

On the Sharpness of a Superconvergence Estimate in Connection with One-Dimensional Galerkin Methods

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The present paper studies some aspects of approximation theory in the context of one-dimensional Galerkin methods. The phenomenon of superconvergence at the knots is well-known. Indeed, for smooth solutions the rate of convergence at these points is $\mathcal{O}(h^{2r})$ instead of $\mathcal{O}(h^{r+1})$, where r is the degree of the finite element space. In order to achieve a corresponding result for less smooth functions, we apply K-functional techniques to a Jackson-type inequality and estimate the relevant error by a modulus of continuity. Furthermore, this error estimate requires no additional assumptions on the solution, and it turns out that it is sharp in connection with general Lipschitz classes. The proof of the sharpness is based upon a quantitative extension of the uniform boundedness principle in connection with some ideas of Douglas and Dupont [*Numer. Math.* **22**] (1974) 99–109. Here it is crucial to design a sequence of test functions such that a Jackson–Bernstein-type inequality and a resonance condition are satisfied simultaneously.

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1 INTRODUCTION

We consider the two point boundary value problem

$$\begin{aligned} -(a(x)u'(x))' + b(x)u(x) &= f(x), \quad x \in (0, 1), \\ u(0) = u(1) &= 0. \end{aligned}$$

The corresponding weak problem is to find a solution $u \in W_0^{1,2}(0, 1)$ satisfying

$$a(u, v) = (f, v)_{L^2(0,1)} \quad \text{for all } v \in W_0^{1,2}(0, 1), \quad (1.1)$$

where $a(u, v) := \int_0^1 [a(x)u'(x)v'(x) + b(x)u(x)v(x)] dx$. Here $W^{s,2}(0, 1)$ denotes the Sobolev space (cf. [1]) of those real-valued functions which possess weak derivatives up to the order s belonging to the Hilbert space $L^2(0, 1)$ of square integrable functions on $(0, 1)$. Therefore, $W^{s,2}(0, 1)$ equipped with the inner product

$$(u, v)_{s,2,(0,1)} := \sum_{k=0}^s (u^{(k)}, v^{(k)})_{L^2(0,1)} := \sum_{k=0}^s \int_0^1 u^{(k)}(x)v^{(k)}(x) dx$$

and the norm $\|u\|_{s,2,(0,1)} := [\sum_{k=0}^s \|u^{(k)}\|_{L^2(0,1)}^2]^{1/2}$ is a Hilbert space as well. Besides, we use the semi-norms $|u|_{j,2,(0,1)} := \|u^{(j)}\|_{L^2(0,1)}$. Let $W_0^{1,2}(0, 1)$ be the closure of $C_{00}^\infty(0, 1)$ in $W^{1,2}(0, 1)$, where $C_{00}^\infty(0, 1)$ is the set of all infinitely often differentiable functions with support in $(0, 1)$. Moreover, similar to $W^{s,2}(0, 1)$ let $W^{s,\infty}(0, 1)$ be the Sobolev space of functions with weak derivatives up to the order s in the space of essentially bounded functions $L^\infty(0, 1)$.

We assume that the function $a(x)$ is Lipschitz continuous on $[0, 1]$ and that $b(x) \in L^\infty(0, 1)$. Therefore, $a(\cdot, \cdot)$ is bounded, i.e., $|a(u, v)| \leq C\|u\|_{1,2,(0,1)}\|v\|_{1,2,(0,1)}$, $u, v \in W_0^{1,2}(0, 1)$. In order to ensure that $a(\cdot, \cdot)$ is $W_0^{1,2}$ -elliptic, i.e., there exists a constant $c > 0$ such that $a(u, u) \geq c\|u\|_{1,2,(0,1)}^2$, $u \in W_0^{1,2}(0, 1)$, we assume $a(x) > \kappa > 0$ and $b(x) \geq 0$ a.e. Further, let $f \in L^2(0, 1)$. Now the representation theorem of F. Riesz assures the unique solvability of problem (1.1). Indeed, the solution u does not only belong to $W_0^{1,2}(0, 1)$ but even to $W^{2,2}(0, 1)$ (cf. [12, p. 200]).

For a discretization via the finite element method we consider the equidistant partitions ($h = 1/n$, $n \in \mathbb{N}$; \mathbb{N} denotes the set of

natural numbers)

$$\mathcal{T}_h := \{[jh, (j+1)h]: 0 \leq j < n\}.$$

The finite element spaces V_h of degree r are now given by

$$V_h = V_h(r) := \{v \in C_0[0, 1]: v \in \mathcal{P}_r[jh, (j+1)h], 0 \leq j < n\},$$

where $C[0, 1]$ is the space of functions continuous on $[0, 1]$ and $C_0[0, 1]$ the subspace of those functions u satisfying $u(0) = u(1) = 0$. By $\mathcal{P}_r[c, d]$ we denote the set of all algebraic polynomials $v(x) = \sum_{k=0}^r \alpha_k x^k$ with degree at most r restricted to $[c, d]$. Obviously, $V_h \subset W_0^{1,2}(0, 1)$. The discretization of problem (1.1) now reads

$$a(u_h, v) = (f, v)_{L^2(0,1)} \quad \text{for all } v \in V_h. \quad (1.2)$$

The theorem of F. Riesz again guarantees the existence of a unique solution $u_h \in V_h$. Furthermore, there still exists a unique solution if we replace the right hand side of (1.2) by an arbitrary functional on $W_0^{1,2}(0, 1)$. The Ritz projection $P_h : W_0^{1,2}(\Omega) \rightarrow V_h$ is therefore well defined via

$$a(P_h u, v) = a(u, v) \quad \text{for all } u \in W_0^{1,2}(0, 1), v \in V_h. \quad (1.3)$$

Due to the ellipticity of $a(\cdot, \cdot)$ the linear operator P_h is bounded independently of h which means that the finite element method is stable.

The aim of this paper is to discuss the error $u - u_h$, arising from problem (1.2), at the knots jh of the partitions. For smooth solutions u one can prove convergence of order $\mathcal{O}(h^r)$ on the whole interval $[0, 1]$ whereas in these special points jh the rate increases to $\mathcal{O}(h^{2r})$ (see [9,10], cf. [3]). But in general we cannot expect the solutions to be sufficiently smooth to ensure these Jackson-type inequalities. Section 2 is therefore concerned with an error bound under minimal smoothness conditions. To this end, we apply K-functional techniques to estimate the error by a modulus of continuity. It turns out that this error bound is sharp in connection with general Lipschitz classes. This is worked out in Section 3 as a consequence of a quantitative extension of the uniform boundedness principle. To establish the relevant resonance condition, we proceed along some ideas of Douglas and Dupont [10].

Let us mention that the article of Křížek and Neittaanmäki [17] as well as the book of Wahlbin [20] give a detailed survey of the field of superconvergence.

2 A DIRECT ESTIMATE

It is well known that Cea's lemma yields the inequality (cf. [4, p. 113])

$$\begin{aligned} \|u - u_h\|_{1,2,(0,1)} &\leq C \inf_{v \in V_h} \|u - v\|_{1,2,(0,1)} \\ &\leq C \|u - \Pi_h u\|_{1,2,(0,1)}, \end{aligned}$$

where $\Pi_h: C[0, 1] \rightarrow \{v \in C[0, 1]: v \in \mathcal{P}_r[jh, (j+1)h], 0 \leq j < n\}$ is the global Lagrange interpolation operator for the equidistant knots $jh + kh/r$, $0 \leq k \leq r$, $0 \leq j < n$. Thereby, without loss of generality, we assume the function u to be continuous. Using affine transformations in connection with a reference element, one immediately obtains the following Jackson-type inequality for $1 \leq j \leq r$, $0 \leq i < n$ (cf. [4, p. 125]):

$$\|u - \Pi_h u\|_{1,2,(ih,(i+1)h)} \leq Ch^j |u|_{j+1,2,(ih,(i+1)h)}, \quad u \in W^{j+1,2}(ih, (i+1)h). \quad (2.1)$$

Thus, in connection with the K-functional $K(\delta, u; s) := \inf_{v \in W^{s,2}(0,1)} [\|u - v\|_{L^2(0,1)} + \delta |u|_{s,2,(0,1)}]$ we conclude the estimate ($r > 1$)

$$\begin{aligned} \|u - u_h\|_{1,2,(0,1)} &\leq C \inf_{v \in W^{r+1,2}(0,1)} \left[\|u - v - \Pi_h(u - v)\|_{1,2,(0,1)} \right. \\ &\quad \left. + \|v - \Pi_h v\|_{1,2,(0,1)} \right] \\ &\leq C \inf_{v \in W^{r+1,2}(0,1)} \left[h |u - v|_{2,2,(0,1)} + h^r |v|_{r+1,2,(0,1)} \right] \\ &= Ch \inf_{w \in W^{r-1,2}(0,1)} \left[\|u^{(2)} - w\|_{L^2(0,1)} + h^{r-1} |w|_{r-1,2,(0,1)} \right] \\ &= ChK(h^{r-1}, u^{(2)}; r-1). \end{aligned}$$

Now, the K-functional $K(h^{r-1}, u; r-1)$ is equivalent to a modulus of continuity of order $r-1$ (see [16]) as defined by

$$\omega_{r-1}(\delta, u, L^2(0, 1)) := \sup_{|\nu| \leq \delta} \|\Delta_\nu^{r-1} u\|_{L^2(0, 1 - [r-1]\nu) \cap (-[r-1]\nu, 1)},$$

where $\Delta_\nu^{r-1}u(x) := \sum_{j=0}^{r-1} (-1)^{r-1-j} \binom{r-1-j}{j} u(x + j\nu)$. Thus, there exists a constant C (which does not depend on h or u) such that for solutions u and u_h of (1.1) and (1.2), respectively, one obtains

$$\|u - u_h\|_{1,2,(0,1)} \leq Ch\omega_{r-1}(h, u^{(2)}, L^2(0, 1)). \tag{2.2}$$

Since $u \in W^{2,2}(0,1)$, the right hand side is well defined. Moreover, applying Nitsche's trick, one has

$$\begin{aligned} \|u - u_h\|_{L^2(0,1)} &\leq Ch\|u - u_h\|_{1,2,(0,1)} \\ &\leq Ch^2\omega_{r-1}(h, u^{(2)}, L^2(0, 1)). \end{aligned} \tag{2.3}$$

This global L^2 -estimate is sharp in connection with Lipschitz classes (see [11]). Concerning a sup-norm error bound one can proceed in the same manner using a Jackson-type estimate developed in [21]. But here we are primarily interested in the error at the knots jh , $0 < j < n$. From [9,10] we quote that for smooth coefficients $a(x)$ and $b(x)$ one has

$$|(u - u_h)(jh)| \leq Ch^r \|u - u_h\|_{1,2,(0,1)}, \quad 0 \leq j \leq n. \tag{2.4}$$

Therefore, one obtains (cf. (2.2))

$$|(u - u_h)(jh)| \leq Ch^{r+1}\omega_{r-1}(h, u^{(2)}, L^2(0, 1)), \tag{2.5}$$

where the constant C is independent of j , h , and u . The main aim of this paper is to discuss the sharpness of this estimate. For the sake of completeness and since it is needed in our considerations in the next section, we sketch a proof of the error bound (2.4) restricting ourselves to the case $a(x) \in W^{r,\infty}(0, 1)$ and $b(x) \equiv 0$. The Green's function $G : [0, 1]^2 \rightarrow \mathbb{R}$ of problem (1.1) is then given by (cf. [19, p. 265f])

$$G(x, y) := \frac{1}{\int_0^1 (1/a(t)) dt} \begin{cases} \int_0^x (1/a(t)) dt \int_y^1 (1/a(t)) dt & \text{for } 0 \leq x \leq y, \\ \int_x^1 (1/a(t)) dt \int_0^y (1/a(t)) dt & \text{for } y < x \leq 1, \end{cases}$$

thus $G(\cdot, y) \in W_0^{1,2}(0, 1)$ for each fixed $y \in [0, 1]$, and there holds true

$$a(u, G(\cdot, y)) = u(y) \quad \text{for all } u \in W_0^{1,2}(0, 1), \quad y \in [0, 1].$$

Evidently, $G(\cdot, y) \in W^{r+1,2}(0, y) \cap W^{r+1,2}(y, 1)$ and

$$|G(\cdot, y)|_{r+1,2,(0,y)}^2 + |G(\cdot, y)|_{r+1,2,(y,1)}^2 \leq Cy(1-y), \quad (2.6)$$

where the constant C is independent of y . Therefore, one has (cf. (1.3))

$$\begin{aligned} |u(jh) - P_h u(jh)| &= |a(u - P_h u, G(\cdot, jh))| \\ &= \inf_{v \in V_h} |a(u - P_h u, G(\cdot, jh) - v)| \\ &\leq C \|u - P_h u\|_{1,2,(0,1)} \inf_{v \in V_h} \|G(\cdot, jh) - v\|_{1,2,(0,1)} \\ &\leq C \|u - P_h u\|_{1,2,(0,1)} \|G(\cdot, jh) - \Pi_h G(\cdot, jh)\|_{1,2,(0,1)}. \end{aligned}$$

Taking (2.6) into consideration, we have (cf. (2.1))

$$\begin{aligned} &\|G(\cdot, jh) - \Pi_h G(\cdot, jh)\|_{1,2,(0,1)} \\ &\leq Ch^r \left[|G(\cdot, jh)|_{r+1,2,(0,jh)}^2 + |G(\cdot, jh)|_{r+1,2,(jh,1)}^2 \right]^{1/2} \\ &\leq Ch^r \sqrt{jh(1-jh)} \end{aligned}$$

which establishes the error bound (2.5) in the case $b \equiv 0$:

$$|(u - P_h u)(jh)| \leq C \sqrt{jh(1-jh)} h^r \|u - P_h u\|_{1,2,(0,1)}. \quad (2.7)$$

One may further note that the Green's function belongs to V_h if $1/a(x)$ is a piecewise polynomial of degree less than r . Then the error vanishes in the knots, $u(jh) - u_h(jh) = 0$, $0 \leq j \leq n$, since $G(\cdot, jh) - \Pi_h G(\cdot, jh) = 0$.

3 SHARPNESS

We establish the sharpness of (2.5) in terms of counterexamples for general Lipschitz classes, determined by abstract moduli of continuity, i.e., by functions ω (e.g. $\omega(\delta) = \delta^\nu$ with $0 \leq \nu \leq 1$), continuous on $[0, \infty)$ such that, for $0 < \delta_1, \delta_2$,

$$0 = \omega(0) < \omega(\delta_1) \leq \omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2). \quad (3.1)$$

The main proposition of this article is the following result.

THEOREM 3.1 *There exists a function $a(x) \in W^{r,\infty}(0, 1)$ with $a(x) > \kappa > 0$, such that if one discusses problems (1.1), (1.2) in connection with the inner product $a(u, v) = \int_0^1 a(x)u'(x)v'(x) dx$, then there holds true the following assertion for finite element spaces of degree $r > 1$: for each $0 < \delta_0 < 1/2$ and every modulus satisfying (3.1) and*

$$\lim_{\delta \rightarrow 0+} \omega(\delta)/\delta = \infty \tag{3.2}$$

there exists a counterexample $u_\omega \in W_0^{1,2}(0, 1) \cap W^{2,2}(0, 1)$ which is a solution of problem (1.1) with a suitable inhomogeneity $f_\omega \in L^2(0, 1)$, determined by u_ω , such that on the one hand $(\delta \rightarrow 0+, h = 1/n \rightarrow 0+)$

$$\omega_{r-1}(\delta, u_\omega^{(2)}, L^2(0, 1)) = \mathcal{O}(\omega(\delta^{r-1})),$$

but on the other hand (cf. (2.5))

$$|u_\omega(\nu) - (u_\omega)_h(\nu)| \neq o(h^{r+1}\omega(h^{r-1}))$$

for each $\nu \in \mathbb{B} := \{j/2^n : 1 \leq j < 2^n, n \in \mathbb{N}\} \cap (\delta_0, 1 - \delta_0)$. In particular, u_ω is a (common) counterexample independent of the points $\nu \in \mathbb{B}$.

The proof of Theorem 3.1 is much more sophisticated than a discussion of the sharpness of (2.3) because we have to establish a lower estimate for a much smaller error. The rest of this section deals with this proof. In the context of approximation theory such negative results are often obtained on the basis of quantitative extensions of the uniform boundedness principle developed by Dickmeis, Nessel and van Wickeren (cf. [5]–[8]).

For a (real) Banach space X with norm $\|\cdot\|_X$ let X^\sim be the set of non-negative-valued sublinear bounded functionals T on X , i.e., T maps X into \mathbb{R} , the set of real numbers, such that for all $f, g \in X, \nu \in \mathbb{R}$

$$Tf \geq 0, \quad T(f + g) \leq Tf + Tg, \quad T(\nu f) = |\nu|Tf,$$

$$\|T\|_{X^\sim} := \sup\{Tf : \|f\|_X \leq 1\} < \infty.$$

THEOREM 3.2 *Suppose that for a family of remainders $\{T_{n,\nu} : n \in \mathbb{N}, \nu \in \mathbb{B}_n\} \subset X^\sim$, where $(\mathbb{B}_n)_{n \in \mathbb{N}}$ is a sequence of non-empty index sets, and for a measure of smoothness $\{S_\delta : \delta > 0\} \subset X^\sim$ there are test elements*

$g_n \in X$ such that the following inequalities hold true ($\delta > 0, n \rightarrow \infty$):

$$\|g_n\|_X \leq C_1 \quad \text{for all } n \in \mathbb{N}, \quad (3.3)$$

$$S_\delta g_n \leq C_2 \min \left\{ 1, \frac{\sigma(\delta)}{\varphi_n} \right\} \quad \text{for all } n \in \mathbb{N}, \quad \delta > 0, \quad (3.4)$$

$$\|T_{n,\nu}\|_{X^\sim} \leq C_{3,n} \quad \text{for all } \nu \in \mathbb{B}_n, \quad n \in \mathbb{N}, \quad (3.5)$$

$$T_{n,\nu} g_j \leq C_{4,\nu} C_{5,j} \varphi_n \quad \text{for all } 1 \leq j \leq n-1, \quad \nu \in \mathbb{B}_n, \quad n \in \mathbb{N}, \quad (3.6)$$

$$T_{n,\nu} g_n \geq C_{6,\nu} > 0 \quad \text{for all } \nu \in \mathbb{B}_n. \quad (3.7)$$

Here $\sigma(\delta)$ is a function, strictly positive on $(0, \infty)$, and $(\varphi_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ is a strictly decreasing sequence with $\lim_{n \rightarrow \infty} \varphi_n = 0$. Then for each modulus ω satisfying (3.1) and (3.2) there exists a (strictly increasing) subsequence $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ and a counterexample $f_\omega \in X$ such that ($\delta \rightarrow 0+$, $n \rightarrow \infty$)

$$\begin{aligned} S_\delta f_\omega &= \mathcal{O}(\omega(\sigma(\delta))), \\ T_{n,\nu} f_\omega &\neq o(\omega(\varphi_n)) \end{aligned}$$

for each $\nu \in \mathbb{B} := \limsup_{k \rightarrow \infty} \mathbb{B}_{n_k} := \bigcap_{k=1}^\infty \bigcup_{j=k}^\infty \mathbb{B}_{n_j}$.

For a proof, further comments, and applications to approximation theory see [5,13,14] and the literature cited there.

Proof of Theorem 3.1 To apply Theorem 3.2, we specify the quantities according to ($h = 2^{-n}$, $n \in \mathbb{N}$)

$$X = W_0^{1,2}(0, 1) \cap W^{2,2}(0, 1), \quad \|\cdot\|_X = \|\cdot\|_{2,2,(0,1)},$$

$$\varphi_n = \frac{1}{2^{(r-1)n}} = h^{r-1}, \quad \sigma(\delta) = \delta^{r-1},$$

$$S_\delta u = \omega_{r-1}(\delta, u^{(2)}, L^2(0, 1)),$$

$$\mathbb{B}_n := \left\{ \frac{j}{2^n} : 1 \leq j \leq 2^n - 1 \right\} \cap (\delta_0, 1 - \delta_0),$$

$$T_{n,\nu} u = 2^{(r+1)n} |(u - P_h u)(\nu)| = h^{-(r+1)} |(u - P_h u)(\nu)|, \quad \nu \in \mathbb{B}_n.$$

Indeed, $S_\delta, T_{n,\nu} \in X^\sim$ (cf. (2.2), (2.7)), and we note that $\mathbb{B}_n \subset \mathbb{B}_{n+1} \subset \dots \subset \mathbb{B} = \bigcup_{n=1}^\infty \mathbb{B}_n$. The crucial point is to find a suitable sequence of test elements and to show the resonance condition (3.7). At first, we will

construct a sequence $(\tilde{g}_n)_{n \in \mathbb{N}}$ which is indeed suitable in connection with (3.7). But these functions are not smooth enough to satisfy the Jackson–Bernstein-type inequality (3.4). Therefore we have to smooth them. This will be done using a partition of unity. The stability of the finite element method assures that for the smoothed functions $(g_n)_{n \in \mathbb{N}}$ the resonance condition still remains valid.

Thus let us start with the sequence $(\tilde{g}_n)_{n \in \mathbb{N}}$ which is defined by

$$\tilde{g}_n(x) := \frac{1}{h^{r-1}} \int_0^x \tilde{f}_n(t) dt,$$

where $(0 \leq j \leq 2^n - 1)$

$$\tilde{f}_n(x) := \begin{cases} 0 & x = 0, \\ (x - jh)^r & x \in (jh, (j + 1)h] \text{ for } j \leq 2^{n-1} - 1, \\ -((j + 1)h - x)^r & x \in (jh, (j + 1)h] \text{ for } j \geq 2^{n-1}, \text{ i.e., } jh \geq 1/2. \end{cases}$$

The function \tilde{f}_n is odd with respect to the point $1/2$, i.e., $\tilde{f}_n(x) = -\tilde{f}_n(1 - x)$ a.e. Obviously, \tilde{g}_n is piecewise polynomial of degree $r + 1$ and satisfies $\tilde{g}_n(0) = \tilde{g}_n(1) = 0$ such that $\tilde{g}_n \in W_0^{1,2}(0, 1)$. Moreover,

$$\|\tilde{f}_n\|_{L^\infty(0,1)} = h^r, \tag{3.8}$$

where $\|\cdot\|_{L^\infty(0,1)}$ denotes the essential supremum. The next step is to smooth \tilde{f}_n with a suitable partition of unity. To this end, given an arbitrary $0 < \varepsilon \leq 1/4$, there exists an infinitely often differentiable function φ_ε with (cf. [15, p.35])

$$\varphi_\varepsilon(x) = \begin{cases} 1: & x \leq 1/2 - \varepsilon, \\ 0: & x \geq 1/2 + \varepsilon, \end{cases} \quad \varphi_\varepsilon(x) = 1 - \varphi_\varepsilon(1 - x), \quad x \in \mathbb{R},$$

and $0 \leq \varphi_\varepsilon(x) \leq 1$. With the aid of this function we define $f_{n,\varepsilon}$ as follows:

(a) for $x \in [(j + 1/2)h, (j + 3/2)h]$, $0 \leq j \leq 2^{n-1} - 2$:

$$f_{n,\varepsilon}(x) := \varphi_\varepsilon\left(\frac{x - jh}{2h}\right)(x - jh)^r + \left(1 - \varphi_\varepsilon\left(\frac{x - jh}{2h}\right)\right)(x - (j + 1)h)^r;$$

(b) for $x \in [(j+1/2)h, (j+3/2)h]$, $2^{n-1} \leq j \leq 2^n - 2$:

$$f_{n,\varepsilon}(x) := -\varphi_\varepsilon\left(\frac{x-jh}{2h}\right)((j+1)h-x)^r \\ - \left(1 - \varphi_\varepsilon\left(\frac{x-jh}{2h}\right)\right)((j+2)h-x)^r;$$

(c) for $x \in [(2^{n-1}-1/2)h, (2^{n-1}+1/2)h] = [(j+1/2)h, (j+3/2)h]$ with $j = 2^{n-1} - 1$:

$$f_{n,\varepsilon}(x) := \varphi_\varepsilon\left(\frac{x-jh}{2h}\right)(x-jh)^r \\ - \left(1 - \varphi_\varepsilon\left(\frac{x-jh}{2h}\right)\right)((j+2)h-x)^r;$$

(d) for $x \in [0, h/2] \cup [1-h/2, 1]$:

$$f_{n,\varepsilon}(x) := \tilde{f}_n(x).$$

Note that $f_{n,\varepsilon}$ is infinitely often differentiable and

$$\|f_{n,\varepsilon}\|_{C[0,1]} := \sup_{x \in [0,1]} |f_{n,\varepsilon}(x)| \leq \left(\frac{3}{2}h\right)^r. \quad (3.9)$$

Furthermore, due to the definitions of f_n and φ_ε the function $f_{n,\varepsilon}$ is odd with respect to $1/2$. Therefore, the resonance elements

$$g_n(x) = g_{n,\varepsilon}(x) := \frac{1}{h^{r-1}} \int_0^x f_{n,\varepsilon}(t) dt$$

satisfy the boundary condition $g_{n,\varepsilon}(0) = g_{n,\varepsilon}(1) = 0$, and they are infinitely often differentiable. In particular, $(g_{n,\varepsilon})_{n \in \mathbb{N}} \subset X$. The parameter ε will be fixed later. Straightforward calculations yield ($1 \leq k \leq r+1$)

$$|g_{n,\varepsilon}|_{k,2,(0,1)} = \frac{1}{h^{r-1}} \|f_{n,\varepsilon}^{(k-1)}\|_{L^2(0,1)} \\ \leq \frac{1}{h^{r-1}} \|f_{n,\varepsilon}^{(k-1)}\|_{C[0,1]} \leq C_\varepsilon h^{2-k}, \quad (3.10)$$

where the constant C_ε is independent of h . Furthermore, Poincaré's inequality (cf. [1, p. 159]) yields

$$\begin{aligned} \|g_{n,\varepsilon}\|_{2,2,(0,1)} &\leq \|g_{n,\varepsilon}\|_{1,2,(0,1)} + |g_{n,\varepsilon}|_{2,2,(0,1)} \\ &\leq C(|g_{n,\varepsilon}|_{1,2,(0,1)} + |g_{n,\varepsilon}|_{2,2,(0,1)}) \leq C_\varepsilon. \end{aligned}$$

Therefore, condition (3.3) is established. Concerning (3.4) we have (cf. (3.10))

$$\omega_{r-1}(\delta, g_{n,\varepsilon}^{(2)}, L^2(0,1)) \leq \begin{cases} C|g_{n,\varepsilon}|_{2,2,(0,1)} \leq C_\varepsilon, \\ C\delta^{r-1}|g_{n,\varepsilon}|_{r+1,2,(0,1)} \leq C_\varepsilon\delta^{r-1}h^{2-(r+1)} = C_\varepsilon\sigma(\delta)/\varphi_n. \end{cases}$$

Concerning the crucial resonance condition (3.7), we first investigate how much the function $g_{n,\varepsilon}$ differs from \tilde{g}_n . One observes that $\tilde{f}_n(x) = f_{n,\varepsilon}(x)$ outside the balls $S(2\varepsilon h, jh) = \{x: |x - jh| < 2\varepsilon h\}$. Once again by Poincaré's inequality

$$\begin{aligned} \|\tilde{g}_n - g_{n,\varepsilon}\|_{1,2,(0,1)} &\leq C|\tilde{g}_n - g_{n,\varepsilon}|_{1,2,(0,1)} \\ &= \frac{C}{h^{r-1}} \|\tilde{f}_n - f_{n,\varepsilon}\|_{L^2(0,1)} \\ &= \frac{C}{h^{r-1}} \left[\sum_{j=0}^{2^n-2} \int_{(j+1)h-2\varepsilon h}^{(j+1)h+2\varepsilon h} |\tilde{f}_n(x) - f_{n,\varepsilon}(x)|^2 dx \right]^{1/2} \\ &\leq \frac{C}{h^{r-1}} [(2^n - 1)4\varepsilon h]^{1/2} \|\tilde{f}_n - f_{n,\varepsilon}\|_{L^\infty(0,1)} \\ &\leq \frac{2C\sqrt{\varepsilon}}{h^{r-1}} (\|\tilde{f}_n\|_{L^\infty(0,1)} + \|f_{n,\varepsilon}\|_{C[0,1]}). \end{aligned}$$

Together with (3.8) and (3.9) this yields

$$\|\tilde{g}_n - g_{n,\varepsilon}\|_{1,2,(0,1)} \leq Ch\sqrt{\varepsilon}.$$

It is important to note that the constant C does not depend on ε and h . Because of the uniform boundedness of the operator P_h we conclude

$$\|P_h(\tilde{g}_n - g_{n,\varepsilon})\|_{1,2,(0,1)} \leq C\|\tilde{g}_n - g_{n,\varepsilon}\|_{1,2,(0,1)} \leq Ch\sqrt{\varepsilon}.$$

Taking the direct estimate (2.7) into consideration, one obtains ($\nu \in \mathbb{B}_n$)

$$\begin{aligned}
& |(\tilde{g}_n - P_h \tilde{g}_n)(\nu)| \\
&= |(g_{n,\varepsilon} - P_h g_{n,\varepsilon})(\nu)| + |(g_{n,\varepsilon} - \tilde{g}_n)(\nu) - P_h(g_{n,\varepsilon} - \tilde{g}_n)(\nu)| \\
&\leq |(g_{n,\varepsilon} - P_h g_{n,\varepsilon})(\nu)| + Ch^r \|g_{n,\varepsilon} - \tilde{g}_n - P_h(g_{n,\varepsilon} - \tilde{g}_n)\|_{1,2,(0,1)} \\
&\leq |(g_{n,\varepsilon} - P_h g_{n,\varepsilon})(\nu)| + Ch^r [\|P_h(g_{n,\varepsilon} - \tilde{g}_n)\|_{1,2,(0,1)} + \|g_{n,\varepsilon} - \tilde{g}_n\|_{1,2,(0,1)}] \\
&\leq |(g_{n,\varepsilon} - P_h g_{n,\varepsilon})(\nu)| + Ch^{r+1} \sqrt{\varepsilon}
\end{aligned}$$

and therefore

$$\begin{aligned}
T_{n,\nu} g_{n,\varepsilon} &= h^{-(r+1)} |(g_{n,\varepsilon} - P_h g_{n,\varepsilon})(\nu)| \\
&\geq h^{-(r+1)} |(\tilde{g}_n - P_h \tilde{g}_n)(\nu)| - C\sqrt{\varepsilon} = T_{n,\nu} \tilde{g}_n - C\sqrt{\varepsilon}.
\end{aligned}$$

In what follows we will prove the crucial inequality

$$T_{n,\nu} \tilde{g}_n \geq c \min\{\nu, 1 - \nu\} \quad (3.11)$$

for $\nu \in \mathbb{B}_n$ and $n \in \mathbb{N}$. Then according to the definition of \mathbb{B}_n one has $\min\{\nu, 1 - \nu\} > \delta_0 > 0$ such that $T_{n,\nu} g_{n,\varepsilon} \geq c\delta_0 - c\delta_0/2 = c\delta_0/2$ for each $\varepsilon \leq \min\{1/4, [c\delta_0/(2C)]^2\}$. Summarizing, we then have found a resonance sequence which satisfies (3.3), (3.4), (3.7). Since the conditions (3.5) and (3.6) follow from the direct estimates (2.2) and (2.7), i.e.,

$$\begin{aligned}
\|T_{n,\nu}\|_{X^\sim} &\leq Ch^{-(r+1)} h^{r+1} = C, \\
T_{n,\nu} g_{j,\varepsilon} &\leq C\omega_{r-1}(h, g_{j,\varepsilon}^{(2)}, L^2(0, 1)) \\
&\leq Ch^{r-1} |g_{j,\varepsilon}|_{r+1,2,(0,1)} = C_{j,\varepsilon} \varphi_n,
\end{aligned}$$

Theorem 3.2 yields a counterexample u_ω which satisfies the assertions of Theorem 3.1. Indeed, this counterexample is a solution of problem (1.1), where the inhomogeneity f_ω is determined by partial integration.

It remains to prove (3.11). In [10] Douglas and Dupont investigate the sharpness of an estimate (2.4) in connection with the case $\omega(\delta) = \delta$ which is excluded here. They are able to present explicitly an elementary counterexample for which the superconvergence error is exactly of order $\mathcal{O}(h^{2r})$. Thereby, the crucial point is a suitable representation for the error which we will apply as well. At this point it becomes necessary

to fix the function $a(x)$ and therefore to determine the inner product $a(\cdot, \cdot)$. Let $a(x) \in W^{r, \infty}(0, 1)$ be even with respect to $1/2$, i.e., $a(x) = a(1 - x)$ a.e., such that the following conditions hold true ($a(x) > \kappa > 0$):

$$\begin{aligned} \frac{1}{a(x)} &\in \mathcal{P}_r[0, 1] \quad \text{with} \quad \left(\frac{1}{a(x)}\right)^{(r)} = r! \quad \text{if } r \text{ is even,} \\ \frac{1}{a(x)} &\in \mathcal{P}_r[0, 1/2] \cap \mathcal{P}_r(1/2, 1] \quad \text{with} \\ \left(\frac{1}{a(x)}\right)^{(r)} &= \begin{cases} r!, & x \in [0, 1/2] \\ -r!, & x \in (1/2, 1] \end{cases} \quad \text{if } r \text{ is odd.} \end{aligned}$$

One may note that a discussion of cases is necessary to assure that $1/a(x)$ and $a(x)$ are even. For example, we can choose

$$\begin{aligned} a(x) &= \frac{1}{(x - (1/2))^r + 1} \quad (r \text{ even}), \\ a(x) &= \begin{cases} \frac{1}{(x - (1/2))^r + 1}, & x \in [0, 1/2] \\ \frac{1}{1 - (x - (1/2))^r}, & x \in (1/2, 1] \end{cases} \quad (r \text{ odd}). \end{aligned}$$

Evidently, $1/a(x)$ is a piecewise polynomial of degree r and not of degree $r - 1$. This is important because otherwise the error vanishes (cf. Section 2).

Due to their construction both the functions $a(x)$ and \tilde{g}_n are even with respect to the point $1/2$. Therefore, the Ritz projection $P_h \tilde{g}_n$ is even, too. This is easy to prove because the partitions \mathcal{T}_h are equidistant. In other words, the error $e_n(x) := \tilde{g}_n(x) - P_h \tilde{g}_n(x)$ is even and e'_n is odd. This is necessary to establish the error representation (cf. [10])

$$e_n(\nu) = \int_0^\nu a(x) e'_n(x) \left[\frac{1}{a(x)} - z(x) \right] dx \tag{3.12}$$

$$\begin{aligned} &= - \int_\nu^1 a(x) e'_n(x) \left[\frac{1}{a(x)} - z(x) \right] dx \\ &\quad \text{for all } z \in \mathcal{P}_{r-1, h}, \nu \in \mathbb{B}_n, \end{aligned} \tag{3.13}$$

where

$$\mathcal{P}_{r, h} := \{z : [0, 1] \rightarrow \mathbb{R} : z \in \mathcal{P}_r(jh, (j + 1)h), 0 \leq j \leq n - 1\}.$$

We prove (3.12) (3.13) follows in the same way). To this end, given a function $z \in \mathcal{P}_{r-1,h}$ we define $v(x) := \int_0^x z(t) dt - x \int_0^1 z(t) dt$. Then $v \in \mathcal{P}_{r,h} \cap C[0,1]$ and the construction assures $v(0) = v(1) = 0$ such that $v \in V_h(r)$. By virtue of $\int_0^1 a(x)e'_n(x) = 0$ we obtain (cf. 1.3))

$$\begin{aligned} (ae'_n, z)_{L^2(0,1)} &= (ae'_n, v')_{L^2(0,1)} + \left(ae'_n, \int_0^1 z(t) dt \right)_{L^2(0,1)} \\ &= (ae'_n, v')_{L^2(0,1)} + (ae'_n, 1)_{L^2(0,1)} \int_0^1 z(t) dt \\ &= (ae'_n, v')_{L^2(0,1)} = a(e_n, v) = 0. \end{aligned}$$

Now let $z \in \mathcal{P}_{r-1,h}$ and $\nu \in \mathbb{B}_n$. The function $\tilde{z}(x) := z(x)$ for $x \leq \nu$, $\tilde{z}(x) := 0$ for $x > \nu$, belongs to $\mathcal{P}_{r-1,h}$ as well, and therefore $0 = \int_0^1 a(x)e'_n(x)\tilde{z}(x) dx = \int_0^\nu a(x)e'_n(x)z(x) dx$ such that $(a(x) > \kappa > 0)$

$$e_n(\nu) = \int_0^\nu e'_n(x) dx = \int_0^\nu a(x)e'_n(x) \left[\frac{1}{a(x)} - z(x) \right] dx.$$

Finally, we use the representation (3.12), (3.13) to establish the resonance condition (3.11). The leading coefficient of e'_n is the same as of $h^{1-r}\tilde{f}_n$ and the leading coefficient of $1/a$ has been fixed by the conditions on $a(x)$:

	r even	r odd
Leading coefficient of $1/a(x)$	+ 1	+ 1 if $x \leq 1/2$ - 1 if $x > 1/2$
Leading coefficient of $e'_n(x)$	+ h^{1-r} if $x \leq 1/2$ - h^{1-r} if $x > 1/2$	+ h^{1-r}

For that reason, one can choose $z_n \in \mathcal{P}_{r-1,h}$ such that for $x \in (jh, (j+1)h]$ there holds true

$$e'_n(x) \left[\frac{1}{a(x)} - z_n(x) \right] = \begin{cases} +h^{r-1}[e'_n(x)]^2 & \text{for } x \leq 1/2, \\ -h^{r-1}[e'_n(x)]^2 & \text{for } x > 1/2. \end{cases} \tag{3.14}$$

In connection with (3.12) one obtains for $\nu \leq 1/2$

$$\begin{aligned} T_{n,\nu} \tilde{g}_n &= \frac{1}{h^{r+1}} |e_n(\nu)| \geq \frac{1}{h^{r+1}} \int_0^\nu a(x) e'_n(x) \left[\frac{1}{a(x)} - z_n(x) \right] dx \\ &= \frac{1}{h^{r+1}} \sum_{j=0}^{(\nu/h)-1} \int_{jh}^{(j+1)h} a(x) e'_n(x) \left[\frac{1}{a(x)} - z_n(x) \right] dx \\ &= \frac{h^{r-1}}{h^{r+1}} \sum_{j=0}^{(\nu/h)-1} \int_{jh}^{(j+1)h} a(x) [e'_n(x)]^2 dx \\ &\geq \left(\inf_{x \in [0,1]} a(x) \right) \frac{1}{h^2} \sum_{j=0}^{(\nu/h)-1} \int_{jh}^{(j+1)h} [e'_n(x)]^2 dx. \end{aligned}$$

In view of the positivity of $[e'_n(x)]^2$ the rest of the proof is a routine argument. Indeed, since $P_h \tilde{g}_n \in \mathbb{P}_r(jh, (j+1)h]$, one has $(P_h \tilde{g}_n)' \in \mathbb{P}_{r-1}(jh, (j+1)h]$ and

$$\begin{aligned} T_{n,\nu} \tilde{g}_n &\geq \frac{\kappa}{h^2} \sum_{j=0}^{(\nu/h)-1} \inf_{v \in \mathcal{P}_{r-1}(jh, (j+1)h]} \int_{jh}^{(j+1)h} [\tilde{g}'_n(x) - v(x)]^2 dx \\ &= \frac{\kappa}{h^2} \sum_{j=0}^{(\nu/h)-1} \inf_{v \in \mathcal{P}_{r-1}(jh, (j+1)h]} \int_{jh}^{(j+1)h} \left[\frac{1}{h^{r-1}} (x - jh)^r - v(x) \right]^2 dx \\ &= \frac{\kappa}{h^2} \sum_{j=0}^{(\nu/h)-1} \inf_{v \in \mathcal{P}_{r-1}(jh, (j+1)h]} \int_{jh}^{(j+1)h} \left[\frac{1}{h^{r-1}} x^r - v(x) \right]^2 dx \\ &= \frac{\kappa}{h^2} \frac{1}{h^{2(r-1)}} \sum_{j=0}^{(\nu/h)-1} \inf_{v \in \mathcal{P}_{r-1}(jh, (j+1)h]} \int_{jh}^{(j+1)h} [x^r - v(x)]^2 dx \\ &= \frac{\kappa}{h^2} \frac{1}{h^{2(r-1)}} \sum_{j=0}^{(\nu/h)-1} \inf_{v \in \mathcal{P}_{r-1}(jh, (j+1)h]} h \int_0^1 [(ht + jh)^r - v(ht + jh)]^2 dt \\ &= \frac{\kappa}{h^2} \frac{1}{h^{2(r-1)}} h^{2r+1} \sum_{j=0}^{(\nu/h)-1} \inf_{v \in \mathcal{P}_{r-1}[0,1]} \int_0^1 [t^r - v(t)]^2 dt \geq ch\nu/h \\ &= c\nu > 0. \end{aligned}$$

By virtue of (3.13) (cf. (3.14)) one analogously obtains that $T_{n,\nu} \tilde{g}_n \geq c(1 - \nu)$ in the case $\nu > 1/2$. This finishes the proof of (3.11).

Though specific, the present results are by no means restricted to the particular superconvergence phenomenon under consideration. As a second example let us briefly mention superconvergence at Gauss and Lobatto points. Using well-known Jackson-type inequalities (cf. [2,18]), one can establish intermediate error bounds analogously to (2.5) in terms of moduli of smoothness which are sharp in connection with general Lipschitz classes. Again this can be proved using the resonance principle of Theorem 3.2 (cf. [11]). Here it seems to be natural to build up a resonance sequence by Legendre polynomials because the Gauss and Lobatto points are zeros of these polynomials and their derivatives. Then the resonance condition follows from the fact that the zeros of Legendre polynomials P_k and P_{k+1} are different.

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