\{a_n\}$, and $\{t_n\}$ satisfy the following restrictions:

(a) $0 < a' \leq \alpha_n \leq a < 1$;

(b) $0 < d \leq a_n \leq e < 2\alpha$, $0 < h \leq t_n \leq j < 2\lambda$

for some $a', a, d, e, h, j \in \mathbb{R}$, then the sequence $\{x_n\}$ converges weakly to some point $\overline{x} \in \mathcal{F}$, where $\overline{x} = \lim_{n\to\infty} P_{\mathcal{F}} x_n$.

Proof. Putting $F_1 = F_2 = F$, $A = B$, and $a_n = b_n$ in Theorem 2.1, we see that $y_n = u_n$. From the proof of Theorem 2.1, we can conclude the desired conclusion immediately. □

Corollary 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and $S : C \to C$ a nonexpansive mapping. Assume that $\mathcal{F} := \mathrm{EP}(F, A) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$. Let $\{a_n\}$ be a sequence in $[0, 2\alpha]$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered} x_1 \in C, \\ F(u_n, u) + \langle Ax_n, u - u_n \rangle + \frac{1}{a_n}\langle u - u_n, u_n - x_n \rangle \geq 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S u_n, \quad \forall n \geq 1. \end{gathered} \tag{2.53}$$

Assume that the sequences $\{\alpha_n\}$ and $\{a_n\}$ satisfy the following restrictions:

(a) $0 < a' \leq \alpha_n \leq a < 1$;

(b) $0 < d \leq a_n \leq e < 2\alpha$

for some $a', a, d, e \in \mathbb{R}$, *then the sequence* $\{x_n\}$ *converges weakly to some point* $\overline{x} \in \mathcal{F}$, *where* $\overline{x} = \lim_{n\to\infty} P_{\mathcal{F}} x_n$.

Proof. Putting $T = 0$ in Corollary 2.2, we can conclude the desired conclusion immediately. □

Remark 2.4. Corollary 2.3 is a generalization of Theorem 1.2 in Section 1. More precisely, Corollary 2.3 is reduced to Theorem 1.2 if $A = 0$.

Corollary 2.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping, $B : C \to H$ a β-inverse-strongly monotone mapping, $T : C \to H$ an λ-inverse-strongly monotone mapping, and $S : C \to C$ a nonexpansive mapping. Assume that $\mathcal{F} := VI(C,A) \cap VI(C,B) \cap VI(C,T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$. Let $\{a_n\}$ be a sequence in $[0,2\alpha]$, $\{b_n\}$ a sequence in $[0,2\beta]$, and $\{t_n\}$ a sequence in $[0,2\lambda]$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{aligned} &x_1 \in C, \\ &y_n = \beta_n P_C(x_n - a_n A x_n) + (1-\beta_n) P_C(x_n - b_n B x_n), \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(y_n - t_n T y_n), \quad \forall n \geq 1. \end{aligned} \tag{2.54}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{t_n\}$ satisfy the following restrictions:

(a) $0 < a' \leq \alpha_n \leq a < 1$, $0 < b \leq \beta_n \leq c < 1$;

(b) $0 < d \leq a_n \leq e < 2\alpha$, $0 < f \leq b_n \leq g < 2\beta$, $0 < h \leq t_n \leq j < 2\lambda$

for some $a', a, b, c, d, e, f, g, h, j \in \mathbb{R}$, then the sequence $\{x_n\}$ converges weakly to some point $\overline{x} \in \mathcal{F}$, where $\overline{x} = \lim_{n\to\infty} P_{\mathcal{F}} x_n$.

Proof. Putting $F_1 = F_2 = 0$, we see that

$$\langle A x_n, u - u_n \rangle + \frac{1}{a_n} \langle u - u_n, u_n - x_n \rangle \geq 0 \tag{2.55}$$

is equivalent to

$$u_n = P_C(x_n - a_n A x_n), \quad \forall n \geq 1. \tag{2.56}$$

In the same way, we can obtain that

$$v_n = P_C(x_n - b_n B x_n), \quad \forall n \geq 1. \tag{2.57}$$

From the proof of Theorem 2.1, we can conclude the desired conclusion immediately. □

Remark 2.6. Corollary 2.5 is a generalization of Theorem 1.1 in Section 1. More precisely, Corollary 2.5 is reduced to Theorem 1.1 if $T = 0$, $A = B$, and $a_n = b_n$ for each $n \geq 1$.

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