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On an Inequality Conjectured by T.J. Lyons

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Let n be any positive integer, x and y any positive real numbers. The inequality

$$\alpha \sum_{j=0}^{n} \frac{(\alpha n)!}{(\alpha j)!\,(\alpha(n-j))!}\, x^{\alpha j} y^{\alpha(n-j)} \leq (x+y)^{\alpha n}$$

was conjectured for $0 < \alpha < 1$ by T.J. Lyons, after he had proved it with an extra factor $1/\alpha$ on the right, in a preprint (Imperial College of Science, Technology and Medicine, 1995). Many numerical trials confirmed the conjecture, and none disproved it. The present paper proves it, with strict inequality, for all α in sufficiently small neighbourhoods of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

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In a long preprint entitled "Differential equations driven by rough signals" (Imperial College, London, 1995), Lyons proves a "neo-classical inequality":

$$\frac{1}{p^2} \sum_{j=0}^{n} \frac{a^{j/p} b^{(n-j)/p}}{(j/p)!((n-j)/p)!} \leq \frac{(a+b)^{n/p}}{(n/p)!},$$

where $p \geq 1$, n is a positive integer, $a > 0$ and $b > 0$ [§2.2.3, pp. 38–42]. He also remarks that this inequality appears to hold with the factor $1/p^2$ on the left replaced by $1/p$, on the basis of numerical evidence. The present paper proves this conjecture for a certain infinite set of values of $1/p$.

It is convenient to write α, x and y in a place of $1/p$, a and b respectively, and to use the binomial coefficient notation

$$\binom{w}{z} = \frac{w!}{z!(w-z)!} = \frac{\Gamma(w+1)}{\Gamma(z+1)\Gamma(w-z+1)}$$

for arbitrary real (or complex) w and z. Lyons's *proved* inequality can then be written

$$\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} x^{\alpha j} y^{\alpha(n-j)} \leq \frac{1}{\alpha}(x+y)^{\alpha n},$$

where $0 < \alpha \leq 1$, $x > 0$ and $y > 0$. Under the same conditions, his *conjectured* inequality is the stronger one in which the factor $1/\alpha$ on the right is reduced to 1. Our first, and main, aim in this paper is to prove that

$$\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} x^{\alpha j} y^{\alpha(n-j)} < (x+y)^{\alpha n} \tag{1}$$

for $\alpha = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ Of course the two sides are equal for $\alpha = 1$, by the binomial theorem.

LEMMA 1 *If $0 < \omega < \frac{1}{2}\pi$ and m is a fixed positive real number, then*

$$I(\omega) = \int_0^{\pi/2} \cos^m(\phi - \omega)\,\mathrm{d}\phi$$

is greatest when $\omega = \frac{1}{4}\pi$, with greatest value $2\int_0^{\pi/4} \cos^m \theta \,\mathrm{d}\theta$.

Proof Since

$$I(\omega) = \int_{-\omega}^{(\pi/2)-\omega} \cos^m \theta \,\mathrm{d}\theta,$$

$$I'(\omega) = -\cos^m(\tfrac{1}{2}\pi - \omega) + \cos^m(-\omega) = \cos^m \omega - \sin^m \omega;$$

so $I(\omega)$ is stationary when $\tan^m \omega = 1$, $\omega = \frac{1}{4}\pi$. Also

$$I''(\omega) = -m\cos^{m-1}\omega \,\sin\omega - m\sin^{m-1}\omega\,\cos\omega < 0,$$

so $\mathrm{I}(\omega)$ is concave throughout $0 < \omega < \frac{1}{2}\pi$. Together these facts prove the lemma.

LEMMA 2 *If $\alpha > 0$ and for all positive x and y such that $x+y \leq 1$,*

$$2\alpha \sum_{j=0}^{n} \binom{2\alpha n}{2\alpha j} x^{2\alpha j} y^{2\alpha(n-j)} \leq (x+y)^{2\alpha n}, \tag{2}$$

then, for all positive x and y,

$$\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} x^{\alpha j} y^{\alpha(n-j)} < (x+y)^{\alpha n}. \tag{3}$$

Proof In Legendre's duplication formula, $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$, take $z = \alpha j + \frac{1}{2}$; thus

$$\Gamma(\alpha j + 1) = \frac{\sqrt{\pi}\,\Gamma(2\alpha j + 1)}{2^{2\alpha j}\Gamma(\alpha j + \frac{1}{2})}. \tag{4}$$

Consequently,

$$\begin{aligned}
\binom{\alpha n}{\alpha j} &= \frac{\Gamma(\alpha n + 1)}{\Gamma(\alpha j + 1)\Gamma(\alpha(n-j)+1)} \\
&= \frac{\sqrt{\pi}\,\Gamma(2\alpha n + 1)}{2^{2\alpha n}\Gamma(\alpha n + \frac{1}{2})} \frac{2^{2\alpha j}\Gamma(\alpha j + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(2\alpha j + 1)} \frac{2^{2\alpha(n-j)}\Gamma(\alpha(n-j)+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(2\alpha(n-j)+1)} \\
&= \frac{1}{\sqrt{\pi}} \binom{2\alpha n}{2\alpha j} \frac{\Gamma(\alpha j + \frac{1}{2})\,\Gamma(\alpha(n-j)+\frac{1}{2})}{\Gamma(\alpha n + \frac{1}{2})} \\
&= \binom{2\alpha n}{2\alpha j} \frac{\Gamma(\alpha n + 1)}{\Gamma(\frac{1}{2})\Gamma(\alpha n + \frac{1}{2})} \frac{\Gamma(\alpha j + \frac{1}{2})\,\Gamma(\alpha(n-j)+\frac{1}{2})}{\Gamma(\alpha n + 1)} \\
&= \binom{2\alpha n}{2\alpha j} \frac{\mathrm{B}(\alpha j + \frac{1}{2}, \alpha(n-j)+\frac{1}{2})}{\mathrm{B}(\alpha n + \frac{1}{2}, \frac{1}{2})},
\end{aligned}$$

where B denotes the beta function. This gives, for all positive x and y,

$$\begin{aligned}
\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} x^{\alpha j} y^{\alpha(n-j)} = {} & \frac{\alpha}{\mathrm{B}(\alpha n + \frac{1}{2}, \frac{1}{2})} \sum_{j=0}^{n} \binom{2\alpha n}{2\alpha j} \\
& \times \mathrm{B}(\alpha j + \tfrac{1}{2}, \alpha(n-j)+\tfrac{1}{2}) x^{\alpha j} y^{\alpha(n-j)}
\end{aligned}$$

$$= \frac{\alpha}{\mathrm{B}(\alpha n + \frac{1}{2}, \frac{1}{2})} \sum_{j=0}^{n} \binom{2\alpha n}{2\alpha j}$$
$$\times \int_0^1 u^{\alpha j - 1/2} (1-u)^{\alpha(n-j)-1/2} \mathrm{d}u \cdot x^{\alpha j} y^{\alpha(n-j)}$$
$$= \frac{\alpha}{\mathrm{B}(\alpha n + \frac{1}{2}, \frac{1}{2})} \int_0^1 \sum_{j=0}^{n} \binom{2\alpha n}{2\alpha j} u^{\alpha j - 1/2}$$
$$\times (1-u)^{\alpha(n-j)-1/2} \mathrm{d}u \cdot r^{\alpha j} (1-r)^{\alpha(n-j)} (x+y)^{\alpha n},$$

where $r = x/(x+y)$. This expression is now

$$\frac{\alpha(x+y)^{\alpha n}}{\mathrm{B}(\alpha n + \frac{1}{2}, \frac{1}{2})} \int_0^1 u^{-1/2} (1-u)^{-1/2} \sum_{j=0}^{n} \binom{2\alpha n}{2\alpha j}$$
$$\times (ur)^{\alpha j} \{(1-u)(1-r)\}^{\alpha(n-j)} \, \mathrm{d}u.$$

Now put $u = \cos^2\phi$ and $r = \cos^2\omega$, where ϕ and ω are in $(0, \frac{1}{2}\pi)$. The expression becomes

$$\frac{\alpha(x+y)^{\alpha n}}{\mathrm{B}(\alpha n + \frac{1}{2}, \frac{1}{2})} \int_0^{\pi/2} 2 \sum_{j=0}^{n} \binom{2\alpha n}{2\alpha j} (\cos\phi \, \cos\omega)^{2\alpha j} (\sin\phi \, \sin\omega)^{2\alpha(n-j)} \, \mathrm{d}\phi$$

$$\leq \frac{(x+y)^{\alpha n}}{\mathrm{B}(\alpha n + \frac{1}{2}, \frac{1}{2})} \int_0^{\pi/2} (\cos\phi \, \cos\omega + \sin\phi \, \sin\omega)^{2\alpha n} \, \mathrm{d}\phi$$

using the inequality (2) in the hypothesis. Thus

$$\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} x^{\alpha j} y^{\alpha(n-j)} \leq \frac{(x+y)^{\alpha n}}{\mathrm{B}(\alpha n + \frac{1}{2}, \frac{1}{2})} \int_0^{\pi/2} \cos^{2\alpha n}(\phi - \omega) \, \mathrm{d}\phi$$
$$\leq \frac{2(x+y)^{\alpha n}}{\mathrm{B}(\alpha n + \frac{1}{2}, \frac{1}{2})} \int_0^{\pi/4} \cos^{2\alpha n} \theta \, \mathrm{d}\theta \quad \text{by Lemma 1,}$$
$$= (x+y)^{\alpha n} \int_0^{\pi/4} \cos^{2\alpha n} \theta \, \mathrm{d}\theta \Big/ \int_0^{\pi/2} \cos^{2\alpha n} \theta \, \mathrm{d}\theta$$
$$< (x+y)^{\alpha n}, \quad \text{as required.}$$

THEOREM 1 *If $x > 0$ and $y > 0$ then*

$$\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} x^{\alpha j} y^{\alpha(n-j)} < (x+y)^{\alpha n} \tag{1}$$

for $\alpha = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ and all positive integers n.

Proof If $\alpha = \frac{1}{2}$ the binomial theorem gives that (2) holds. The required inequality (3), for $\alpha = \frac{1}{2}$, then follows by Lemma 2.

Inequality (3) now follows for $\alpha = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ from the case $\alpha = \frac{1}{2}$ by successive applications of Lemma 2 with these values of α.

THEOREM 2 *Inequality* (1) *in Theorem* 1 *holds for all α in sufficiently small neighbourhoods of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ and all n, x* and *y considered therein.*

Proof Since

$$\alpha \sum_{j=0}^{n} \binom{\alpha n}{\alpha j} x^{\alpha j} y^{\alpha(n-j)} / (x+y)^{\alpha n}$$

is a continuous function of α in $0 < \alpha \leq 1$, and is strictly less than 1 at the points $\alpha = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ by Theorem 1, it is strictly less than 1 in sufficiently small neighbourhoods of these points.