Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 865096, 9 pages
doi:10.1155/2010/865096

Research Article

Lyapunov Inequalities for One-Dimensional p-Laplacian Problems with a Singular Weight Function

Inbo Sim[1] and Yong-Hoon Lee[2]

[1] *Department of Mathematics, University of Ulsan, Ulsan 680-749, South Korea*
[2] *Department of Mathematics, Pusan National University, Pusan 609-735, South Korea*

Correspondence should be addressed to Yong-Hoon Lee, yhlee@pusan.ac.kr

Received 8 October 2009; Accepted 24 November 2009

Academic Editor: Yeol Je Cho

We estimate Lyapunov inequalities for a single equation, a cycled system and a coupled system of one-dimensional p-Laplacian problems with weight functions having stronger singularities than L^1.

1. Introduction

The Lyapunov inequality for linear ordinary differential equation

$$\begin{gathered} -u'' = r(t)u, \quad t \in (a,b), \\ u(a) = 0 = u(b), \end{gathered} \tag{L}$$

where $r \in C([a,b],[0,\infty))$, gives a necessary condition for the existence of a positive solution as follows:

$$\frac{4}{b-a} \le \int_a^b r(t)\,dt. \tag{1.1}$$

Lyapunov [1] initiated to estimate the above inequality. Since then, there have been several results to generalize the above linear ordinary differential equation in many directions.

Before stating many efforts, it is worth to mention Hartman and Pinasco's work. Hartman [2] obtained the generalized inequality by using Green's function:

$$(b-a) \leq \int_a^b (t-a)(b-t)r(t)dt. \tag{1.2}$$

In fact, for $a \leq t \leq b$, by the inequality

$$(t-a)(b-t) \leq \frac{(b-a)^2}{4}, \tag{1.3}$$

condition (1.2) is a generalization of condition (1.1).

Pinasco [3] extended linear ordinary differential equations to the following one-dimensional p-Laplacian problem:

$$\begin{gathered} -\varphi_p(u'(t))' = r(t)\varphi_p(u(t)), \quad t \in (a,b), \\ u(a) = 0 = u(b), \end{gathered} \tag{P}$$

where $\varphi_p(x) = |x|^{p-2}x,\ p > 1$ and $r \in C([a,b],(0,\infty))$. He obtained Lyapunov inequality for (P) as follows:

$$\frac{2^p}{(b-a)^{p/q}} \leq \int_a^b r(t)dt, \tag{1.4}$$

where $1/p + 1/q = 1$.

There have been many studies for various types of equations. Among others, one may refer to de Náploi and Pinasco [4] for the case of monotone quasilinear operators which include one-dimensional p-Laplacian as a special case, Parhi and Panigrahi [5] for the case of third order differential equations, Cañada et al. [6] for the case of partial differential equations which have a weight function in L^1, and Clark and Hinton [7] for the case of Hamiltonian systems.

Until now, the most general class of weight functions for the Lyapunov inequalities is $L^1(a,b)$. The purpose of this paper is to get Lyapunov inequalities for single equations as well as systems of one-dimensional p-Laplacian problems with singular weight functions which have a stronger singularities than those of $L^1(a,b)$.

For this purpose, we first give three specific classes of weight functions. The first class can be given as

$$\begin{aligned} \mathcal{A} \triangleq \Bigg\{ r \in C((a,b),[0,\infty)) : &\int_a^{(a+b)/2} \varphi_p^{-1}\left(\int_s^{(a+b)/2} r(\tau)d\tau\right)ds \\ &+ \int_{(a+b)/2}^b \varphi_p^{-1}\left(\int_{(a+b)/2}^s r(\tau)d\tau\right)ds < \infty \Bigg\}. \end{aligned} \tag{1.5}$$

It comes naturally from the study of the existence of positive solutions for p-Laplacian problems. The second one is just the extension of Hartman's condition to p-Laplacian problems given as follows:

$$\mathcal{B} \triangleq \left\{ r \in C((a,b),[0,\infty)) : \int_a^b (s-a)^{p-1}(b-s)^{p-1} r(s)ds < \infty \right\}. \tag{1.6}$$

It is easy to see that $L^1(a,b) \subset \mathcal{A} \cap \mathcal{B}$ and classes $\mathcal{A}$ and $\mathcal{B}$ are equivalent when $p = 2$. It is also known [8] that $\mathcal{B} \subsetneqq \mathcal{A}$ for $p > 2$ and $\mathcal{A} \subsetneqq \mathcal{B}$ for $1 < p < 2$. The third one can be given as

$$\begin{aligned} \mathcal{C} \triangleq \Bigg\{ r \in C((a,b),[0,\infty)) :\ & \text{there are } \alpha,\beta > 0 \text{ such that } \alpha,\beta < p-1 \text{ and} \\ & \int_a^b (s-a)^{\alpha}(b-s)^{\beta} r(s)ds < \infty \Bigg\}. \end{aligned} \tag{1.7}$$

It is obvious to see that $\mathcal{C} \subset \mathcal{B}$ and $\mathcal{C} \subset \mathcal{A}$ (see [8]).

This paper is organized as follows. In Section 2, we show Lyapunov inequality for one-dimensional p-Laplacian problem when a weight function is $r \in \mathcal{A} \cap \mathcal{B}$. In Section 3, we estimate Lyapunov inequality for a cycled system of one-dimensional p-Laplacian problem when a weight function is $r \in \mathcal{C}$. Finally in Section 4, we have Lyapunov inequality for a strongly coupled system of one-dimensional p-Laplacian problem when a weight function is $r \in \mathcal{C}$.

2. Single Equation

Let us consider problem (P). By a solution of (P) we mean that $u \in C[a,b] \cap C^1(a,b)$, $\varphi_p(u')$ is absolutely continuous in any compact subinterval of (a,b), and u satisfies the first equation in (P) in (a,b) and $u(a) = 0 = u(b)$. We assume $r \in \mathcal{A} \cap \mathcal{B}$. It is known that all solutions for (P) are of class $C_0^1[a,b]$ (see [9]).

Theorem 2.1. *Assume $r \in \mathcal{A} \cap \mathcal{B}$. If u is a positive solution for (P), then one has*

$$\frac{(b-a)^{p-1}}{2^{p-2}} \le \int_a^b (t-a)^{p-1}(b-t)^{p-1} r(t)dt. \tag{2.1}$$

Proof. By Hölder's inequality, we get

$$|u(t)| \le \int_a^t |u'(s)|ds \le (t-a)^{(p-1)/p} \left(\int_a^t |u'|^p ds \right)^{1/p}. \tag{2.2}$$

For $a \le t \le (a+b)/2$, noting $t-a \le (2/(b-a))(t-a)(b-t)$, we have

$$|u(t)| \le \left(\frac{2}{b-a}(t-a)(b-t)\right)^{(p-1)/p}\left(\int_a^{(a+b)/2}\left|u'\right|^p ds\right)^{1/p}. \tag{2.3}$$

Thus, we have

$$|u(t)|^p \le \left(\frac{2}{b-a}(t-a)(b-t)\right)^{p-1}\left(\int_a^{(a+b)/2}\left|u'\right|^p ds\right). \tag{2.4}$$

Similarly, by Hölder's inequality, we get

$$|u(t)| \le \int_t^b \left|u'(s)\right| ds \le (b-t)^{(p-1)/p}\left(\int_t^b \left|u'\right|^p ds\right)^{1/p}. \tag{2.5}$$

For $(a+b)/2 \le t \le b$, noting $b-t \le (2/(b-a))(t-a)(b-t)$, we get

$$|u(t)|^p \le \left(\frac{2}{b-a}(t-a)(b-t)\right)^{p-1}\left(\int_{(a+b)/2}^{b}\left|u'\right|^p ds\right). \tag{2.6}$$

Adding (2.4) and (2.6), we have

$$2|u(t)|^p \le \left(\frac{2}{b-a}(t-a)(b-t)\right)^{p-1}\left(\int_a^b\left|u'\right|^p ds\right). \tag{2.7}$$

Multiplying both sides of (2.7) by $r(t)$ and rewriting, we get

$$\frac{(b-a)^{p-1}}{2^{p-2}} r(t)|u(t)|^p \le r(t)((t-a)(b-t))^{p-1}\left(\int_a^b\left|u'\right|^p ds\right). \tag{2.8}$$

Since u is a solution for (P), we have

$$\int_a^b \left|u'\right|^p dt = \int_a^b r(t)|u(t)|^p dt. \tag{2.9}$$

We note that the right-hand side makes sense because u is in $C_0^1[a,b]$. Integrating (2.8) on $[a,b]$ and using (2.9), we have

$$\begin{aligned}\frac{(b-a)^{p-1}}{2^{p-2}}\int_a^b\left|u'\right|^p dt &= \int_a^b \frac{(b-a)^{p-1}}{2^{p-2}} r(t)|u(t)|^p dt \\ &\le \int_a^b r(t)((t-a)(b-t))^{p-1}\left(\int_a^b\left|u'\right|^p ds\right) dt.\end{aligned} \tag{2.10}$$

Therefore, we get

$$\frac{(b-a)^{p-1}}{2^{p-2}} \le \int_a^b r(t)((t-a)(b-t))^{p-1}dt. \tag{2.11}$$

□

Remark 2.2. (i) When $p = 2$, the above result coincides with Hartman's estimate. But Hartman's argument does not work here by lack of Green's function for p-Laplacian.

(ii) If $r \in L^1(a,b)$, for $a \le t \le b$, since $(t-a)(b-t) \le (b-a)^2/4$ and $p/q = p-1$, we have Pinasco's estimate (1.4). Thus our estimate generalizes Pinasco's.

For $(0 \le) r \in C[a,b]$, Pinasco [3] also estimated the lower bounds for eigenvalues $\{\lambda_n\}$ of

$$\begin{gathered} -\varphi_p(u'(t))' = \lambda r(t)\varphi_p(u(t)), \quad t \in (a,b), \\ u(a) = 0 = u(b). \end{gathered} \tag{P_λ}$$

The proof mainly makes use of the nodal property of its corresponding eigenfunctions $\{u_n\}$; that is, u_n has $n-1$ interior zeros in (a,b). Recently, when $r \in \mathcal{A} \cap \mathcal{B}$, Kajikiya et al. [9] showed the existence of eigenvalues $\{\lambda_n\}$ for (P_λ) and its corresponding eigenfunctions also have the nodal property. Employing Pinasco's argument ([3, Theorem 1.1]) with (2.1), for each $n \in \mathbb{N}$, we have

$$\lambda_n \ge \frac{(b-a)^{p-1}}{(2n)^{p-2}\int_a^b r(t)((t-a)(b-t))^{p-1}dt}. \tag{2.12}$$

3. Cycled System

Let us consider a cycled system:

$$\begin{gathered} \varphi_p(u_1'(t))' + r_1(t)\varphi_p(u_2(t)) = 0, \quad t \in (a,b), \\ \varphi_p(u_2'(t))' + r_2(t)\varphi_p(u_3(t)) = 0, \quad t \in (a,b), \\ \cdots \\ \varphi_p(u_{n-1}'(t))' + r_{n-1}(t)\varphi_p(u_n(t)) = 0, \quad t \in (a,b), \\ \varphi_p(u_n'(t))' + r_n(t)\varphi_p(u_1(t)) = 0, \quad t \in (a,b), \\ u_1(a) = \cdots = u_n(a) = 0 = u_1(b) = \cdots = u_n(b). \end{gathered} \tag{CS}$$

We say that $(u_1, u_2, \ldots, u_n)$ is a solution of (CS) if $u_i \in C[a,b] \cap C^1(a,b)$, $\varphi_p(u_i')$ is absolutely continuous in any compact subinterval of (a,b), each u_i satisfies the equations in (CS) in (a,b), and $u_1(a) = \cdots = u_n(a) = 0 = u_1(b) = \cdots = u_n(b)$. We assume that $r_i \in \mathcal{C}$. We note that all solutions for (CS) are of class $C_0^1[a,b]$ (see [10]).

Theorem 3.1. *Assume $r_i \in \mathcal{C}$. If $(u_1, u_2, \ldots, u_n)$ is a positive solution of (CS), then*

$$\int_a^b ((t-a)(b-t))^{p-1} r_1(t)dt \cdots \int_a^b ((t-a)(b-t))^{p-1} r_n(t)dt \geq \frac{\left[(b-a)^{p-1}\right]^n}{\left[2^{p-2}\right]^n}. \tag{3.1}$$

Proof. We only show the case $n = 2$. For the general case, we can prove it by repeating this procedure. As in (2.7), for $i = 1, 2$, we have

$$|u_i(t)|^{p-1} \leq \frac{2^{(p-2)(p-1)/p}}{(b-a)^{(p-1)^2/p}} ((t-a)(b-t))^{(p-1)^2/p} \left(\int_a^b |u_i'|^p ds \right)^{(p-1)/p} \tag{3.2}$$

or

$$|u_i(t)| \leq \frac{2^{(p-2)/p}}{(b-a)^{(p-1)/p}} ((t-a)(b-t))^{(p-1)/p} \left(\int_a^b |u_i'|^p ds \right)^{1/p}. \tag{3.3}$$

Multiplying the first equation of (CS) by u_1 and integrating on $[a, b]$, we have by (3.2) and (3.3) that

$$\begin{aligned} \int_a^b |u_1'|^p dt &\leq \int_a^b r_1(t)|u_2|^{p-1}|u_1| dt \\ &\leq \int_a^b \frac{2^{p-2}}{(b-a)^{p-1}} ((t-a)(b-t))^{p-1} r_1(t) dt \\ &\quad \times \left(\int_a^b |u_2'|^p ds \right)^{(p-1)/p} \left(\int_a^b |u_1'|^p ds \right)^{1/p}. \end{aligned} \tag{3.4}$$

Thus, we have

$$\left(\int_a^b |u_1'|^p dt \right)^{(p-1)/p} \leq \frac{2^{p-2}}{(b-a)^{p-1}} \int_a^b ((t-a)(b-t))^{p-1} r_1(t) dt \left(\int_a^b |u_2'|^p ds \right)^{(p-1)/p}. \tag{3.5}$$

Similarly, for the second equation in (CS), we have

$$\left(\int_a^b |u_2'|^p dt \right)^{(p-1)/p} \leq \frac{2^{p-2}}{(b-a)^{p-1}} \int_a^b ((t-a)(b-t))^{p-1} r_2(t) dt \left(\int_a^b |u_1'|^p ds \right)^{(p-1)/p}. \tag{3.6}$$

Thus, we have

$$\int_a^b ((t-a)(b-t))^{p-1} r_1(t) dt \int_a^b ((t-a)(b-t))^{p-1} r_2(t) dt \geq \frac{\left[(b-a)^{p-1}\right]^2}{\left[2^{p-2}\right]^2}. \tag{3.7}$$

□

Corollary 3.2. *Assume $r_i = r \in \mathcal{C}$, for $i = 1, 2, \dots, n$. If $(u_1, u_2, \dots, u_n)$ is a positive solution of (CS), then one has*

$$\int_a^b ((t-a)(b-t))^{p-1} r(t)dt \geq \frac{(b-a)^{p-1}}{2^{p-2}}. \tag{3.8}$$

4. Strongly Coupled System

Let us consider a strongly coupled system:

$$\begin{gathered}
\varphi_p(u_1'(t))' + r_1(t)(\varphi_p(u_1(t)) + \varphi_p(u_2(t)) + \cdots + \varphi_p(u_n(t))) = 0, \quad t \in (a,b), \\
\varphi_p(u_2'(t))' + r_2(t)(\varphi_p(u_1(t)) + \varphi_p(u_2(t)) + \cdots + \varphi_p(u_n(t))) = 0, \quad t \in (a,b), \\
\cdots \\
\varphi_p(u_n'(t))' + r_n(t)(\varphi_p(u_1(t)) + \varphi_p(u_2(t)) + \cdots + \varphi_p(u_n(t))) = 0, \quad t \in (a,b), \\
u_1(a) = \cdots = u_n(a) = 0 = u_1(b) = \cdots = u_n(b),
\end{gathered} \tag{SCS}$$

where $r_i \in \mathcal{C}$. We can give a definition for a solution of (SCS) as the definition for a solution of (CS) and it is known that all positive solutions for (SCS) are of class $C_0^1[a,b]$ (see [10]). We emphasize that it is only shown for a positive solution so far.

Theorem 4.1. *Assume $r_i \in \mathcal{C}$. If $(u_1, u_2, \dots, u_n)$ is a positive solution of (SCS), then one has*

$$\int_a^b ((t-a)(b-t))^{p-1} r_1(t)dt + \cdots + \int_a^b ((t-a)(b-t))^{p-1} r_n(t)dt \geq \frac{1}{n}\frac{(b-a)^{p-1}}{2^{p-2}}. \tag{4.1}$$

Proof. As in the proof of Theorem 3.1, we only show the case $n = 2$. Multiplying u_1 to the first equation in (SCS) and integrating on $[a,b]$ and using (2.7), (3.2), and (3.3), we have

$$\begin{aligned}
\int_a^b |u_1'|^p dt &\leq \int_a^b r_1(t)|u_1|^p dt + \int_a^b r_1(t)|u_2|^{p-1}|u_1| dt \\
&\leq \frac{2^{p-2}}{(b-a)^{p-1}} \int_a^b ((t-a)(b-t))^{p-1} r_1(t)dt \int_a^b |u_1'|^p ds \\
&\quad + \frac{2^{p-2}}{(b-a)^{p-1}} \int_a^b ((t-a)(b-t))^{p-1} r_1(t)dt \\
&\quad \times \left(\int_a^b |u_2'|^p ds \right)^{(p-1)/p} \left(\int_a^b |u_1'|^p ds \right)^{1/p}.
\end{aligned} \tag{4.2}$$

Similarly, from the second equation of (SCS), we have

$$\begin{aligned}
\int_a^b |u_2'|^p dt &\le \int_a^b r_2(t)|u_2|^p dt + \int_a^b r_2(t)|u_1|^{p-1}|u_2| dt \\
&\le \frac{2^{p-2}}{(b-a)^{p-1}} \int_a^b ((t-a)(b-t))^{p-1} r_2(t) dt \int_a^b |u_2'|^p ds \\
&\quad + \frac{2^{p-2}}{(b-a)^{p-1}} \int_a^b ((t-a)(b-t))^{p-1} r_2(t) dt \\
&\quad \times \left(\int_a^b |u_1'|^p ds \right)^{(p-1)/p} \left(\int_a^b |u_2'|^p ds \right)^{1/p}.
\end{aligned} \tag{4.3}$$

Let us denote $X = \int_a^b |u_1'|^p dt$, $Y = \int_a^b |u_2'|^p dt$, $C_1 = (2^{p-2}/(b-a)^{p-1}) \int_a^b ((t-a)(b-t))^{p-1} r_1(t) dt$, and $C_2 = (2^{p-2}/(b-a)^{p-1}) \int_a^b ((t-a)(b-t))^{p-1} r_2(t) dt$. Then from (4.2) and (4.3), we have

$$\begin{aligned}
X &\le C_1 X + C_1 X^{1/p} Y^{(p-1)/p}, \\
Y &\le C_2 Y + C_2 Y^{1/p} X^{(p-1)/p},
\end{aligned} \tag{4.4}$$

respectively. Equation (4.4) implies

$$\begin{aligned}
X &\le C_1 (X+Y) + C_1 \left(X^{1/p} Y^{(p-1)/p} + Y^{1/p} X^{(p-1)/p} \right), \\
Y &\le C_2 (X+Y) + C_2 \left(X^{1/p} Y^{(p-1)/p} + Y^{1/p} X^{(p-1)/p} \right),
\end{aligned} \tag{4.5}$$

respectively. Therefore, we have

$$X + Y \le (C_1 + C_2)(X+Y) + (C_1 + C_2)\left(X^{1/p} Y^{(p-1)/p} + Y^{1/p} X^{(p-1)/p} \right). \tag{4.6}$$

Since $X^{1/p} Y^{(p-1)/p} + Y^{1/p} X^{(p-1)/p} \le X + Y$ ([11, page 38]), we get

$$X + Y \le 2(C_1 + C_2)(X + Y). \tag{4.7}$$

Hence, we have

$$C_1 + C_2 \ge \frac{1}{2}. \tag{4.8}$$

That is,

$$\int_a^b ((t-a)(b-t))^{p-1} r_1(t) dt + \int_a^b ((t-a)(b-t))^{p-1} r_2(t) dt \ge \frac{1}{2} \frac{(b-a)^{p-1}}{2^{p-2}}. \tag{4.9}$$

□

Corollary 4.2. *Assume $r_i = r \in \mathcal{C}$, for $i = 1, 2, \ldots, n$. If $(u_1, u_2, \ldots, u_n)$ is a positive solution of (SCS), then one has*

$$\int_a^b ((t-a)(b-t))^{p-1} r(t)dt \geq \frac{1}{n^2} \frac{(b-a)^{p-1}}{2^{p-2}}. \tag{4.10}$$

Acknowledgment

The first author was supported by the 2009 Research Fund of the University of Ulsan.

References

[1] A. Lyapunov, *Probleme General de la Stabilite du Mouvement*, Annals of Mathematics Studies, no. 17, Princeton University Press, Princeton, NJ, USA, 1949.
[2] P. Hartman, *Ordinary Differential Equations*, Birkhäuser, Boston, Mass, USA, 2nd edition, 1982.
[3] J. P. Pinasco, "Lower bounds for eigenvalues of the one-dimensional p-Laplacian," *Abstract and Applied Analysis*, no. 2, pp. 147–153, 2004.
[4] P. L. de Nápoli and J. P. Pinasco, "A Lyapunov inequality for monotone quasilinear operators," *Differential and Integral Equations*, vol. 18, no. 10, pp. 1193–1200, 2005.
[5] N. Parhi and S. Panigrahi, "On Liapunov-type inequality for third-order differential equations," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 2, pp. 445–460, 1999.
[6] A. Cañada, J. A. Montero, and S. Villegas, "Lyapunov inequalities for partial differential equations," *Journal of Functional Analysis*, vol. 237, no. 1, pp. 176–193, 2006.
[7] S. Clark and D. Hinton, "A Liapunov inequality for linear Hamiltonian systems," *Mathematical Inequalities & Applications*, vol. 1, no. 2, pp. 201–209, 1998.
[8] J. Byun and I. Sim, "A relation between two classes of indefinite weights in singular one-dimensional p-Laplacian problems," *Mathematical Inequalities & Applications*, vol. 10, no. 4, pp. 889–894, 2007.
[9] R. Kajikiya, Y.-H. Lee, and I. Sim, "One-dimensional p-Laplacian with a strong singular indefinite weight. I. Eigenvalue," *Journal of Differential Equations*, vol. 244, no. 8, pp. 1985–2019, 2008.
[10] E. K. Lee, Y.-H. Lee, and I. Sim, "C^1-regularity of solutions for p-Laplacian problems," *Applied Mathematics Letters*, vol. 22, no. 5, pp. 759–765, 2009.
[11] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalties*, Cambridge University Press, Cambridge, UK, 1978.