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An Integral Operator Inequality with Applications

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Dedicated to the memory of John Meadows Jackson 1907–1998

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Linear integral operators are defined acting in the Lebesgue integration spaces on intervals of the real line. A necessary and sufficient condition is given for these operators to be bounded, and a characterisation is given for the operator bounds. There are applications of the results to integral inequalities; also to properties of the domains of self-adjoint unbounded operators, in Hilbert function spaces, associated with the classical orthogonal polynomials and their generalisations.

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1. INTRODUCTION

Let $\mathbb{R}$ and $\mathbb{C}$ represent the real and complex number fields; let (a, b) and $[\alpha, \beta]$ be open and compact intervals of $\mathbb{R}$ respectively; we identify $\mathbb{R} \equiv (-\infty, \infty)$ and the extended real field $\mathbb{R}^* \equiv [-\infty, \infty]$.

The symbol '$(x \in K)$' is to be read as 'for all x in the set K'.

We use the notations $o(1)$ and $O(1)$ to indicate terms that tend to zero or are bounded, respectively, for some indicated limit process; see [11, p. 1].

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For the interval (a,b), a weight function $w:(a,b)\to\mathbb{R}$ is Lebesgue measurable and satisfies $w(x)\geq 0$ for $x\in(a,b)$.

The index p satisfies $p\in[1,\infty)$; when $p\in(1,\infty)$ the index $q\in(1,\infty)$ and conjugate to p is determined by $p^{-1}+q^{-1}=1$.

For Lebesgue integration we use the standard notations and write

$$\int_a^b f(x)\mathrm{d}x \equiv \int_a^b f. \tag{1.1}$$

Let L denote Lebesgue integration; for $p\geq 1$ and a weight function w the symbol $L^p((a,b):w)$ denotes both the classical integration space and the Banach space of equivalence classes of functions; the norm of f is written $\|f\|_p$. For conjugate indices p,q the spaces $L^p((a,b):w)$ and $L^q((a,b):w)$ are first conjugate spaces of each other; define

$$\langle f,g\rangle_{p,q}:=\int_a^b wfg \quad (f\in L^p((a,b):w) \text{ and } g\in L^q((a,b):w)). \tag{1.2}$$

The norm of a bounded, linear operator A defined on $L^p((a,b):w)$ is denoted by $\|A\|_p$.

In the self-conjugate space $p=q=2$ the symbol $L^2((a,b):w)$ denotes the Hilbert function space with inner-product

$$(f,g)_2 := \int_a^b wf\bar{g} \quad (f,g\in L^2((a,b):w)). \tag{1.3}$$

The use of 'loc' in respect of the interval (a,b) restricts a property to all compact sub-intervals $[\alpha,\beta]\subset(a,b)$; thus $L^p_{\mathrm{loc}}((a,b):w)$ is the set of functions $f:(a,b)\to\mathbb{C}$ such that $f\in L^p([\alpha,\beta]:w)$ for all $[\alpha,\beta]\subset(a,b)$.

Let $f\in L^p((a,b):w)$ and $g\in L^q((a,b):w)$; then the Hölder integral inequality [7, Section 6.9] implies that $wfg\in L^1((a,b):w)$; since $w=w^{1/p}\times w^{1/q}$ and then

$$\left|\int_a^b wfg\right| \leq \int_a^b w^{1/p}|f|w^{1/q}|g| \leq \left\{\int_a^b w|f|^p\right\}^{1/p}\left\{\int_a^b w|g|^q\right\}^{1/q}. \tag{1.4}$$

We are now in a position to state the general results of this paper:

THEOREM 1 *Suppose given the open interval, bounded or unbounded, (a,b) of $\mathbb{R}$, the conjugate indices $p,q\in(1,\infty)$, and the weight function w.*

Let the pair of functions $\varphi, \psi : (a,b) \to \mathbb{C}$ satisfy the three conditions:

1.

$$\varphi \in L^p_{\mathrm{loc}}((a,b) : w) \text{ and } \psi \in L^q_{\mathrm{loc}}((a,b) : w) \tag{1.5}$$

2. *For some $c \in (a,b)$ (and then for all $c \in (a,b)$)*

$$\varphi \in L^p((a,c] : w) \text{ and } \psi \in L^q([c,b) : w). \tag{1.6}$$

3. *For all $[\alpha, \beta] \subset (a,b)$*

$$\int_a^\alpha w|\varphi|^p > 0 \text{ and } \int_\beta^b w|\psi|^q > 0. \tag{1.7}$$

Define the linear operators A and B on $L^p((a,b):w)$ and $L^q((a,b):w)$ respectively by

$$(Af)(x) := \varphi(x) \int_x^b w\psi f \quad (x \in (a,b) \text{ and } f \in L^p((a,b) : w)) \tag{1.8}$$

$$(Bg)(x) := \psi(x) \int_a^x w\varphi g \quad (x \in (a,b) \text{ and } g \in L^q((a,b) : w)); \tag{1.9}$$

then

$$A : L^p((a,b) : w) \to L^p_{\mathrm{loc}}((a,b) : w), \tag{1.10}$$

$$B : L^q((a,b) : w) \to L^q_{\mathrm{loc}}((a,b) : w). \tag{1.11}$$

Define $K(\cdot) : (a,b) \to (0,\infty)$ by

$$K(x) := \left\{ \int_a^x w|\varphi|^p \right\}^{1/p} \left\{ \int_x^b w|\psi|^q \right\}^{1/q} \quad (x \in (a,b)) \tag{1.12}$$

and the number $K \in (0, \infty]$

$$K := \sup\{K(x) :\ x \in (a,b)\}. \tag{1.13}$$

Then a necessary and sufficient condition that A, respectively B, is a bounded linear operator on $L^p((a,b):w)$, respectively on $L^q((a,b):w)$, into

$L^p((a,b):w)$, respectively into $L^q((a,b):w)$, is that the number K is finite, i.e.

$$K \in (0,\infty). \tag{1.14}$$

Proof See Sections 2 and 3 below.

COROLLARY 1 *Let all the conditions of Theorem* 1 *hold and suppose that* (1.14) *is satisfied, so that the linear operators A and B are bounded in the spaces $L^p((a,b):w)$ and $L^q((a,b):w)$ respectively, then the following inequalities hold*

$$\|Af\|_p \le p^{1/p} q^{1/q} K \|f\|_p \quad (f \in L^p((a,b):w)), \tag{1.15}$$

$$\|Bg\|_q \le p^{1/p} q^{1/q} K \|g\|_q \quad (g \in L^q((a,b):w)), \tag{1.16}$$

where the number K is defined by (1.13) *and* (1.14).

In general, for all self-conjugate indices p and q, the number $p^{1/p}\, q^{1/q}\, K$ given in (1.15) *and* (1.16) *is best possible for these inequalities to hold.*

Proof See Section 3 below.

COROLLARY 2 *The bounded operator B, respectively A, is the conjugate operator of A, respectively B, between the first conjugate spaces $L^q((a,b):w)$ and $L^p((a,b):w)$ respectively, i.e.*

$$\langle Af, g\rangle_{p,q} = \langle f, Bg\rangle_{p,q} \quad (f \in L^p((a,b):w) \text{ and } g \in L^q((a,b):w)). \tag{1.17}$$

The operators A and B have the same norm, i.e.

$$\|A\|_p = \|B\|_q. \tag{1.18}$$

Proof See Section 4 below.

COROLLARY 3 *Let the self-conjugate case $p=q=2$ hold; let the bounding condition* (1.14) *be satisfied; let the set $\{\varphi,\psi\}$ be restricted by*

$$\varphi, \psi : (a,b) \to \mathbb{R}. \tag{1.19}$$

Then the linear operators A and B are both bounded in the Hilbert space $L^2((a,b):w)$ and are the adjoints of each other, i.e.

$$A^* = B \quad \textit{and} \quad B^* = A. \tag{1.20}$$

Proof See Section 4 below.

Remark 1 The original motivation for these results came from the work of Titchmarsh; see the definition of the resolvent function Φ in [12, Chapter II], and the remarks in [1, Section 1].

The first proof of these results seems to have been given by Chisholm and Everitt [1]; this paper was followed by the application of Stuart [10] to the measure of non-compactness of integral operators. Extensions of these results may be found in the books of Maz'ja [9], and Edmunds and Evans [2].

There is an interesting connection in the recent paper of Toland and Williams [13].

There are many recent applications to the theory of orthogonal polynomials; see the papers by Everitt and Littlejohn [3], and Everitt and Marić [5]; also the doctoral theses of Loveland [8] and Wellman [14].

A suitable choice of the interval (a,b) and the set $\{\varphi, \psi\}$ leads to many interesting integral inequalities. In particular the integral inequality of Hardy [6], see also [7, Section 9.8], is a special case of the inequality (1.16).

We discuss some of these applications and inequalities in Sections 5 and 6 below.

Remark 2 The condition (1.7) is to avoid the case when one or both of the functions φ and ψ are null in some neighbourhood of a^+ and b^-, respectively.

Remark 3 In this paper we give, in the spirit of [7, Section 1.7], elementary proofs of these results. We give a new improved form of the proofs first given in [1], and discuss additional examples and applications.

In Section 2 we give an essential lemma; the proofs of Theorem 1 and Corollary 1 are given in Section 3; the proof of Corollaries 2 and 3 appear in Section 4; some critical examples are given in Section 5; an application of Theorem 1 to a property of the Legendre differential operator is given in Section 6.

2. A LEMMA

We require

LEMMA 1 *Suppose given the interval (a, b), conjugate indices p, q, and a weight w; let $f \in L^p((a, b) : w)$ and $g \in L^q_{\mathrm{loc}}((a, b) : w)$; for some $c \in (a, b)$ let $g \notin L^q([c, b) : w)$; then*

$$\lim_{x \to b^-} \int_c^x wfg . \left\{ \int_c^x w|g|^q \right\}^{-1/q} = 0. \tag{2.1}$$

There is a similar result at a^+ if $g \notin L^q((a, c] : w)$.

Proof Let $\varepsilon > 0$ be given; choose d close to b^- so that $\left\{ \int_d^b w|f|^p \right\}^{1/p} < \varepsilon$; then for $x > d$, using the Hölder inequality (1.4),

$$\left| \int_c^x wfg \right| . \left\{ \int_c^x w|g|^q \right\}^{-1/q} \leq \left| \int_c^d wfg \right| . \left\{ \int_c^x w|g|^q \right\}^{-1/q} + \varepsilon.$$

On letting $x \to b^-$ and using $\lim_{x \to b^-} \int_c^x w|g|^q = +\infty$, we obtain

$$\limsup_{x \to b^-} \left| \int_c^x wfg \right| . \left\{ \int_c^x w|g|^q \right\}^{-1/q} \leq \varepsilon$$

and the result follows.

3. PROOF OF THEOREM 1 AND COROLLARY 1

It is clear from the conditions (1.5) and (1.6) and then the definitions (1.8) and (1.9) that if we impose further the conditions that φ and ψ satisfy

$$\varphi \in L^p([c, b) : w) \quad \text{and} \quad \psi \in L^q((a, c] : w), \tag{3.1}$$

then it follows that the required boundedness of the operators A and B is satisfied. Note also that in this case the condition (1.14) follows from (3.1).

The most difficult case for the proof of Theorem 1 is when both of the conditions (3.1) are not satisfied, i.e.

$$\varphi \notin L^p([c,b):w) \quad \text{and} \quad \psi \notin L^q((a,c]:w). \tag{3.2}$$

We give the proof when (3.2) holds; when only one of the conditions in (3.2) is satisfied the proof follows similar lines.

The results (1.10) and (1.11) follow from the given conditions and the definitions of the operators A and B.

We begin with the sufficiency of the condition (1.14) to prove the boundedness of the operators A and B, i.e. we assume that

$$\begin{aligned}\sup&\left[\left\{\int_a^x w|\varphi|^p\right\}^{1/p}\left\{\int_x^b w|\psi|^q\right\}^{1/q} : \ x\in(a,b)\right]\\ &= K < +\infty \quad (x\in(a,b)).\end{aligned} \tag{3.3}$$

To prove the required result for A take monotonic sequences $\{a_n \in \mathbb{R} : n \in \mathbb{N}_0\}$ and $\{b_n \in \mathbb{R} : n \in \mathbb{N}_0\}$ such that

$$a < a_{n+1} < a_n < b_n < b_{n+1} < b \quad (n \in \mathbb{N}_0)$$

and

$$\lim_{n\to\infty} a_n = a \quad \text{and} \quad \lim_{n\to\infty} b_n = b.$$

From a standard property of the Lebesgue integral

$$\left|\int_x^b w\psi f\right| \le \int_x^b w|\psi f|;$$

from this inequality and on integration by parts we obtain

$$\begin{aligned}\int_{a_n}^{b_n} w|Af|^p &\le \int_{a_n}^{b_n} w(x)|\varphi(x)|^p\left\{\int_x^b w|\psi f|\right\}^p \mathrm{d}x\\ &= \left[\int_a^x w|\varphi|^p.\left\{\int_x^b w|\psi f|\right\}^p\right]_{a_n}^{b_n}\\ &\quad + p\int_{a_n}^{b_n}\left\{\int_a^x w|\varphi|^p\right\}\cdot\left\{\int_x^b w|\psi f|\right\}^{p-1} w(x)|\psi(x)f(x)|\mathrm{d}x.\end{aligned} \tag{3.4}$$

From (3.3) the first part of the first term on the right-hand side of (3.4) is dominated by

$$\left\{K\int_{b_n}^{b} w|\psi f|\right\}^{p}\cdot\left\{\int_{b_n}^{b} w|\psi|^{q}\right\}^{-p/q}$$

and this is seen to be $o(1)$ for large n on using the Hölder inequality (1.4) and the given conditions on the functions ψ and f. The second part of the first term is dominated by

$$\left\{K\int_{a_n}^{b} w|\psi f|\right\}^{p}\cdot\left\{\int_{a_n}^{b} w|\psi|^{q}\right\}^{-p/q}$$

and this is seen to be $o(1)$ for large n on using the Hölder inequality, the condition (3.2) and the result of Lemma 1.

Thus, on using again (3.3),

$$\begin{aligned}\int_{a_n}^{b_n} w|Af|^{p} &\le o(1) + pK^{p}\int_{a_n}^{b_n}\left\{\int_{x}^{b} w|\psi|^{q}\right\}^{-p/q} \\ &\quad\times\left\{\int_{x}^{b} w|\psi f|\right\}^{p-1} w(x)|\psi(x)f(x)|\,\mathrm{d}x \\ &= o(1) + pK^{p}I_n\ (\text{say})\end{aligned}\tag{3.5}$$

as $n\to\infty$.

On integration by parts, again as $n\to\infty$,

$$\begin{aligned}I_n &= p^{-1}\left[-\left\{\int_{x}^{b} w|\psi f|\right\}^{p}\cdot\left\{\int_{x}^{b} w|\psi|^{q}\right\}^{-p/q}\right]_{a_n}^{b_n} \\ &\quad + q^{-1}\int_{a_n}^{b_n}\left\{\int_{x}^{b} w|\psi f|\right\}^{p}\left\{\int_{x}^{b} w|\psi|^{q}\right\}^{-(p/q)-1} w(x)|\psi(x)|^{q}\,\mathrm{d}x \\ &= o(1) + q^{-1}J_n\ (\text{say})\end{aligned}\tag{3.6}$$

where the $o(1)$ term is obtained on using the same arguments as for (3.5).

Now from the definition of J_n in (3.6) it follows that

$$J_{n+1}\ge J_n\ge 0\quad (n\in\mathbb{Z});$$

then, without loss of generality, we can assume that either $J_n = 0$ $(n \in \mathbb{N}_0)$ or, for some $\delta > 0$,

$$J_n \geq \delta > 0 \quad (n \in \mathbb{N}_0). \tag{3.7}$$

In the first case, on letting $n \to +\infty$ in (3.5) and (3.6) we obtain

$$\int_a^b w|Af|^p = 0,$$

i.e. $Af \in L^p((a,b):w)$.

Assume then that (3.7) holds. Since (3.6) is an equality, and not an inequality, we can rewrite this result to give

$$J_n = o(1) + qI_n \quad (n \to +\infty). \tag{3.8}$$

From the definition (3.5) of I_n and an application of the Hölder inequality (1.4) we obtain, using $q(p-1) = p$,

$$\begin{aligned} I_n &\leq \left\{\int_{a_n}^{b_n} w|f|^p\right\}^{1/p} \left\{\int_{a_n}^{b_n} \left\{\int_x^b w|\psi f|\right\}^p \left\{\int_x^b w|\psi|^q\right\}^{-p} w(x)|\psi(x)|^q \mathrm{d}x\right\}^{1/q} \\ &\leq \|f\|_p J_n^{1/q} \quad (n \in \mathbb{N}_0). \end{aligned} \tag{3.9}$$

Hence from (3.7)–(3.9) we obtain

$$J_n \leq o(1) + q\|f\|_p J_n^{1/q} \quad (n \to +\infty)$$

and then

$$\begin{aligned} J_n^{1/p} &\leq o(1)\delta^{-1/q} + q\|f\|_p \\ &\leq o(1) + q\|f\|_p \quad (n \to +\infty). \end{aligned} \tag{3.10}$$

Thus from (3.5) and (3.6)

$$\begin{aligned} \int_{a_n}^{b_n} w|Af|^p &\leq o(1) + pK^p I_n \\ &\leq o(1) + pK^p(o(1) + q^{-1}J_n) \\ &\leq o(1) + pq^{-1}K^p J_n \quad (n \to +\infty). \end{aligned}$$

Hence from (3.10)

$$\left\{\int_{a_n}^{b_n} w|Af|^p\right\}^{1/p} \le o(1) + (p/q)^{1/p} K J_n^{1/p}$$
$$\le o(1) + (p/q)^{1/p} qK\|f\|_p \quad (n \to +\infty)$$

and this yields, on letting $n \to \infty$,

$$\|Af\|_p \le q(p/q)^{1/p} K\|f\|_p$$

and since f is arbitrary this last result holds for all $f \in L^p((a,b):w)$.

Finally then, since $q(p/q)^{1/p} = p^{1/p}\, q^{1/q}$, we obtain

$$\|Af\|_p \le p^{1/p} q^{1/q} K\|f\|_p \quad (f \in L^p((a,b):w)). \tag{3.11}$$

A similar argument shows that

$$\|Bg\|_q \le p^{1/p} q^{1/q} K\|g\|_q \quad (g \in L^q((a,b):w)). \tag{3.12}$$

These last two results show that the condition (1.14) is sufficient to establish the required boundedness properties of the operators A and B of Theorem 1, and the inequalities (1.15) and (1.16) of Corollary 1.

To show that the condition (1.14) is necessary for the boundedness result of Theorem 1 define, for all $\alpha \in (a,b)$, the function $f_\alpha : (a,b) \to \mathbb{C}$ by

$$f_\alpha(x) := \begin{cases} 0 & (x \in (a,\alpha)), \\ \bar{\psi}(x)|\psi(x)|^{q-2} & (x \in [\alpha, b)); \end{cases}$$

then it may be seen that $f_\alpha \in L^p((a,b):w)$.

Now suppose that the operator A is bounded on $L^p((a,b):w)$, say $\|Af\|_p \le k\|f\|_p$ for all $f \in L^p((a,b):w)$; then

$$\int_a^b w(x)|Af_\alpha(x)|^p \mathrm{d}x$$
$$= \left|\int_\alpha^b w|\psi|^q\right|^p \int_a^\alpha w|\varphi|^p + \int_\alpha^b w(x)|\varphi(x)|^p \left\{\int_x^b w|\psi|^q\right\} \mathrm{d}x$$
$$\le k^p \int_a^b w|f_\alpha|^p = k^p \int_\alpha^b w|\psi|^q \quad (\alpha \in (a,b)).$$

This result gives

$$\left|\int_\alpha^b w|\psi|^q\right|^p \int_a^\alpha w|\varphi|^p \leq k^p \int_\alpha^b w|\psi|^q \quad (\alpha \in (a,b))$$

and hence

$$\left\{\int_\alpha^b w|\psi|^q\right\}^{p-1} \int_a^\alpha w|\varphi|^p \leq k^p;$$

thus

$$\left\{\int_a^\alpha w|\varphi|^p\right\}^{1/p} \left\{\int_\alpha^b w|\psi|^q\right\}^{1/q} \leq k \quad (\alpha \in (a,b)). \tag{3.13}$$

Thus the condition (1.14) is seen to be necessary if the operator is bounded in $L^p((a,b):w)$.

There is a similar proof of (3.13) if it is assumed that the operator B is bounded on $L^q((a,b):w)$.

The Hardy integral inequality [6], see also [7, Section 9.8], shows that, in general, the number $p^{1/p}q^{1/q}K$ is best possible for the inequalities (1.15) and (1.16) to hold; this example, and other examples are discussed in Section 5 below.

4. PROOF OF COROLLARIES 2 AND 3

Proof of Corollary 2 From the definitions given in Section 1 and on integration by parts

$$\begin{aligned}
\langle Af, g\rangle_{p,q} &= \int_a^b w(x)\varphi(x)\left\{\int_x^b w\psi f\right\}g(x)\mathrm{d}x \\
&= \left[\int_a^x w\varphi g \int_x^b w\psi f\right]_a^b + \int_a^b \left\{\int_a^x w\varphi g\right\} w(x)\psi(x)f(x)\mathrm{d}x \\
&= \int_a^b w(x)f(x)\psi(x)\left\{\int_a^x w\varphi g\right\}\mathrm{d}x \\
&= \langle f, Bg\rangle_{p,q} \quad (f \in L^p((a,b):w) \text{ and } g \in L^q((a,b):w))
\end{aligned}$$

since the integrated term $[\cdots]_a^b$ is seen to be zero on using the methods given in Section 3. This gives the proof of (1.17).

The proof of (1.18) follows from standard procedures in functional analysis.

Proof of Corollary 3 Suppose that the functions φ and $\psi : (a, b) \to \mathbb{R}$; then with $(\cdot\,, \cdot)_2$ as the inner-product for $L^2((a, b) : w)$, and integrating by parts,

$$\begin{aligned}(Af, g)_2 &= \int_a^b w(x)\varphi(x)\left\{\int_x^b w\psi f\right\}\bar{g}(x)\,\mathrm{d}x \\ &= \left[\int_a^x w\varphi\bar{g}\int_x^b w\psi f\right]_a^b + \int_a^b\left\{\int_a^x w\varphi\bar{g}\right\}w(x)\psi(x)f(x)\,\mathrm{d}x \\ &= \int_a^b\left\{\int_a^x w\varphi\bar{g}\right\}w(x)\psi(x)f(x)\,\mathrm{d}x \\ &= (f, Bg)_2 \quad (f, g \in L^2((a, b) : w))\end{aligned}$$

since the integrated term $[\cdots]_a^b$ is seen to be zero on using the methods given in Section 3. This gives the proof of (1.20).

5. EXAMPLES

Example 1 Let $a = 0$ and $b = +\infty$, *with* $\varphi(x) = 1$, $\psi(x) = 1/x$ *and* $w(x) = 1$ *for all* $x \in (0, +\infty)$; *then*

$$K(x) = \frac{1}{(q-1)^{1/q}} \quad (x \in (0, \infty)) \quad \text{and} \quad Kp^{1/p}q^{1/q} = p = \frac{q}{q-1} \tag{5.1}$$

and, from the definitions of the operators A and B, we obtain the inequalities

$$\begin{aligned}\|Af\|_p &\le p\|f\|_p \qquad (f \in L^p(0, +\infty)), \\ \|Bg\|_q &\le \frac{q}{q-1}\|g\|_q \quad (g \in L^q(0, +\infty)).\end{aligned}$$

Equivalently these inequalities may be written in the form

$$\int_0^{+\infty}\left|\int_x^{+\infty}\frac{1}{t}f(t)\,\mathrm{d}t\right|^p\mathrm{d}x \le p^p\int_0^{+\infty}|f(x)|^p\,\mathrm{d}x \quad (f \in L^p(0, +\infty))$$

and

$$\int_0^{+\infty} \frac{1}{x^q}\left|\int_0^x g(t)\,\mathrm{d}t\right|^q \le \left(\frac{q}{q-1}\right)^q \int_0^{+\infty} |g(x)|^q\,\mathrm{d}x \quad (g \in L^q(0,+\infty)). \tag{5.2}$$

In all these inequalities the bounds shown are best possible, and the only case of equality is when $f(x) = g(x) = 0$ *for almost all* $x \in (0, \infty)$.

Proof A computation shows that the results given in (5.1) are valid and the inequalities then follow from Theorem 1 and Corollary 1.

The inequality (5.2) is Hardy's integral inequality; see [7, Section 9.8, Theorem 327].

The proof that the bounds are best possible and that the null function is the only case of equality, follows the analysis given in [7, Section 9.8]. It is of interest to note that the very general form of the inequalities in Theorem 1 and Corollary 1 precludes the possibility of determining, in the general case, the best possible bounds and the cases of equality; such detail has to be left for analysis in individual and special cases; Hardy's integral inequality is an example of this procedure.

Example 2 Let $a = -\infty$ *and* $b = +\infty$, *with* $\varphi(x) = \exp(x), \psi(x) = \exp(-x)$ *and* $w(x) = 1$ *for all* $x \in (-\infty, +\infty)$; *then*

$$K(x) = p^{-1/p}q^{-1/q} \quad (x \in (-\infty, +\infty)) \quad \textit{and} \quad Kp^{1/p}q^{1/q} = 1 \tag{5.3}$$

and, from the definitions of the operators A and B, we obtain the inequalities

$$\begin{aligned} \|Af\|_p &\le \|f\|_p \quad (f \in L^p(-\infty, +\infty)), \\ \|Bg\|_q &\le \|g\|_q \quad (g \in L^q(-\infty, +\infty)). \end{aligned}$$

Equivalently these inequalities may be written in the form

$$\begin{aligned} &\int_{-\infty}^{+\infty} \exp(px)\left|\int_x^{+\infty} \exp(-t)f(t)\,\mathrm{d}t\right|^p \mathrm{d}x \\ &\quad\le \int_{-\infty}^{+\infty} |f(x)|^p\,\mathrm{d}x \quad (f \in L^p(-\infty, +\infty)) \end{aligned}$$

and

$$\int_{-\infty}^{+\infty} \exp(-qx)\left|\int_{-\infty}^{x} \exp(t)g(t)\,\mathrm{d}t\right|^q \mathrm{d}x$$
$$\leq \int_{-\infty}^{+\infty} |g(x)|^q\,\mathrm{d}x \quad (g \in L^q(-\infty, +\infty)).$$

In all these inequalities the bound 1 *is best possible and the only case of equality is the null function.*

Proof A computation shows that the results given in (5.3) are valid and the inequalities then follow from Theorem 1 and Corollary 1.

To prove that the bound 1 is best possible substitute in the first integral inequality $f = f_X$ where

$$f_X(x) = 1 \quad (x \in [-X, X])$$

and f_X is zero elsewhere on $(-\infty, \infty)$. A computation then shows that

$$\lim_{X\to+\infty} \frac{\|Af_X\|_p}{\|f_X\|_p} = 1$$

if

$$\lim_{X\to+\infty} (2X)^{-1} \int_{-X}^{X} [1 - \exp(-(X - x))]^p\,\mathrm{d}x = 1. \tag{5.4}$$

To prove this result let $\Phi : (0, +\infty) \to (0, \infty)$ be defined by

$$\Phi(X) := (2X)^{-1} \int_{-X}^{X} [1 - \exp(-(X - x))]^p\,\mathrm{d}x \quad (X \in (0, +\infty)).$$

Then

$$\Phi(X) \leq (2X)^{-1} \int_{-X}^{X} 1\,\mathrm{d}x = 1 \quad (X \in (0, +\infty))$$

so that

$$\limsup_{X\to+\infty} \Phi(X) \leq 1. \tag{5.5}$$

Now let $\delta \in (0,1)$; then

$$\begin{aligned}\Phi(X) &\geq (2X)^{-1}\int_{-\delta X}^{\delta X}[1-\exp(-(X-x))]^p\,\mathrm{d}x\\ &\geq (2X)^{-1}\int_{-\delta X}^{\delta X}[1-\exp(-(X-\delta X))]^p\,\mathrm{d}x\\ &= [1-\exp(-X(1-\delta))]^p(2X)^{-1}\int_{-\delta X}^{\delta X}1\,\mathrm{d}x \quad (X\in(0,+\infty))\end{aligned}$$

and from this last result it follows that

$$\liminf_{X\to+\infty}\Phi(X)\geq\delta;$$

thus

$$\liminf_{X\to+\infty}\Phi(X)\geq 1 \tag{5.6}$$

The required result (5.4) now follows from (5.5) and (5.6).

There is a similar proof concerning the best bound of 1 for the second integral inequality of this Example 2.

The proof that the only case of equality is the null function follows the lines of the corresponding proof for Example 3 below and is omitted.

Example 3 *Let* $a=-1$ *and* $b=1$, *with* $\varphi(x)=(1-x)^{-1}$, $\psi(x)=(1+x)^{-1}$ *and* $w(x)=1$ *for all* $x\in(-1,1)$; *then for all* $x\in(-1,1)$

$$\begin{aligned}K(x)=&\frac{1}{\{2(p-1)\}^{1/p}}\frac{1}{\{2(q-1)\}^{1/q}}\left\{\frac{2^{p-1}-(1-x)^{p-1}}{1+x}\right\}^{1/p}\\ &\times\left\{\frac{2^{q-1}-(1+x)^{q-1}}{1-x}\right\}^{1/q}.\end{aligned}$$

The following results hold:

1. *For* $p\neq 2$, *i.e.* $q\neq 2$, *the function* $K(\cdot)$ *is continuous on* $[-1,1]$ *with* $K(-1)=K(1)=0$, *and the bounding condition* (1.14) *is satisfied; thus the operator and integral inequalities are valid.*
2. *For* $p=q=2$

$$K(x)=2^{-1}\quad(x\in[-1,1])$$

and the bounding constant $p^{1/p}\, q^{1/q}\, K = 1$; *the integral inequalities take the form*

$$\int_{-1}^{1} \frac{1}{(1-x)^2} \left| \int_{x}^{1} \frac{1}{1+t} f(t)\,\mathrm{d}t \right|^2 \mathrm{d}x \leq \int_{-1}^{1} |f(x)|^2\,\mathrm{d}x \quad (f \in L^2(-1,1))$$

and

$$\int_{-1}^{1} \frac{1}{(1+x)^2} \left| \int_{-1}^{x} \frac{1}{1-t} g(t)\,\mathrm{d}t \right|^2 \mathrm{d}x \leq \int_{-1}^{1} |g(x)|^2\,\mathrm{d}x \quad (g \in L^2(-1,1)).$$

The bound 1 *in these two inequalities is best possible and the only case of equality is the null function.*

Proof These results follow from a computation for the function $K(\cdot)$ as defined in (1.12), and then on a calculation with the explicit form of $K(\cdot)$.

In Case 2 to prove that the bound 1 is best possible in the first of the two inequalities, substitute, with $\varepsilon \in (0,1/2)$, the function f_ε defined by

$$f_\varepsilon(x) := \begin{cases} (1+x)^{\varepsilon-1/2} & (x \in (-1,0]), \\ 0 & (x \in (0,1)); \end{cases}$$

then $f_\varepsilon \in L^2(-1,1)$. Then the right-side of the inequality gives

$$\int_{-1}^{0} \frac{1}{(1+x)^{1-2\varepsilon}}\,\mathrm{d}x = \frac{1}{2\varepsilon}(1+O(\varepsilon)) \quad (\varepsilon \to 0^+).$$

The left-hand side reduces, after a calculation, to

$$\begin{aligned} &\frac{4}{(1-2\varepsilon)^2} \int_{-1}^{0} \frac{1}{(1-x)^2} \frac{1}{(1+x)^{1-2\varepsilon}}\,\mathrm{d}x + O(1) \\ &= \frac{1}{(1-2\varepsilon)^2} \int_{-1}^{0} \frac{1}{(1+x)^{1-2\varepsilon}}\,\mathrm{d}x \\ &\quad + \frac{4}{(1-2\varepsilon)^2} \int_{-1}^{0} (3-x)(1+x)^{2\varepsilon}\,\mathrm{d}x + O(1) \\ &= \frac{1}{(1-2\varepsilon)^2} \frac{1}{2\varepsilon}(1+O(\varepsilon)) \quad (\varepsilon \to 0^+). \end{aligned}$$

Thus the quotient (left-hand side)/(right-hand side) tends to the limit 1 as $\varepsilon \to 0^+$, and the bound 1 is best-possible for the inequality.

There is a similar argument for the second of the two inequalities.

To prove that the only case of equality is the null function start by noting that a consequence of the first inequality of Case 2 is that

$$\int_{-1}^{1} \frac{1}{(1-x)^2} \left| \int_x^1 \frac{1}{1+t} f(t)\,\mathrm{d}t \right|^2 \mathrm{d}x \leq \int_{-1}^{1} \frac{1}{(1-x)^2} \left\{ \int_x^1 \frac{1}{1+t} |f(t)|\,\mathrm{d}t \right\}^2 \mathrm{d}x$$
$$\leq \int_{-1}^{1} |f(x)|^2\,\mathrm{d}x \tag{5.7}$$

since $f \in L^2(-1,1)$ if and only if $|f| \in L^2(-1,1)$. Thus it is sufficient to argue with the case of equality when

$$f(x) \geq 0 \quad (x \in (-1,1)). \tag{5.8}$$

Thus, on integration by parts,

$$\int_{-1}^{1} \frac{1}{(1-x)^2} \left\{ \int_x^1 \frac{1}{1+t} f(t)\,\mathrm{d}t \right\}^2 \mathrm{d}x$$
$$= \left[\int_{-1}^{x} \frac{1}{(1-t)^2}\,\mathrm{d}t . \left\{ \int_x^1 \frac{1}{1+t} f(t)\,\mathrm{d}t \right\}^2 \right]_{-1}^{1}$$
$$+ 2 \int_{-1}^{1} \left\{ \int_{-1}^{x} \frac{1}{(1-t)^2}\,\mathrm{d}t \right\} \frac{f(x)}{1+x} \left\{ \int_x^1 \frac{1}{1+t} f(t)\,\mathrm{d}t \right\} \mathrm{d}x. \tag{5.9}$$

From the formula for $K(\cdot)$ above with $p=q=2$

$$\int_{-1}^{x} \frac{1}{(1-t)^2}\,\mathrm{d}t \int_x^1 \frac{1}{(1+t)^2}\,\mathrm{d}t = \frac{1}{4} \quad (x \in (-1,1). \tag{5.10}$$

Thus the integrated term in (5.9) becomes

$$\left[\frac{1}{4} \left\{ \int_x^1 \frac{1}{(1+t)^2}\,\mathrm{d}t \right\}^{-1} \left\{ \int_x^1 \frac{1}{1+t} f(t)\,\mathrm{d}t \right\}^2 \right]_{-1}^{1}$$

and this expression is seen to be zero by the methods of Section 3.

Hence (5.9) becomes, noting that $\int_x^1 (1+t)^2\,\mathrm{d}t = (1-x)\times \{2(1+x)\}^{-1}$ $(x \in (-1,1))$,

$$\int_{-1}^{1} \frac{1}{(1-x)^2}\left\{\int_x^1 \frac{1}{1+t} f(t)\,\mathrm{d}t\right\}^2 \mathrm{d}x$$
$$= \int_{-1}^{1} \frac{1}{(1-x)} f(x) \left\{\int_x^1 \frac{1}{1+t} f(t)\,\mathrm{d}t\right\} \mathrm{d}x.$$

At this stage it is important to note that this last result is an equality to which the Cauchy–Schwarz inequality is applied to give

$$\int_{-1}^{1} \frac{1}{(1-x)^2}\left\{\int_x^1 \frac{1}{1+t} f(t)\,\mathrm{d}t\right\}^2 \mathrm{d}x$$
$$< \left\{\int_{-1}^{1} \frac{1}{(1-x)^2}\left\{\int_x^1 \frac{1}{1+t} f(t)\,\mathrm{d}t\right\}^2 \mathrm{d}x\right\}^{1/2}$$
$$\times \left\{\int_{-1}^{1} |f(x)|^2\,\mathrm{d}x\right\}^{1/2} \tag{5.11}$$

unless, for some $\alpha \in \mathbb{R}$,

$$f(x) = \frac{\alpha}{1-x}\int_x^1 \frac{1}{1+t} f(t)\,\mathrm{d}t \quad (\text{almost all } x \in (-1,1)). \tag{5.12}$$

From (5.8) it follows that $\alpha \geq 0$; if $\alpha = 0$ then f is null on $(-1,1)$, so suppose that $\alpha > 0$.

Now since $t(1+t)f(t) \in L^1_{\mathrm{loc}}(-1,1)$ it follows from (5.12) that $f \in C(-1,1)$, and then that $f \in C^{(1)}(-1,1)$. Differentiating (5.12) then gives

$$((1-x)f(x))' = (1-x)f'(x) - f(x) = \alpha(1+x)^{-1} f(x) \quad (x \in (-1,1))$$

i.e.

$$f'(x) = \frac{(1+x-\alpha)}{1-x^2} f(x) \quad (x \in (-1,1)). \tag{5.13}$$

This last result is a first-order, linear differential equation for f that is regular on $(-1,1)$ but is singular at ± 1. Since $f \geq 0$ on $(-1,1)$ and

$f \in C(-1,1)$ either f is null on $(-1,1)$, or for some point $x_0 \in (-1,1)$ it is the case that $f(x_0) > 0$. In the second case apply standard methods of solution to (5.13) to give

$$f(x) = f(x_0)\frac{1}{\sqrt{1-x^2}}\exp\left((1-\alpha)\int_{x_0}^{x}\frac{1}{1-t^2}\,\mathrm{d}t\right) \quad (x \in (-1,1)). \tag{5.14}$$

If $\alpha \in (0,1]$ then (5.14) implies that, for $k > 0$, near 1^-

$$f(x)^2 \geq k(1-x^2)^{-1}$$

and this result gives $f \notin L^2(-1,1)$; if $\alpha \in (1,\infty)$ then, similarly, near -1^+ the same result holds and again $f \notin L^2(-1,1)$. This contradiction on the hypothesis $f \in L^2(-1,1)$ implies that f is null on $(-1,1)$.

Returning now to (5.11) we have either f is null on $(-1,1)$ or f is not null and

$$\left\{\int_{-1}^{1}\frac{1}{(1-x)^2}\left\{\int_x^1\frac{1}{1+t}f(t)\,\mathrm{d}t\right\}^2\mathrm{d}x\right\}^{1/2} > 0$$

in which case we can cancel this factor from both sides and then, on squaring the result, obtain, noting again (5.7),

$$\int_{-1}^{1}\frac{1}{(1-x)^2}\left|\int_x^1\frac{1}{1+t}f(t)\,\mathrm{d}t\right|^2\mathrm{d}x < \int_{-1}^{1}|f(x)|^2\,\mathrm{d}x \quad (f \in L^2(-1,1))$$

unless f is null on $(-1,1)$.

There is a similar proof for the case of equality in the second inequality of Case 2.

This result completes the proof for Example 3.

In Case 1 it is not known if there is an explicit formula for the upper bound of $K(\cdot)$ on the interval $[-1,1]$, in terms of the parameters p and q, nor if there are any cases of equality other than the null function.

6. AN APPLICATION

There is an important connection between orthogonal polynomials, defined on the real line and generated as solutions of ordinary differential equations with a spectral parameter, and the spectral theory of self-adjoint operators in Hilbert function spaces; for a survey of the main results, up to 1990, see [4]. The operator and integral inequalities of [1] and the results in this paper have played a significant rôle in the determination of the domains of the unbounded self-adjoint operators in this connection; see the results in [8, 14]. For an example the case of the Legendre differential operator is considered as an application of Theorem 1.

The Legendre differential expression $M: D(M) \subset L^2(-1,1) \to L^2(-1,1)$ is defined by

$$M[f] := -((1-x^2)f'(x))' \quad (x \in (-1,1) \text{ and } f \in D(M))$$

where

$$\begin{aligned} D(M) := \{ f : (-1,1) \to \mathbb{C} \colon & \text{(i) } f, f' \in AC_{\mathrm{loc}}(-1,1), \\ & \text{(ii) } f, M[f] \in L^2(-1,1) \}. \end{aligned}$$

In general the elements of $D(M)$ have singular behaviour near the end-points ± 1. For example the function $l(x) := \ln((1+x)/(1-x))$ $(x \in (-1,1))$ is a member of $D(M)$ but has logarithmic singularities near ± 1, and $l'(x) = 2/(1-x^2)$ so that $l' \notin L^1(-1,1)$.

The self-adjoint operator S in $L^2(-1,1)$ associated with the Legendre polynomials is defined by, see [3, 4],

$$D(S) := \{ f \in D(M) \colon \lim_{x \to \pm 1} (1-x^2)f'(x) = 0 \}$$

and

$$Sf := M[f] \quad (f \in D(S)).$$

The elements of this domain $D(S)$, in comparison with $D(M)$, enjoy remarkably smooth properties; from the results in [5]:

COROLLARY 4 *Let* $f \in D(S)$ *then:*

(i) $f \in AC[-1,1]$
(ii) $f' \in L^2(-1,1)$.

Proof The idea for this proof is taken, in part, from the forthcoming paper [5]; see also [3, pp. 58,59].

Let $f \in D(S)$; then $M[f] \in L^2[0,1)$ so that $M[f] \in L^1[0,1)$; hence for $0 < x < \xi < 1$

$$\int_x^\xi M[f](t)\,\mathrm{d}t = (1-\xi^2)f'(\xi) - (1-x^2)f'(x).$$

In this result let $\xi \to 1^-$ to obtain, from the definition of $D(S)$,

$$\int_x^1 M[f](t)\,\mathrm{d}t = -(1-x^2)f'(x) \quad (x \in [0,1)),$$

i.e.

$$f'(x) = \frac{1}{x^2-1}\int_x^1 M[f](t)\,\mathrm{d}t \quad (x \in [0,1)).$$

To this last result apply Theorem 1 with $p=q=2$, $a=0, b=1$ and $\varphi(x) = (1-x^2)^{-1}$, $\psi(x) = 1$ for all $x \in (0,1)$. In this case

$$K(x)^2 = \int_0^x \frac{1}{(1-t^2)^2}\,\mathrm{d}t \int_x^1 1^2\,\mathrm{d}t \quad (x \in (0,1))$$

and it may be seen that $K(\cdot)$ is bounded on $(0,1)$.

Hence an application of Theorem 1 gives

$$f' \equiv Af \in L^2(0,1) \quad (f \in D(S)).$$

A similar argument shows that $f' \in L^2(-1,0)$; thus $f' \in L^2(-1,1)$.

Since $f' \in L^2(-1,1)$ implies that $f' \in L^1(-1,1)$ it follows that $f \in AC[-1,1]$.

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