Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 838740, 4 pages
doi:10.1155/2010/838740

Research Article

Sharp Becker-Stark-Type Inequalities for Bessel Functions

Ling Zhu

Department of Mathematics, Zhejiang Gongshang University, Hangzhou, Zhejiang 310018, China

Correspondence should be addressed to Ling Zhu, zhuling0571@163.com

Received 22 January 2010; Accepted 23 March 2010

Academic Editor: Wing-Sum Cheung

We extend the Becker-Stark-type inequalities to the ratio of two normalized Bessel functions of the first kind by using Kishore formula and Rayleigh inequality.

1. Introduction

In 1978, Becker and Stark [1] (or see Kuang [2, page 248]) obtained the following two-sided rational approximation for $(\tan x)/x$.

Theorem 1.1. *Let* $0 < x < \pi/2$; *then*

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}. \tag{1.1}$$

Furthermore, 8 *and* π^2 *are the best constants in* (1.1).

In recent paper [3], we obtained the following further result.

Theorem 1.2. *Let* $0 < x < \pi/2$; *then*

$$\frac{\pi^2 + \left(4\left(8 - \pi^2\right)/\pi^2\right)x^2}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2 + \left(\left(\pi^2/3\right) - 4\right)x^2}{\pi^2 - 4x^2}. \tag{1.2}$$

Furthermore, $\alpha = 4(8 - \pi^2)/\pi^2$ *and* $\beta = (\pi^2/3) - 4$ *are the best constants in* (1.2).

Moreover, the following refinement of the Becker-Stark inequality was established in [3].

Theorem 1.3. *Let $0 < x < \pi/2$, and $N \geq 0$ be a natural number. Then*

$$\frac{P_{2N}(x) + \alpha x^{2N+2}}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{P_{2N}(x) + \beta x^{2N+2}}{\pi^2 - 4x^2} \tag{1.3}$$

holds, where $P_{2N}(x) = a_0 + a_1 x^2 + \cdots + a_N x^{2N}$, and

$$a_n = \frac{2^{2n+2}(2^{2n+2}-1)\pi^2}{(2n+2)!}|B_{2n+2}| - \frac{4\cdot 2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where B_{2n} are the even-indexed Bernoulli numbers. Furthermore, $\beta = a_{N+1}$ and $\alpha = (8 - a_0 - a_1(\pi/2)^2 - \cdots - a_N(\pi/2)^{2N})/(\pi/2)^{2N+2}$ are the best constants in (1.3).

Our aim of this paper is to extend the tangent function to Bessel functions. To achieve our goal, let us recall some basic facts about Bessel functions. Suppose that $\nu > -1$ and consider the normalized Bessel function of the first kind $\mathcal{J}_\nu : \mathbb{R} \to (-\infty, 1]$, defined by

$$\mathcal{J}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} J_\nu(x) = \sum_{n\geq 0} \frac{(-1/4)^n}{n!(\nu+1)_n} x^{2n}, \tag{1.5}$$

where, $(\nu+1)_n = \Gamma(\nu+1+n)/\Gamma(\nu+1)$ is the well- known Pochhammer (or Appell) symbol, and $J_\nu(x)$ defined by [4, page 40]

$$J_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(\nu+1+n)} \left(\frac{x}{2}\right)^{2n+\nu}, \quad x \in \mathbb{R}. \tag{1.6}$$

Particularly for $\nu = 1/2$ and $\nu = -1/2$, respectively, the function $\mathcal{J}_\nu$ reduces to some elementary functions, like [4, page 54] $\mathcal{J}_{1/2}(x) = \sin x/x$ and $\mathcal{J}_{-1/2}(x) = \cos x$. In view of that $\tan x/x = (\mathcal{J}_{1/2}(x)/\mathcal{J}_{-1/2}(x))$, in Section 3 we shall extend the result of Theorem 1.3 to the ratio of two normalized Bessel functions of the first kind $\mathcal{J}_{\nu+1}(x)$ and $\mathcal{J}_\nu(x)$.

2. Some Lemmas

In order to prove our main result in next section, each of the following lemmas will be needed.

Lemma 2.1 (Kishore Formula, see [5, 6]). *Let $\nu > -1$, $j_{\nu,n}$ be the nth positive zero of the Bessel function of the first kind of order ν, and $x \in \mathbb{R}$. Then*

$$\frac{x}{2}\frac{J_{\nu+1}(x)}{J_\nu(x)} = \sum_{m=0}^{\infty} \sigma_\nu^{(2m)} x^{2m}, \tag{2.1}$$

where $m \in \{1, 2, 3, \ldots\}$, and $\sigma_\nu^{(2m)} = \sum_{n=1}^{\infty} j_{\nu,n}^{-2m}$ is the Rayleigh function of order $2m$, which showed in [4, page 502].

Lemma 2.2 (Rayleigh Inequality [5, 6]). *Let $\nu > -1$, and $j_{\nu,n}$ be the nth positive zero of the Bessel function of the first kind of order ν, $m \in \{1,2,3,\ldots\}$, and $\sigma_\nu^{(2m)} = \sum_{n=1}^{\infty} j_{\nu,n}^{-2m}$ is the Rayleigh function of order $2m$. Then*

$$j_{\nu,1}^2 < \frac{\sigma_\nu^{(2m)}}{\sigma_\nu^{(2m+2)}}, \tag{2.2}$$

$$\sigma_\nu^{(2)} = \sum_{n=1}^{\infty} j_{\nu,n}^{-2} = \frac{1}{4(\nu+1)} \tag{2.3}$$

hold.

Lemma 2.3. *Let $\nu > -1$, $\mathcal{J}_\nu(x)$ be the normalized Bessel function of the first kind of order ν, $j_{\nu,n}$ the nth positive zero of the Bessel function of the first kind of order ν, $m \in \{1,2,3,\cdots\}$, $\sigma_\nu^{(2m)} = \sum_{n=1}^{\infty} j_{\nu,n}^{-2m}$ the Rayleigh function of order $2m$, and $0 < |x| < j_{\nu,1}$. Then*

$$E(x) \triangleq \left(j_{\nu,1}^2 - x^2\right)\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} = j_{\nu,1}^2 + 4(\nu+1)\sum_{m=1}^{\infty} A_m x^{2m}, \tag{2.4}$$

where $A_m = j_{\nu,1}^2 \sigma_\nu^{(2m+2)} - \sigma_\nu^{(2m)} < 0$.

Proof. By Lemma 2.1 and (2.3) in Lemma 2.2, we have

$$\begin{aligned}
E(x) &= \left(j_{\nu,1}^2 - x^2\right)\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} \\
&= \left(j_{\nu,1}^2 - x^2\right)\frac{2(\nu+1)}{x}\frac{J_{\nu+1}(x)}{J_\nu(x)} \\
&= \left(j_{\nu,1}^2 - x^2\right)\frac{4(\nu+1)}{x^2}\sum_{m=1}^{\infty}\sigma_\nu^{(2m)} x^{2m} \\
&= 4(\nu+1)\left(j_{\nu,1}^2 - x^2\right)\sum_{m=1}^{\infty}\sigma_\nu^{(2m)} x^{2m-2} \\
&= 4(\nu+1) j_{\nu,1}^2 \sum_{m=1}^{\infty}\sigma_\nu^{(2m)} x^{2m-2} - 4(\nu+1)\sum_{m=1}^{\infty}\sigma_\nu^{(2m)} x^{2m} \\
&= 4(\nu+1) j_{\nu,1}^2 \left[\sigma_\nu^{(2)} + \sum_{m=2}^{\infty}\sigma_\nu^{(2m)} x^{2m-2}\right] - 4(\nu+1)\sum_{m=1}^{\infty}\sigma_\nu^{(2m)} x^{2m} \\
&= j_{\nu,1}^2 + 4(\nu+1)\sum_{m=1}^{\infty}\left[j_{\nu,1}^2\sigma_\nu^{(2m+2)} - \sigma_\nu^{(2m)}\right]x^{2m} \\
&\triangleq j_{\nu,1}^2 + 4(\nu+1)\sum_{m=1}^{\infty} A_m x^{2m},
\end{aligned} \tag{2.5}$$

where $A_m = j_{\nu,1}^2 \sigma_\nu^{(2m+2)} - \sigma_\nu^{(2m)} < 0$, which follows from (2.2) in Lemma 2.2. □

3. Main Result and Its Proof

Theorem 3.1. *Let* $\nu > -1$, $\mathcal{J}_\nu(x)$ *be the normalized Bessel function of the first kind of order* ν, $j_{\nu,n}$ *the* n*th positive zero of the Bessel function of the first kind of order* ν, $m \in \{1,2,3,\ldots\}$, $\sigma_\nu^{(2m)} = \sum_{n=1}^{\infty} j_{\nu,n}^{-2m}$ *the Rayleigh function of order* $2m$, $N \geq 0$ *a natural number, and* $0 < |x| < j_{\nu,1}$. *Let* $\lambda = (1 - (j_{\nu,1}^2/4(\nu+1)) - \sum_{m=1}^{N} A_m j_{\nu,1}^{2m})/j_{\nu,1}^{2N+2}$, *and* $\mu = A_{N+1}$. *Then*

$$\frac{R_{2N}(x) + 4(\nu+1)\lambda x^{2N+2}}{j_{\nu,1}^2 - x^2} < \frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} < \frac{R_{2N}(x) + 4(\nu+1)\mu x^{2N+2}}{j_{\nu,1}^2 - x^2} \tag{3.1}$$

holds, where $R_{2N}(x) = j_{\nu,1}^2 + 4(\nu+1)\sum_{m=1}^{N} A_m x^{2m}$ *and*

$$A_n = j_{\nu,1}^2 \sigma_\nu^{(2n+2)} - \sigma_\nu^{(2n)}, \quad n \in \{1,2,3,\ldots\}. \tag{3.2}$$

Furthermore, λ *and* μ *are the best constants in* (3.1).

Proof of Theorem 3.1. Let

$$H(x) = \frac{\left(\left(E(x) - j_{\nu,1}^2\right)/4(\nu+1)\right) - \sum_{m=1}^{N} A_m x^{2m}}{x^{2N+2}}. \tag{3.3}$$

Then by Lemma 2.3, we have

$$H(x) = \frac{\sum_{n=N+1}^{+\infty} A_n x^{2n}}{x^{2N+2}} = \sum_{k=0}^{+\infty} A_{N+1+k} x^{2k}. \tag{3.4}$$

Since $A_n < 0$ for $n \in \mathbb{N}^+$ by Lemma 2.3, $H(x)$ is decreasing on $(0, j_{\nu,1})$.

At the same time, in view of that $\lim_{x \to j_{\nu,1}^-} E(x) = 4(\nu+1)$ we have that $\lambda = \lim_{x \to j_{\nu,1}^-} H(x) = (1-(j_{\nu,1}^2/4(\nu+1)) - \sum_{m=1}^{N} A_m j_{\nu,1}^{2m})/j_{\nu,1}^{2N+2}$ by (3.3), and $\mu = \lim_{x \to 0^+} H(x) = A_{N+1}$ by (3.4), so λ and μ are the best constants in (3.1). □

Remark 3.2. Let $\nu = -1/2$ in Theorem 3.1; we obtain Theorem 1.3.

References

[1] M. Becker and E. L. Stark, "On a hierarchy of quolynomial inequalities for tanx," *University of Beograd Publikacije Elektrotehnicki Fakultet. Serija Matematika i fizika*, no. 602–633, pp. 133–138, 1978.

[2] J. C. Kuang, *Applied Inequalities*, Shandong Science and Technology Press, Jinan, China, 3rd edition, 2004.

[3] L. Zhu and J. K. Hua, "Sharpening the Becker-Stark inequalities," *Journal of Inequalities and Applications*, vol. 2010, Article ID 931275, 4 pages, 2010.

[4] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, UK, 1995.

[5] N. Kishore, "The Rayleigh function," *Proceedings of the American Mathematical Society*, vol. 14, pp. 527–533, 1963.

[6] Á. Baricz and S. Wu, "Sharp exponential Redheffer-type inequalities for Bessel functions," *Publicationes Mathematicae Debrecen*, vol. 74, no. 3-4, pp. 257–278, 2009.