

Research Article

On the Fermionic p -adic Integral Representation of Bernstein Polynomials Associated with Euler Numbers and Polynomials

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The purpose of this paper is to give some properties of several Bernstein type polynomials to represent the fermionic p -adic integral on \mathbb{Z}_p . From these properties, we derive some interesting identities on the Euler numbers and polynomials.

1. Introduction

Throughout this paper, let p be an odd prime number. The symbol, \mathbb{Z}_p , \mathbb{Q}_p , and \mathbb{C}_p denote the ring of p -adic integers, the field of p -adic rational numbers, the complex number field and the completion of algebraic closure of \mathbb{Q}_p , respectively.

Let \mathbb{N} be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-\nu_p(p)} = 1/p$. Note that $\mathbb{Z}_p = \{x \mid |x|_p \leq 1\} = \lim_{\leftarrow N} \mathbb{Z}/p^N\mathbb{Z}_p$.

When one talks of q -extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we normally assume $|q| < 1$, and if $q \in \mathbb{C}_p$, we always assume $|1 - q|_p < 1$.

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and write $f \in UD(\mathbb{Z}_p)$, if the difference quotient $F_f(x, y) = (f(x) - f(y))/(x - y)$ has a limit $f'(a)$ as $(x, y) \rightarrow (a, a)$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p -adic q -integral on \mathbb{Z}_p is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \quad (1.1)$$

(see [1]). In the special case $q = 1$ in (1.1), the integral

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x), \quad (1.2)$$

is called the fermionic p -adic invariant integral on \mathbb{Z}_p (see [2]). From (1.2), we note

$$I_{-1}(f_1) = -I_{-1}(f) + 2f(0), \quad (1.3)$$

where $f_1(x) = f(x + 1)$.

Moreover, for $n \in \mathbb{N}$, let $f_n(x) = f(x + n)$. Then we note that

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \quad (1.4)$$

It is well known that the Euler polynomials are defined by

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (1.5)$$

(see [1–15]). In the special case, $x = 0$, and $E_n(0) = E_n$ are called the n th Euler numbers.

Let $f(x) = e^{tx}$. Then, by (1.3), (1.4), and (1.5), we see that

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \quad (1.6)$$

Let $C[0, 1]$ denote the set of continuous functions on $[0, 1]$. For $f \in C[0, 1]$, Bernstein introduced the following well-known linear positive operator in the field of real numbers \mathbb{R} :

$$\mathbb{B}_n(f : x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x), \quad (1.7)$$

where $\binom{n}{k} = (n(n-1) \cdots (n-k+1))/k! = n!/k!(n-k)!$ (see [3, 4, 7, 10, 11, 14]). Here, $\mathbb{B}_n(f : x)$ is called the Bernstein operator of order n for f .

For $k, n \in \mathbb{Z}_+$, the Bernstein polynomial of degree n is defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{for } x \in [0, 1]. \quad (1.8)$$

For example, $B_{0,1}(x) = 1 - x$, $B_{1,1}(x) = x$, $B_{0,2}(x) = (1-x)^2$, $B_{1,2}(x) = 2x - 2x^2$, $B_{2,2}(x) = x^2, \dots$, and $B_{k,n}(x) = 0$ for $n < k$, $B_{k,n}(x) = B_{n-k,n}(1-x)$.

In this paper, we study the properties of Bernstein polynomials in the p -adic number field. For $f \in UD(\mathbb{Z}_p)$, we give some properties of several type Bernstein polynomials

to represent the fermionic p -adic invariant integral on \mathbb{Z}_p . From those properties, we derive some interesting identities on the Euler polynomials.

2. Fermionic p -adic Integral Representation of Bernstein Polynomials

By (1.5) and (1.6), we see that

$$\frac{2}{e^t + 1} e^{(1-x)t} = \sum_{n=0}^{\infty} E_n(1-x) \frac{t^n}{n!}. \quad (2.1)$$

We also have that

$$\frac{2}{e^t + 1} e^{(1-x)t} = \frac{2}{1 + e^{-t}} e^{-xt} = \sum_{n=0}^{\infty} E_n(x) \frac{(-1)^n}{n!} t^n. \quad (2.2)$$

From (2.1) and (2.2), we note that $E_n(1-x) = (-1)^n E_n(x)$. It is easy to show that

$$E_n(2) = 2 - \sum_{l=0}^n \binom{n}{l} E_l = 2 + E_n, \quad \text{for } n > 0. \quad (2.3)$$

By (1.5), (1.6), (2.1), (2.2), and (2.3), we see that for $n > 0$,

$$\begin{aligned} \int_{\mathbb{Z}_p} (1-x)^n d\mu_{-1}(x) &= (-1)^n \int_{\mathbb{Z}_p} (x-1)^n d\mu_{-1}(x) = \int_{\mathbb{Z}_p} (x+2)^n d\mu_{-1}(x) \\ &= 2 + \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x). \end{aligned} \quad (2.4)$$

Therefore, we obtain the following theorem.

Theorem 2.1. For $n \in \mathbb{N}$, one has

$$\int_{\mathbb{Z}_p} (1-x)^n d\mu_{-1}(x) = 2 + \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x). \quad (2.5)$$

Theorem 2.1 is important to derive our main result in this paper.

Taking the fermionic p -adic integral on \mathbb{Z}_p for one Bernstein polynomial in (1.8), we get

$$\begin{aligned}
 \int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} \binom{n}{k} x^k (1-x)^{n-k} d\mu_{-1}(x) \\
 &= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} \int_{\mathbb{Z}_p} x^{n-j} d\mu_{-1}(x) \\
 &= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^{n-k-j} E_{n-j} \\
 &= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{k+j}.
 \end{aligned} \tag{2.6}$$

Therefore, we obtain the following proposition.

Proposition 2.2. For $k, n \in \mathbb{Z}_+$, one is

$$\int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-1}(x) = \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{k+j}. \tag{2.7}$$

It is known that $B_{k,n}(x) = B_{n-k,n}(1-x)$. Thus, one has

$$\begin{aligned}
 \int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x) d\mu_{-1}(x) \\
 &= \binom{n}{n-k} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \int_{\mathbb{Z}_p} (1-x)^{n-j} d\mu_{-1}(x).
 \end{aligned} \tag{2.8}$$

By (2.8) and Theorem 2.1, we see that for $n > k$,

$$\begin{aligned}
 \int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-1}(x) &= \binom{n}{k} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \left(2 + \int_{\mathbb{Z}_p} x^{n-j} d\mu_{-1}(x) \right) \\
 &= \binom{n}{k} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} (2 + E_{n-j}) \\
 &= \begin{cases} 2 + E_n & \text{if } k = 0, \\ \binom{n}{k} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} E_{n-j} & \text{if } k > 0. \end{cases}
 \end{aligned} \tag{2.9}$$

From (2.9), we obtain the following theorem.

Theorem 2.3. For $n, k \in \mathbb{Z}_+$ with $n > k$, we have

$$\int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-1}(x) = \begin{cases} 2 + E_n & \text{if } k = 0, \\ \binom{n}{k} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} E_{n-j} & \text{if } k > 0. \end{cases} \tag{2.10}$$

By Proposition 2.2 and Theorem 2.3, we obtain the following corollary.

Corollary 2.4. For $n, k \in \mathbb{Z}_+$ with $n > k$, we have

$$\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{k+j} = \begin{cases} 2 + E_n & \text{if } k = 0, \\ \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} E_{n-j} & \text{if } k > 0. \end{cases} \tag{2.11}$$

For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k$, fermionic p -adic invariant integral for multiplication of two Bernstein polynomials on \mathbb{Z}_p can be given by the following relation:

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} \binom{n}{k} x^k (1-x)^{n-k} \binom{m}{k} x^k (1-x)^{m-k} d\mu_{-1}(x) \\ &= \binom{n}{k} \binom{m}{k} \int_{\mathbb{Z}_p} x^{2k} (1-x)^{n+m-2k} d\mu_{-1}(x) \\ &= \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} \int_{\mathbb{Z}_p} (1-x)^{n+m-j} d\mu_{-1}(x) \\ &= \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} \left(2 + \int_{\mathbb{Z}_p} x^{n+m-j} d\mu_{-1}(x) \right) \\ &= \begin{cases} 2 + E_{n+m} & \text{if } k = 0, \\ \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j} & \text{if } k > 0. \end{cases} \end{aligned} \tag{2.12}$$

Therefore, we obtain the following theorem.

Theorem 2.5. For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k$, one has

$$\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)d\mu_{-1}(x) = \begin{cases} 2 + E_{n+m} & \text{if } k = 0, \\ \binom{n}{k} \binom{m}{k} \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j} & \text{if } k > 0. \end{cases} \quad (2.13)$$

For $m, n, k \in \mathbb{Z}_+$, one has

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)d\mu_{-1}(x) &= \binom{n}{k} \binom{m}{k} \int_{\mathbb{Z}_p} x^{2k}(1-x)^{n+m-2k} d\mu_{-1}(x) \\ &= \binom{n}{k} \binom{m}{k} \sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j \int_{\mathbb{Z}_p} x^{j+2k} d\mu_{-1}(x) \quad (2.14) \\ &= \binom{n}{k} \binom{m}{k} \sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j E_{j+2k}. \end{aligned}$$

Thus, we obtain the following proposition.

Proposition 2.6. For $m, n, k \in \mathbb{Z}_+$, one has

$$\int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)d\mu_{-1}(x) = \binom{n}{k} \binom{m}{k} \sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j E_{j+2k}. \quad (2.15)$$

By Theorem 2.5 and Proposition 2.6, we obtain the following corollary.

Corollary 2.7. For $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k$, one has

$$\sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j E_{j+2k} = \begin{cases} 2 + E_{n+m} & \text{if } k = 0, \\ \sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j} & \text{if } k > 0. \end{cases} \quad (2.16)$$

In the same manner, multiplication of three Bernstein polynomials can be given by the following relation:

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)B_{k,s}(x)d\mu_{-1}(x) \\
 &= \binom{n}{k} \binom{m}{k} \binom{s}{k} \sum_{j=0}^{n+m+s-3k} \binom{n+m+s-3k}{j} (-1)^j \int_{\mathbb{Z}_p} x^{j+3k} d\mu_{-1}(x) \quad (2.17) \\
 &= \binom{n}{k} \binom{m}{k} \binom{s}{k} \sum_{j=0}^{n+m+s-3k} \binom{n+m+s-3k}{j} (-1)^j E_{j+3k},
 \end{aligned}$$

where $m, n, s, k \in \mathbb{Z}_+$ with $m + n + s > 3k$.

For $m, n, s, k \in \mathbb{Z}_+$ with $m + n + s > 3k$, by the symmetry of Bernstein polynomials, we see that

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)B_{k,s}(x)d\mu_{-1}(x) \\
 &= \binom{n}{k} \binom{m}{k} \binom{s}{k} \sum_{j=0}^{3k} \binom{3k}{j} (-1)^{3k-j} \int_{\mathbb{Z}_p} (1-x)^{n+m+s-j} d\mu_{-1}(x) \\
 &= \binom{n}{k} \binom{m}{k} \binom{s}{k} \sum_{j=0}^{3k} \binom{3k}{j} (-1)^{3k-j} \left(2 + \int_{\mathbb{Z}_p} x^{n+m+s-j} \mu_{-1}(x) \right) \quad (2.18) \\
 &= \begin{cases} 2 + E_{n+m+s} & \text{if } k = 0, \\ \binom{n}{k} \binom{m}{k} \binom{s}{k} \sum_{j=0}^{3k} \binom{3k}{j} (-1)^{3k-j} E_{n+m+s-j} & \text{if } k > 0. \end{cases}
 \end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.8. For $m, n, s, k \in \mathbb{Z}_+$ with $m + n + s > 3k$, one has

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} B_{k,n}(x)B_{k,m}(x)B_{k,s}(x)d\mu_{-1}(x) \\
 &= \begin{cases} 2 + E_{n+m+s} & \text{if } k = 0, \\ \binom{n}{k} \binom{m}{k} \binom{s}{k} \sum_{j=0}^{3k} \binom{3k}{j} (-1)^{3k-j} E_{n+m+s-j} & \text{if } k > 0. \end{cases} \quad (2.19)
 \end{aligned}$$

By (2.17) and Theorem 2.8, we obtain the following corollary.

Corollary 2.9. For $m, n, s, k \in \mathbb{Z}_+$ with $m + n + s > 3k$, one has

$$\begin{aligned} & \sum_{j=0}^{n+m+s-3k} \binom{n+m+s-3k}{j} (-1)^j E_{j+3k} \\ &= \begin{cases} 2 + E_{n+m+s} & \text{if } k = 0, \\ \sum_{j=0}^{3k} \binom{3k}{j} (-1)^{3k-j} E_{n+m+s-j} & \text{if } k > 0. \end{cases} \end{aligned} \quad (2.20)$$

Using the above theorems and mathematical induction, we obtain the following theorem.

Theorem 2.10. Let $s \in \mathbb{N}$. For $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \dots + n_s > sk$, the multiplication of the sequence of Bernstein polynomials $B_{k,n_1}(x), \dots, B_{k,n_s}(x)$ with different degrees under fermionic p -adic invariant integral on \mathbb{Z}_p can be given as

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^s B_{k,n_i}(x) \right) d\mu_{-1}(x) = \begin{cases} 2 + E_{n_1+n_2+\dots+n_s} & \text{if } k = 0, \\ \left(\prod_{i=1}^s \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk-j} E_{n_1+n_2+\dots+n_s-j} & \text{if } k > 0. \end{cases} \quad (2.21)$$

We also easily see that

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^s B_{k,n_i}(x) \right) d\mu_{-1}(x) = \left(\prod_{i=1}^s \binom{n_i}{k} \right) \sum_{j=0}^{n_1+\dots+n_s-sk} \binom{n_1+\dots+n_s-sk}{j} (-1)^j E_{j+sk}. \quad (2.22)$$

By Theorem 2.10 and (2.22), we obtain the following corollary.

Corollary 2.11. Let $s \in \mathbb{N}$. For $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \dots + n_s > sk$, one has

$$\sum_{j=0}^{n_1+\dots+n_s-sk} \binom{n_1+\dots+n_s-sk}{j} (-1)^j E_{j+sk} = \begin{cases} 2 + E_{n_1+n_2+\dots+n_s} & \text{if } k = 0, \\ \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk-j} E_{n_1+n_2+\dots+n_s-j} & \text{if } k > 0. \end{cases} \quad (2.23)$$

Let $m_1, \dots, m_s, n_1, \dots, n_s, k \in \mathbb{Z}_+$ with $m_1 n_1 + \dots + m_s n_s > (m_1 + \dots + m_s)k$. By the definition of $B_{k, n_s}^{m_s}(x)$, we easily get

$$\begin{aligned} & \int_{\mathbb{Z}_p} \left(\prod_{i=1}^s B_{k, n_i}^{m_i}(x) \right) d\mu_{-1}(x) \\ &= \left(\prod_{i=1}^s \binom{n_i}{k}^{m_i} \right) \sum_{j=0}^{k \sum_{i=1}^s m_i} (-1)^{k \sum_{i=1}^s m_i - j} \int_{\mathbb{Z}_p} (1-x)^{\sum_{i=1}^s n_i m_i - j} d\mu_{-1}(x) \\ &= \left(\prod_{i=1}^s \binom{n_i}{k}^{m_i} \right) \sum_{j=0}^{k \sum_{i=1}^s m_i} \binom{k \sum_{i=1}^s m_i}{j} (-1)^{k \sum_{i=1}^s m_i - j} (2 + E_{\sum_{i=1}^s n_i m_i - j}) \quad (2.24) \\ &= \begin{cases} 2 + E_{m_1 n_1 + \dots + m_s n_s} & \text{if } k = 0, \\ \left(\prod_{i=1}^s \binom{n_i}{k}^{m_i} \right) \sum_{j=0}^{k \sum_{i=1}^s m_i} \binom{k \sum_{i=1}^s m_i}{j} (-1)^{k \sum_{i=1}^s m_i - j} E_{\sum_{i=1}^s n_i m_i - j} & \text{if } k > 0. \end{cases} \end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.12. Let $s \in \mathbb{N}$. For $m_1, \dots, m_s, n_1, \dots, n_s, k \in \mathbb{Z}_+$ with $m_1 n_1 + \dots + m_s n_s > (m_1 + \dots + m_s)k$, one has

$$\begin{aligned} & \int_{\mathbb{Z}_p} \left(\prod_{i=1}^s B_{k, n_i}^{m_i}(x) \right) d\mu_{-1}(x) \\ &= \begin{cases} 2 + E_{m_1 n_1 + \dots + m_s n_s} & \text{if } k = 0, \\ \left(\prod_{i=1}^s \binom{n_i}{k}^{m_i} \right) \sum_{j=0}^{k \sum_{i=1}^s m_i} \binom{k \sum_{i=1}^s m_i}{j} (-1)^{k \sum_{i=1}^s m_i - j} E_{\sum_{i=1}^s n_i m_i - j} & \text{if } k > 0. \end{cases} \quad (2.25) \end{aligned}$$

By simple calculation, we easily get

$$\begin{aligned} & \int_{\mathbb{Z}_p} \left(\prod_{i=1}^s B_{k, n_i}^{m_i}(x) \right) d\mu_{-1}(x) \\ &= \left(\prod_{i=1}^s \binom{n_i}{k}^{m_i} \right) \sum_{j=0}^{\sum_{i=1}^s n_i m_i - k \sum_{i=1}^s m_i} \binom{\sum_{i=1}^s n_i m_i - k \sum_{i=1}^s m_i}{j} (-1)^j E_{k \sum_{i=1}^s m_i - j}, \quad (2.26) \end{aligned}$$

where $m_1, \dots, m_s, n_1, \dots, n_s, k \in \mathbb{Z}_+$ for $s \in \mathbb{N}$. By Theorem 2.12 and (2.26), we obtain the following corollary.

Corollary 2.13. Let $s \in \mathbb{N}$. For $m_1, \dots, m_s, n_1, \dots, n_s, k \in \mathbb{Z}_+$ with $m_1 n_1 + \dots + m_s n_s > (m_1 + \dots + m_s)k$, one has

$$\begin{aligned} & \sum_{j=0}^{\sum_{i=1}^s n_i m_i - k} \sum_{i=1}^s m_i \binom{\sum_{i=1}^s n_i m_i - k}{j} (-1)^j E_{k, \sum_{i=1}^s m_i - j} \\ &= \begin{cases} 2 + E_{m_1 n_1 + \dots + m_s n_s} & \text{if } k = 0, \\ \sum_{j=0}^k \binom{\sum_{i=1}^s m_i}{j} (-1)^k E_{\sum_{i=1}^s m_i - j, \sum_{i=1}^s n_i m_i - j} & \text{if } k > 0. \end{cases} \end{aligned} \quad (2.27)$$

The fermionic p -adic invariant integral of multiplication of $(n + 1)$ Bernstein polynomials, the n th degree Bernstein polynomials $B_{i,n}(x)$ with $i = 0, 1, \dots, n$ and with multiplicity m_0, m_1, \dots, m_n on \mathbb{Z}_p , respectively, can be given by

$$\begin{aligned} \int_{\mathbb{Z}_p} \left(\prod_{i=0}^n B_{i,n}^{m_i}(x) \right) d\mu_{-1}(x) &= \left(\prod_{i=0}^n \binom{n}{i}^{m_i} \right) \int_{\mathbb{Z}_p} x^{\sum_{i=1}^n i m_i} (1-x)^{n \sum_{i=0}^n m_i - \sum_{i=1}^n i m_i} d\mu_{-1}(x) \\ &= \frac{\left(\prod_{i=1}^n \binom{n}{i}^{m_i} \right)}{\binom{n \sum_{i=0}^n m_i}{\sum_{i=1}^n i m_i}} \int_{\mathbb{Z}_p} B_{\sum_{i=1}^n i m_i, n \sum_{i=0}^n m_i}(x) d\mu_{-1}(x), \end{aligned} \quad (2.28)$$

where $m_0, m_1, \dots, m_n \in \mathbb{Z}_+$ with $n \in \mathbb{Z}_+$.

Assume that $nm_0 + nm_1 + \dots + nm_n > m_1 + 2m_2 + \dots + nm_n$. Then one has

$$\begin{aligned} & \int_{\mathbb{Z}_p} \left(\prod_{i=0}^n B_{i,n}^{m_i}(x) \right) d\mu_{-1}(x) \\ &= \begin{cases} 2 + E_{nm_0 + nm_1 + \dots + nm_n} & \text{if } \sum_{i=1}^n i m_i = 0, \\ \left(\prod_{i=0}^n \binom{n}{i}^{m_i} \right) \sum_{j=0}^{\sum_{i=1}^n m_i} \binom{\sum_{i=1}^n i m_i}{j} (-1)^{\sum_{i=1}^n i m_i - j} E_{n \sum_{i=0}^n m_i - \sum_{i=1}^n i m_i} & \text{if } \sum_{i=1}^n i m_i > 0. \end{cases} \end{aligned} \quad (2.29)$$

Therefore, we obtain the following theorem.

Theorem 2.14. Let $n \in \mathbb{Z}_+$.

(i) For $m_0, m_1, \dots, m_n \in \mathbb{Z}_+$ with $n \sum_{i=0}^n m_i > \sum_{i=1}^n im_i$, one has

$$\int_{\mathbb{Z}_p} \left(\prod_{i=0}^n B_{i,n}^{m_i}(x) \right) d\mu_{-1}(x) = \begin{cases} 2 + E_{nm_0+nm_1+\dots+nm_n} & \text{if } \sum_{i=1}^n im_i = 0, \\ \left(\prod_{i=0}^n \binom{n}{i}^{m_i} \right) \sum_{j=0}^{\sum_{i=1}^n m_i} \binom{\sum_{i=1}^n im_i}{j} (-1)^{\sum_{i=1}^n im_i-j} E_{n \sum_{i=0}^n m_i - \sum_{i=1}^n im_i} & \text{if } \sum_{i=1}^n im_i > 0. \end{cases} \quad (2.30)$$

(ii) For $m_0, m_1, \dots, m_n \in \mathbb{Z}_+$, one has

$$\int_{\mathbb{Z}_p} \left(\prod_{i=0}^n B_{i,n}^{m_i}(x) \right) d\mu_{-1}(x) = \left(\prod_{i=0}^n \binom{n}{i}^{m_i} \right) \sum_{j=0}^{n \sum_{i=0}^n m_i - \sum_{i=1}^n im_i} \binom{n \sum_{i=0}^n m_i - \sum_{i=1}^n im_i}{j} (-1)^j E_{\sum_{i=1}^n im_i+j}. \quad (2.31)$$

By Theorem 2.14, we obtain the following corollary.

Corollary 2.15. For $n, m_0, m_1, \dots, m_n \in \mathbb{Z}_+$ with $n \sum_{i=0}^n m_i > \sum_{i=1}^n im_i$, one has

$$\sum_{j=0}^{n \sum_{i=0}^n m_i - \sum_{i=1}^n im_i} \binom{n \sum_{i=0}^n m_i - \sum_{i=1}^n im_i}{j} (-1)^j E_{\sum_{i=1}^n im_i+j} = \begin{cases} 2 + E_{nm_0+nm_1+\dots+nm_n} & \text{if } \sum_{i=1}^n im_i = 0, \\ \sum_{j=0}^{\sum_{i=1}^n m_i} \binom{\sum_{i=1}^n im_i}{j} (-1)^{\sum_{i=1}^n im_i-j} E_{n \sum_{i=0}^n m_i - \sum_{i=1}^n im_i} & \text{if } \sum_{i=1}^n im_i > 0. \end{cases} \quad (2.32)$$

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