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Some Inequalities for the Hersch–Pfluger Distortion Function

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The authors obtain some new functional inequalities for the Hersch–Pfluger distortion function in the theory of plane quasiconformal mappings, thus solving some recent conjectures.

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1 INTRODUCTION

In 1952, Hersch and Pfluger [10] generalized the classical Schwarz lemma for analytic functions to the class $QC_K(B)$ of K-quasiconformal mappings of the unit disk B into itself with the origin fixed, for $K \geq 1$. They showed that there is a strictly increasing function $\varphi_K : [0, 1] \to [0, 1]$ such that

$$|f(z)| \leq \varphi_K(|z|), \tag{1.1}$$

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for each $f \in QC_K(B)$ and $z \in B$ (Cf. [12, Theorem 3.1, p. 64]). This distortion function is defined by [12, p. 63]

$$\begin{cases} \varphi_K(r) = \mu^{-1}(\mu(r)/K), \\ \mu(r) = \dfrac{\pi}{2}\dfrac{\mathcal{K}(r')}{\mathcal{K}(r)}, \\ \mathcal{K} = \mathcal{K}(r) = \displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{1 - r^2 \sin^2 x}}, \end{cases} \tag{1.2}$$

for $r \in (0, 1)$ and $K \in (0, \infty)$, $\varphi_K(0) = \varphi_K(1) - 1 = 0$, where $r' = \sqrt{1 - r^2}$ and $\mu(r)$ is the modulus of the Grötzsch ring $B \backslash [0, r]$, $0 < r < 1$. We also denote $\mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r')$. These are called the complete elliptic integrals of the first kind [7,8]. For later reference, we recall that the complete elliptic integrals of the second kind are defined by

$$\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 x}\, dx, \ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \tag{1.3}$$

$r' = \sqrt{1 - r^2}$, $0 \le r \le 1$ (cf. [8, p. 17]).

It is well known that the Hersch–Pfluger function $\varphi_K(r)$ plays a very important role in quasiconformal theory [1,4,12,13,22,23]. It has also found applications in other mathematical fields such as number theory. In number theory, with different notation, $\varphi_K(r)$ occurs in Ramanujan's work on modular equations and singular values of elliptic integrals [5,7,22,24]. Ramanujan's modular equations provide numerous algebraic identities satisfied by $\varphi_K(r)$ (see 3.2 below for further references).

In recent papers [13,15,16,18,22], many new properties and applications were obtained for $\varphi_K(r)$. However, some open problems on this function are still to be settled. Among them, the following two conjectures appear in [22, Conjectures 2.19]:

(C1) For $K \in [1, \infty)$ and $r \in (0, 1)$,

$$\text{th}\big(d(K)\text{arth}(r^{1/K})\big) \le \varphi_K(r) \le \text{th}\big(c(K)\text{arth}(r^{1/K})\big), \tag{1.4}$$

where $c(K) = \max\{K, 4^{1-1/K}\}$ and $d(K) = \min\{K, 4^{1-1/K}\}$. The equalities hold iff $K = 1$ or $K = 2$.

(C2) For $K\in[1,\infty)$ and $r\in(0,1)$,

$$\varphi_K(r)\geq \mathrm{th}\Big(2^{2-1/K}\,\mathrm{arth}\big(A(r)^{1/K}\big)\Big), \tag{1.5}$$

where $A(r)=r/(1+r')$, $r'=\sqrt{1-r^2}$.

We observe that $\varphi_K(r)$ satisfies the conditions $\varphi_K(0)=\varphi_K(1)-1=\varphi_1(r)-r=\lim_{K\to\infty}\varphi_K(r)-1=0$, for $r\in(0,1)$, $K>1$. The lower and upper bounds in (1.4) also satisfy the same boundary conditions.

On the other hand, some properties of $\varphi_K(r)$, especially the sharp bounds for $\varphi_K(r)$, depend on those of the function $m(r)+\log r$, where

$$m(r)\equiv\frac{2}{\pi}(r')^2\mathcal{K}(r)\mathcal{K}'(r), \tag{1.6}$$

for $r\in(0,1)$. Some properties of the function $m(r)$ appear in [2,11,19]. We need still better estimates for $m(r)$ to accomplish our results.

The main purpose of this paper is to prove that conjectures (C_1) and (C_2) are true, and to obtain some new properties for the function $m(r)+\log r$, including sharp lower and upper bounds, from which sharp bounds for $\varphi_K(r)$ follow. Hence, the explicit upper bound in the quasiconformal Schwarz lemma is further sharpened.

Throughout this paper, we let $r'=\sqrt{1-r^2}$ whenever $r\in[0,1]$. We let th denote the hyperbolic tangent function and let arth denote its inverse.

We now state some of our main results.

THEOREM 1 (1) *The function*

$$F(r)\equiv\sqrt{r}\,\frac{m(r)+\log r}{(1-r)\mathrm{arth}\sqrt{r}} \tag{1.7}$$

is strictly increasing from $(0,1)$ *onto* $(\log 4, 2)$. *In particular,*

$$\frac{(1-r)\mathrm{arth}\sqrt{r}}{\sqrt{r}}\log 4<m(r)+\log r<2\frac{(1-r)\mathrm{arth}\sqrt{r}}{\sqrt{r}}<2(1-r)^{2/3}, \tag{1.8}$$

for all $r\in(0,1)$.

(2) *The function* $G(r) \equiv r(m(r)+\log r)/[(r')^2(\operatorname{arth} r)]$ *is strictly decreasing from* $(0,1)$ *onto* $(1,\log 4)$. *In particular,*

$$(r')^2 \frac{\operatorname{arth} r}{r} < m(r) + \log r < (r')^2 \frac{\operatorname{arth} r}{r} \log 4 < (r')^{4/3} \log 4, \qquad (1.9)$$

for all $r \in (0,1)$.

(3) *The function*

$$f(r) \equiv m(r) + \log r - 2\frac{(1-r)\operatorname{arth}\sqrt{r}}{\sqrt{r}}$$

is strictly increasing from $(0,1)$ *onto* $(\log 4 - 2, 0)$.

(4) *The function*

$$g(r) \equiv m(r) + \log r - \frac{(r')^2 \operatorname{arth} r}{r}$$

is strictly decreasing from $(0,1)$ *onto* $(0, \log 4 - 1)$.

(5) *The function*

$$h(r) \equiv \frac{m(r)}{\log(1/r)}$$

is strictly increasing from $(0,1)$ *onto* $(1,\infty)$.

Theorem 1 improves the known bounds and some other properties of the function $m(r) + \log r$. Our next result is an application of Theorem 1 to the function $\varphi_K(r)$.

THEOREM 2 *For each* $r \in (0,1)$, *define the functions f and g on* $[1,\infty)$ *by*

$$f(K) = \varphi_K(r)\left(\mathrm{e}^{a(r)}/r\right)^{1/K}, \qquad g(K) = \varphi_{1/K}(r)\left(\mathrm{e}^{b(r)}/r\right)^{K}, \qquad (1.10)$$

where $a(r) = \min\{2c(\sqrt{r}), c(r)\log 4\}$, $b(r) = \max\{c(r), c(\sqrt{r})\log 4\}$ *and* $c(r) \equiv ((r')^2 \operatorname{arth} r)/r$. *Then*

(1) f *is strictly decreasing from* $[1,\infty)$ *onto* $(1, \mathrm{e}^{a(r)}]$. *In particular, for* $r \in (0,1)$ *and* $K \in (1,\infty)$,

$$\varphi_K(r) < \mathrm{e}^{a(r)(1-1/K)} r^{1/K} < 4^{(r')^{4/3}(1-1/K)} r^{1/K}. \qquad (1.11)$$

(2) *g is strictly decreasing from* $[1,\infty)$ *onto* $(0, e^{b(r)}]$. *In particular, for* $r \in (0,1)$ *and* $K \in (1,\infty)$,

$$\varphi_{1/K}(r) < e^{b(r)(1-K)} r^K \leq 4^{c(\sqrt{r})(1-K)} r^K < 4^{(1-r)(1-K)} r^K. \tag{1.12}$$

Remark 1 The proof of Theorem 2 implies some previously known bounds of $\varphi_K(r)$ such as [11]

$$\varphi_K(r) < r\,\exp((1-1/K)m(r)), \tag{1.13}$$

where $m(r)$ is as in (1.6). This upper bound follows, for instance, from Corollary 1(1) if we set $a(r) = m(r) + \log r$ there. The bounds in (1.12) significantly improve the well-known inequality [12]

$$\varphi_{1/K}(r) < r^K, \quad r \in (0,1), \ K \in (1,\infty). \tag{1.14}$$

THEOREM 3 (1) *For* $r \in (0,1)$, *define the function* f *on* $[1,\infty)$ *by* $f(K) = (1/K)\operatorname{arth}(\varphi_K(r^K))$. *Then, there exists a unique* $K_1 \in (1,2)$ *such that* f *is strictly increasing from* $[1, K_1]$ *onto* $[\operatorname{arth} r, f(K_1)]$ *and strictly decreasing from* $[K_1,\infty)$ *onto* $(0, f(K_1)]$, *with* $f(2) = \operatorname{arth} r = f(1)$.

(2) *For* $r \in (0,1)$, *define the function* F *on* $[1,\infty)$ *by* $F(K) = 4^{(1/K)-1}\operatorname{arth}(\varphi_K(r^K))$. *Then there exists a unique* $K_2 \in (1,2)$ *such that* F *is strictly decreasing from* $[1, K_2]$ *onto* $[F(K_2), \operatorname{arth} r]$, *and strictly increasing from* $[K_2,\infty)$ *onto* $[F(K_2), \frac{1}{4}\operatorname{arth}\mu^{-1}(\log(1/r)))$ *with* $F(2) = \operatorname{arth} r = F(1)$.

(3) *For* $r \in (0,1)$ *and* $K \in [1,\infty)$,

$$\operatorname{th}\big(d(K)\operatorname{arth}(r^{1/K})\big) \leq \varphi_K(r) \leq \operatorname{th}(c(K)\operatorname{arth}(r^{1/K})), \tag{1.15}$$

where $c(K) = \max\{K, 4^{1-1/K}\}$ *and* $d(K) = \min\{K, 4^{1-1/K}\}$. *The equalities hold iff* $K = 1$ *or* 2. *Thus, the Conjecture* (C1) *is true.*

THEOREM 4 *For* $r \in (0,1)$ *and* $K \in [1,\infty)$,

(1) $\operatorname{th}(2^{2-1/K}\operatorname{arth}(A(r)^{1/K})) \leq \varphi_K(r) \leq \operatorname{th}(2K\operatorname{arth}(A(r)^{1/K}))$,

(2) $\{\operatorname{th}((1/K)\operatorname{arth}(\sqrt{r}))\}^{2K} \leq \varphi_{1/K}(r) \leq \{\operatorname{th}\big(2^{(1/K)-1}\operatorname{arth}(\sqrt{r})\big)\}^{2K}$,

where $A(r) = r/(1+r')$. *Equalities hold iff* $K = 1$. *In particular, Conjecture* (C2) *is true.*

The next result [4, Theorem 1.25] is a monotone analogue of l'Hospital rule and is very useful in our proofs.

LEMMA 1 *For $a, b \in R$ with $a < b$, let $f, g : [a, b) \to R$, be differentiable functions such that $g'(x) \neq 0$, for all $x \in (a, b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so is $(f(x) - f(a))/(g(x) - g(a))$. Analogous result holds on $(a, b]$.*

2 PRELIMINARY RESULTS

In this section, we obtain results which are needed for the proofs of the theorems stated in Section 1. We shall frequently employ the following well-known identities, which include some formulas [2, Lemma 2.1, 7,9] for the derivatives of the special functions defined in formulas (1.2) and (1.3):

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}r} = \frac{\mathcal{E} - (r')^2\mathcal{K}}{r(r')^2}, \qquad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}r} = \frac{\mathcal{E} - \mathcal{K}}{r}; \tag{2.1}$$

$$\begin{cases} \mathcal{K}\left(\dfrac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), & \mathcal{K}'\left(\dfrac{2\sqrt{r}}{1+r}\right) = \dfrac{1}{2}(1+r)\mathcal{K}'(r), \\ \mathcal{E}\left(\dfrac{2\sqrt{r}}{1+r}\right) = \dfrac{2\mathcal{E}(r) - (r')^2\mathcal{K}(r)}{1+r}, & \mathcal{E}'\left(\dfrac{2\sqrt{r}}{1+r}\right) = \dfrac{\mathcal{E}'(r) + r\mathcal{K}'(r)}{1+r}; \end{cases} \tag{2.2}$$

$$\mathcal{K}\mathcal{E}' + \mathcal{K}'\mathcal{E} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2}; \tag{2.3}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}r} = -\frac{\pi^2}{4r(r')^2\mathcal{K}(r)^2}; \tag{2.4}$$

$$\frac{\partial s}{\partial r} = \frac{s(s')^2\mathcal{K}(s)^2}{Kr(r')^2\mathcal{K}(r)^2}, \qquad \frac{\partial s}{\partial K} = \frac{2}{\pi K}s(s')^2\mathcal{K}(s)\mathcal{K}'(s), \tag{2.5}$$

where $s = \varphi_K(r)$, $0 < r < 1$, $0 < K < \infty$.

LEMMA 2 *The function*

$$f(r) \equiv \mathcal{E}(r)\mathcal{E}'(r) \tag{2.6}$$

is strictly increasing on $[0, 1/\sqrt{2}]$ and decreasing on $[1/\sqrt{2}, 1]$ with $f([0, 1]) = [\pi/2, c]$, where $c = \mathcal{E}(1/\sqrt{2})^2 = 1.824\ldots$

Proof Differentiation yields

$$\frac{1}{r}f'(r) = \mathcal{E}(r)\frac{\mathcal{K}'(r) - \mathcal{E}'(r)}{(r')^2} - \mathcal{E}'(r)\frac{\mathcal{K}(r) - \mathcal{E}(r)}{r^2}, \tag{2.7}$$

which, by (1.2) and (1.3), is strictly decreasing from $(0,1)$ onto $(-\infty,\infty)$ with $r=1/\sqrt{2}$ as its unique zero and hence, the result follows.

LEMMA 3 (1) *Let* $c>0$. *Then the function* $f(r)\equiv(r')^c\,(\operatorname{arth} r)/r$ *is decreasing on* $(0,1)$ *iff* $c\geq\frac{2}{3}$.

(2) *The function* $g(r)\equiv(\operatorname{arth} r)/[r+\log(1+r)]$ *is strictly increasing from* $(0,1)$ *onto* $(\frac{1}{2},\infty)$.

(3) *The function* $h(r)\equiv\operatorname{arth} r-\operatorname{arth}(r^2)$ *is strictly increasing and concave, while* $k(r)\equiv r\operatorname{arth} r-\operatorname{arth}(r^2)$ *is strictly increasing and convex from* $[0,1)$ *onto* $[0,\frac{1}{2}\log 2)$. *In particular, for* $r\in(0,1)$,

$$\max\left\{\frac{1}{r}\operatorname{arth}(r^2),\ \operatorname{arth}(r^2)+\frac{1}{2}r\log 2\right\}<\operatorname{arth} r<\operatorname{arth}(r^2)+\frac{1}{2}\log 2. \tag{2.8}$$

(4) *The function* $H(r)\equiv[(1+r^2)\operatorname{arth} r-r]/r^3$ *is strictly increasing and convex from* $(0,1)$ *onto* $(\frac{4}{3},\infty)$.

Proof (1) Differentiation gives

$$\frac{r^2(r')^{(2-c)}}{cr^2+(r')^2}f'(r)=\frac{r}{cr^2+(r')^2}-\operatorname{arth} r,$$

which is negative iff $c\geq[r-(r')^2\operatorname{arth} r]/(r^2\operatorname{arth} r)$, for all $r\in(0,1)$.

Now,

$$\frac{\mathrm{d}}{\mathrm{d}r}[r^2\operatorname{arth} r]\bigg/\frac{\mathrm{d}}{\mathrm{d}r}[r-(r')^2\operatorname{arth} r]=1+[r/(2(r')^2\operatorname{arth} r)].$$

Next,

$$\frac{\mathrm{d}}{\mathrm{d}r}[(r')^2\operatorname{arth} r]\bigg/\frac{\mathrm{d}}{\mathrm{d}r}[r]=1-2r\operatorname{arth} r,$$

which is clearly decreasing. Hence the result follows by Lemma 1.

(2) The function g has the form 0/0 at $r=0$. Denote $g_1(r)=\operatorname{arth} r$, $g_2(r)=r+\log(1+r)$. Then $g_1'(r)/g_2'(r)=1/[(1-r)(2+r)]$, which is clearly increasing, so that $g=g_1/g_2$ is also increasing by Lemma 1. The limiting values are clear.

(3) Differentiation gives

$$h'(r) = \frac{1-r}{(1+r)(1+r^2)}, \qquad k'(r) = \operatorname{arth} r - \frac{r}{1+r^2}, \tag{2.9}$$

from which the assertions follow.

(4) This follows easily from differentiation.

LEMMA 4 (1) *The function* $f(r) \equiv \mathcal{K}(r) + \log r'$ *is strictly decreasing from* $(0,1)$ *onto* $(\log 4, \pi/2)$.

(2) *The function* $g(r) \equiv \mathcal{K}(r) - \operatorname{arth} r$ *is decreasing from* $(0,1)$ *onto* $(\log 2, \pi/2)$.

(3) *The function* $F(r) \equiv [\mathcal{K}(r) - (\pi/2)]/\log(1/r')$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4, 1)$. *In particular, for* $r \in (0,1)$,

$$(\pi/4)\log(1/r') + (\pi/2) < \mathcal{K}(r) < \log(1/r') + (\pi/2). \tag{2.10}$$

Proof The assertion (1) was proved in [2, Theorem 2.2]. The result (2) follows from (1). For (3), let $G(r) \equiv \mathcal{K}(r) - \pi/2$ and $h(r) = \log(1/r')$. Then

$$G'(r)/h'(r) = \frac{\mathcal{E}(r) - (r')^2\mathcal{K}(r)}{r^2},$$

which, by (1.2) and (1.3), is strictly increasing on $(0,1)$ and, hence, the result follows from Lemma 1.

LEMMA 5 (1) *The function* $g(r) \equiv r^2(\mathcal{K}' - \mathcal{E}')/(r')^2$ *is strictly increasing from* $(0,1)$ *onto* $(0, \pi/4)$.

(2) *The function* $h(r) \equiv \mathcal{E}'/\sqrt{r}$ *is strictly decreasing from* $(0,1)$ *onto* $(\pi/2, \infty)$.

(3) *The function* $F(r) \equiv (\sqrt{r})\mathcal{K}'$ *is strictly increasing from* $(0,1)$ *onto* $(0, \pi/2)$.

(4) *For all* $r \in (0,1)$,

$$\mathcal{E}'(r)[\mathcal{E}(r) - (r')^2\mathcal{K}(r)] < \frac{\pi}{4}\left(1 + r^2 - \frac{(r')^2 \operatorname{arth} r}{r}\right). \tag{2.11}$$

Proof (1) This follows from [18, Theorem 2.1(6)].

(2) This follows from [3, Theorem 1.3].

(3) This follows from [2, Theorem 2.2(3)].

(4) Let

$$f(r) \equiv \frac{4}{\pi}\mathcal{E}'(\mathcal{E}-(r')^2\mathcal{K}) - r^2 + \frac{(r')^2 \operatorname{arth} r}{r} - 1, \quad \text{for } r \in (0,1).$$

Then,

$$\begin{aligned} \frac{1}{r}f'(r) = f_1(r) \equiv & \frac{4}{\pi(r')^2}(\mathcal{K}'-\mathcal{E}')(\mathcal{E}-(r')^2\mathcal{K}) \\ & + \frac{4}{\pi}\mathcal{K}\mathcal{E}' + \frac{r-(1+r^2)\operatorname{arth} r}{r^3} - 2. \end{aligned} \tag{2.12}$$

By [3, Theorem 2.1(7)] and [18, Theorem 2.1(6)], the function

$$\frac{\mathcal{K}'-\mathcal{E}'}{(r')^2}(\mathcal{E}-(r')^2\mathcal{K}) = \frac{(r^2)(\mathcal{K}'-\mathcal{E}')}{(r')^2}\cdot\frac{\mathcal{E}-(r')^2\mathcal{K}}{r^2}$$

is strictly increasing on (0, 1). Hence, by Lemma 3(4),

$$\begin{aligned} f_1(r) \le & \frac{4}{\pi}\left\{\frac{\mathcal{K}'(b)-\mathcal{E}'(b)}{(b')^2}[\mathcal{E}(b)-(b')^2\mathcal{K}(b)] + \mathcal{K}(b)\mathcal{E}'(b)\right\} \\ & + \frac{a-(1+a^2)\operatorname{arth} a}{a^3} - 2 \equiv f_2(a,b), \end{aligned} \tag{2.13}$$

for $r \in (a,b] \subset (0,1)$. Since

$$f_2(0+, \sin 38^\circ) = -0.001\ldots, \quad f_2(\sin 38^\circ, \sin 42^\circ) = -0.05\ldots,$$

it follows from (2.12) and (2.13) that

$$f(r) < f(0+) = 0, \quad \text{for } r \in (0, \sin 42^\circ]. \tag{2.14}$$

Next, by (1.2) and (1.3), we have

$$\begin{aligned} f'(r) = rf_1(r) > & \frac{4r}{\pi(r')^2}(\mathcal{K}'-\mathcal{E}')(\mathcal{E}-(r')^2\mathcal{K}) + \frac{4}{\pi}r\mathcal{E}'\log 2 \\ & - \left(1 - \frac{4}{\pi}\frac{r^3}{1+r^2}\mathcal{E}'\right)\frac{(1+r^2)\operatorname{arth} r}{r^2} - 2r + \frac{1}{r} \\ > f_3(r) \equiv & \, r(\mathcal{E}-(r')^2\mathcal{K}) + \frac{4}{\pi}r\mathcal{E}'\log 2 \\ & - \frac{4}{\pi}\frac{r'(1+r^2)\operatorname{arth} r}{r^2}\cdot\frac{\frac{\pi}{4} - \frac{r^3}{1+r^2}\mathcal{E}'}{r'} - 1, \end{aligned} \tag{2.15}$$

for $r \in (0,1)$.

Let

$$f_4(r) = \frac{\pi}{4} - \frac{r^3}{1+r^2}\mathcal{E}' \quad \text{and} \quad f_5(r) \equiv r'.$$

Then $f_4(1) = f_5(1) = 0$ and

$$\frac{f_4'(r)}{f_5'(r)} = f_6(r) \equiv \frac{r^3 r'}{1+r^2}\left\{\frac{3+r^2}{1+r^2}\frac{\mathcal{E}'}{r^2} + \frac{\mathcal{K}' - \mathcal{E}'}{(r')^2}\right\}.$$

By (1.2), (1.3), and part (2), the function

$$\frac{3+r^2}{1+r^2}\frac{\mathcal{E}'}{r^2} + \frac{\mathcal{K}' - \mathcal{E}'}{(r')^2}$$

is strictly decreasing on (0, 1). Since

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^3 r'}{1+r^2}\right) = \frac{r^2(3 - 2r^4 - 3r^2)}{r'(1+r^2)^2},$$

$r^3 r'/(1+r^2)$ is strictly decreasing on $\left[\frac{1}{2}\sqrt{\sqrt{33}-3}, 1\right)$. Since $\frac{1}{2}\sqrt{\sqrt{33}-3} < \sin 56°$, f_6 is strictly decreasing on [sin 56°, 1), and hence, so is $f_4(r)/f_5(r)$ by Lemma 1.

By Lemma 3(1), $r'(1+r^2)(\operatorname{arth} r)/r^2$ is decreasing on (0, 1). Therefore, f_3 is strictly increasing on [sin 56°, 1). Since $f_3(\sin 64°) = 0.06\ldots > 0$, it follows from (2.15) that f is strictly increasing on [sin 64°, 1) and, hence,

$$f(r) < f(1-) = 0, \quad \text{for } r \in [\sin 64°, 1). \tag{2.16}$$

On the other hand, it is clear that

$$f(r) < f_7(a,b) \equiv \frac{4}{\pi}\mathcal{E}'(b)[\mathcal{E}(b) - (b')^2\mathcal{K}(b)] + \frac{(a')^2 \operatorname{arth} a}{a} - a^2 - 1, \tag{2.17}$$

for $r \in [a, b] \subset (0, 1)$. Computation gives

$f_7(\sin 42°, \sin 46°) = -0.01\ldots$, $f_7(\sin 46°, \sin 50°) = -0.01\ldots$,
$f_7(\sin 50°, \sin 54°) = -0.003\ldots$, $f_7(\sin 54°, \sin 57°) = -0.02\ldots$,
$f_7(\sin 57°, \sin 60°) = -0.01\ldots$, $f_7(\sin 60°, \sin 62°) = -0.04\ldots$,
$f_7(\sin 62°, \sin 64°) = -0.03\ldots$

Hence, it follows from (2.17) that

$$f(r) < 0, \quad \text{for } r \in [\sin 42°, \sin 64°]. \tag{2.18}$$

Finally, the inequality (2.11) follows from (2.14), (2.16) and (2.18).

THEOREM 5 (1) *The function* $f(r) \equiv [(1+r^2)\mathcal{E}(r) - (1+r^4)\mathcal{K}(r)]/r^2$ *is strictly decreasing from* $(0,1)$ *onto* $(-\infty, \pi/4)$, *and has a unique zero* $r_0 \in (\sin 34°, \sin 35°)$.

(2) *The function* $g(r) \equiv r^{-2}\{\mathcal{K}'(r)[\mathcal{E}(r) - (r')^2\mathcal{K}(r)] + \mathcal{E}(r)\mathcal{E}'(r) - \pi/2\}$ *is strictly decreasing from* $(0,1)$ *onto* $(\pi/2, \infty)$.

(3) *The function* $F(r) \equiv (1+r^2)[m(r) + \log r]$ *is strictly decreasing from* $(0,1)$ *onto* $(0, \log 4)$.

(4) *The function* $G(r) = F(r)/(r')^2 - (4/\pi)\mathcal{K}(r)\mathcal{E}'(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(-1, \log 4 - 2)$. *In particular, for all* $r \in (0,1)$,

$$\frac{4}{\pi}\mathcal{K}(r)\mathcal{E}'(r) - 1 < \frac{(1+r^2)}{(r')^2}(m(r) + \log r) < \frac{4}{\pi}\mathcal{K}(r)\mathcal{E}'(r) + \log 4 - 2. \tag{2.19}$$

Proof (1) Since

$$f(r) = \mathcal{E}(r) - r^2\mathcal{K}(r) - \frac{\mathcal{K}(r) - \mathcal{E}(r)}{r^2}$$

it follows from (1.2) and (1.3) that f is decreasing on $(0,1)$ with $f(0+) = \pi/4$ and $f(1-) = -\infty$. Hence, f has a unique zero $r_0 \in (0,1)$. Since $f(\sin 34°) = 0.0001\ldots > 0$ and $f(\sin 35°) = -0.01\ldots < 0$, $r_0 \in (\sin 34°, \sin 35°)$,

(2) Let $g_1(r) \equiv \mathcal{K}'(\mathcal{E} - (r')^2\mathcal{K}) + \mathcal{E}\mathcal{E}' - (\pi/2)$ and $g_2(r) \equiv r^2$. Then

$$g_1'(r)/g_2'(r) = \mathcal{E}(\mathcal{K}' - \mathcal{E}')/(r')^2,$$

which is strictly decreasing on $(0,1)$ by [3, Theorem 2.1(7)], and hence, so is g by Lemma 1. The limiting values are clear.

(3) By differentiation,

$$F'(r)/(2r\mathcal{K}') = F_1(r) \equiv \frac{2}{\pi}f(r) - (\log(1/r))/\mathcal{K}',$$

where f is as in part (1). Since $(\log(1/r))/\mathcal{K}'$ is strictly decreasing on $(0,1)$ by Lemma 4(1),

$$F_1(r) \leq F_2(a,b) \equiv (2f(a)/\pi) + (\log b)/\mathcal{K}'(b),$$

for $r \in (a,b] \subset (0,1)$ by part (1). By computation, we get

$$F_2(0+, \sin 14°) \leq 0.5 + (\log \sin 14°)/2.83275 = -0.00097\ldots,$$
$$F_2(\sin 14°, \sin 21°) = -0.002\ldots, \quad F_2(\sin 21°, \sin 27°) = -0.04\ldots,$$
$$F_2(\sin 27°, \sin 35°) = -0.08\ldots$$

Hence, $F_1(r) < 0$ for $r \in (0, \sin 35°]$. On the other hand, by part (1), $F_1(r) < 0$ for $r \in [\sin 35°, 1)$. Hence, the result follows.

(4) By differentiation,

$$-\frac{\pi(r')^4}{4r}G'(r) = G_1(r) \equiv -\pi \log r - (r')^2\mathcal{K}'\mathcal{E} + (r')^2\frac{2\mathcal{E}\mathcal{E}' - \pi}{2r^2},$$
$$r^3 G_1'(r) = G_2(r) \equiv \pi(r')^2 - 2\mathcal{E}(\mathcal{E}' - r^2\mathcal{K}') - (rr')^2 \times \left(\frac{(\mathcal{K}-\mathcal{E})(\mathcal{E}' - r^2\mathcal{K}')}{r^2} + \mathcal{K}'\mathcal{E}\right),$$
$$\frac{1}{2r}G_2'(r) = G_3(r) \equiv -\pi + 2(\mathcal{K}-\mathcal{E})(\mathcal{E}' - r^2\mathcal{K}') + 2r^2\mathcal{K}'\mathcal{E} + \frac{(r')^2}{r^2}\mathcal{E}'(\mathcal{K}-\mathcal{E}).$$

Using Legendre's relation (2.3) we can write

$$G_3(r) = 2(r')^2\mathcal{K}'(\mathcal{K}-\mathcal{E}) - 2\mathcal{E}(\mathcal{E}' - r^2\mathcal{K}') + \frac{(r')^2}{r^2}\mathcal{E}'(\mathcal{K}-\mathcal{E})$$

and, hence,

$$\frac{G_2'(r)}{2r(r')^2} = G_4(r) \equiv (2r^2\mathcal{K}' + \mathcal{E}')\frac{\mathcal{K}-\mathcal{E}}{r^2} - 2\mathcal{E}\frac{\mathcal{E}' - r^2\mathcal{K}'}{(r')^2},$$

which is strictly increasing from $(0,1)$ onto $(-3\pi/4, \infty)$ by (1.2), (1.3), and Lemma 5(3). Therefore, G_4 has a unique zero $r_0 \in (0,1)$ such that $G_4(r) < 0$ for $r \in (0, r_0)$ and $G_4(r) > 0$ for $r \in (r_0, 1)$. This implies that G_2 is strictly decreasing on $(0, r_0]$ and increasing on $[r_0, 1)$. Since

$G_2(0+) = G_2(1-) = 0,$

$$G_2(r) < 0 \quad \text{for all } r \in (0,1)$$

so that G_1 is strictly decreasing on $(0,1)$. Clearly, $G_1(1) = 0$. Hence $G_1(r) > 0$ for all $r \in (0,1)$, and the monotonicity of G follows. The limiting values of G are clear.

Our next result shows how the bounds for $\varphi_K(r)$ are related to the functions $m(r) + \log r$ and $\mu(r) + \log r$.

THEOREM 6 *Let $a(r)$ and $b(r)$ be functions on $(0,1)$. For each $r \in (0,1)$, define the functions f and g on $[1,\infty)$ by*

$$f(K) = \varphi_K(r)\left(e^{a(r)}/r\right)^{1/K} \quad \text{and} \quad g(K) = \varphi_{1/K}(r)\left(e^{b(r)}/r\right)^{K}. \tag{2.20}$$

Then

(1) *f is decreasing on $[1,\infty)$ iff*

$$m(r) + \log r \le a(r), \quad \text{for all } r \in (0,1), \tag{2.21}$$

(2) *g is decreasing on $[1,\infty)$ iff*

$$m(r) + \log r \ge b(r), \quad \text{for all } r \in (0,1), \tag{2.22}$$

(3) *g is increasing on $[1,\infty)$ iff*

$$\mu(r) + \log r \le b(r), \quad \text{for all } r \in (0,1). \tag{2.23}$$

Proof For (1), put $s = \varphi_K(r)$. By logarithmic differentiation, we get

$$\frac{K^2 f'(K)}{f(K)} = \frac{2}{\pi}(s'\mathcal{K}(s))^2\frac{\mathcal{K}'(r)}{\mathcal{K}(r)} + \log r - a(r),$$

which is strictly decreasing in K on $[1,\infty)$ by Lemma 5(3). From this formula, one can easily see that the result is true.

For (2) and (3), put $u = \varphi_{1/K}(r)$. Then,

$$\frac{g'(K)}{g(K)} = -\left\{\frac{2}{\pi}(u')^2\mathcal{K}(u)^2\frac{\mathcal{K}'(r)}{\mathcal{K}(r)} + \log r\right\} + b(r), \tag{2.24}$$

which is decreasing in K. Hence (2) and (3) follow.

From Theorem 6, the next result follows immediately.

COROLLARY 1 *Let* $a(r)$ *and* $b(r)$ *be functions on* $(0, 1)$. *Then*

(1) *The inequality*

$$\varphi_K(r) < r^{1/K} e^{(1-1/K)a(r)}$$

holds for all $r \in (0, 1)$ *and* $K \in (1, \infty)$, *iff* (2.21) *holds.*

(2) *The inequality*

$$\varphi_{1/K}(r) < r^K e^{(1-K)b(r)}$$

holds for all $r \in (0, 1)$ *and* $K \in (1, \infty)$, *iff* (2.22) *holds.*

(3) *The inequality*

$$\varphi_{1/K}(r) > r^K e^{(1-K)b(r)}$$

holds for all $r \in (0, 1)$ *and* $K \in (1, \infty)$, *iff* (2.23) *holds.*

Proof The "if" parts are clear. We need prove the "only if" part of (1), since the others are similar. Denote $s = \varphi_K(r)$. In (1), taking logarithm, raising to power $K/(K-1)$ and letting $K \to 1$, we get

$$\lim[K \log s - \log r]/(K-1) \leq a(r).$$

By l'Hospital rule and (2.5), we get

$$\begin{aligned} \lim[K \log s - \log r]/(K-1) &= \lim[\log s + K(1/s)(\mathrm{d}s/\mathrm{d}K)] \\ &= \lim[\log s + m(s)] = m(r) + \log r, \end{aligned} \tag{2.25}$$

hence (2.21) follows.

By Corollary 1, in order to obtain the bounds for $\varphi_K(r)$, $K > 0$, $0 < r < 1$, we need only to obtain bounds for $m(r) + \log r$ and $\mu(r) + \log r$.

LEMMA 6 *For* $r \in (0, 1)$, *as a function of* K, $s = s(K) \equiv \varphi_K(r^K)$ *is strictly increasing from* $[1, \infty)$ *onto* $[r, \mu^{-1}(\log(1/r)))$.

Proof Since $\mu(s) = \mu(r^K)/K$, we get making use of (2.5)

$$\frac{\mathrm{d}s}{\mathrm{d}K} = \frac{s(s')^2 \mathcal{K}(s)^2}{(1 - r^{2K})\mathcal{K}(r^K)^2 K^2}\left[m(r^K) + \log r^K\right], \tag{2.26}$$

which is positive by Theorem 5(3), yielding the monotonicity of $s(K)$.

Clearly, $s(1) = r$. By l'Hospital's rule,

$$\lim_{K\to\infty} s(K) = \lim_{K\to\infty} \mu^{-1}(\mu(s)) = \mu^{-1}\left(\lim_{K\to\infty} \frac{\mu(r^K)}{K}\right) = \mu^{-1}(\log(1/r)). \tag{2.27}$$

LEMMA 7 (1) *The function*

$$f(r) \equiv \mu(r) + \frac{\pi^2 \log r}{4(r')^2 \mathcal{K}(r)^2}$$

is strictly decreasing and concave from $(0,1)$ *onto* $(0, \log 4)$. *In particular, for* $r \in (0,1)$,

$$\begin{aligned}
&\frac{1}{r'}\log(1/r) + (1-r)\log 4 \\
&\quad < \frac{\pi^2}{4(r')^2\mathcal{K}(r)^2}\log(1/r) + (1-r)\log 4 \\
&\quad < \mu(r) < \frac{\pi^2}{4(r')^2\mathcal{K}(r)^2}\log(1/r) + \log 4.
\end{aligned} \tag{2.28}$$

(2) *The function* $g(r) \equiv [\mathcal{E}\mathcal{E}' - \mathcal{K}\mathcal{E}' + r^2\mathcal{K}\mathcal{K}']/r^2$ *is strictly decreasing from* $(0,1)$ *onto* $(\pi/2, \infty)$.

Proof (1) Differentiation gives

$$f'(r) = -\frac{\pi^2}{2}\frac{\mathcal{K} - \mathcal{E}}{r^2(r')^2\mathcal{K}^3} \cdot \frac{r\log(1/r)}{(r')^2}.$$

It is easy to verify that $(r\log(1/r))/(r')^2$ is strictly increasing on $(0,1)$ by Lemma 1. Hence, by (1.2), (1.3), and Lemma 5(3), we see that f' is negative and strictly decreasing on $(0,1)$. Therefore the monotonicity and concavity of f follow.

Clearly, $f(1-) = 0$, and the second and third inequalities in (2.28) hold. The first inequality in (2.28) follows from Lemma 5(3). By l'Hospital's rule,

$$\lim_{r\to 0} \frac{1}{r^2}\left[\frac{\pi^2}{4} - (r')^2\mathcal{K}(r)^2\right] = \frac{\pi^2}{8}.$$

Hence

$$f(0+) = \lim_{r\to 0}\left\{[\mu(r)+\log r] + \frac{(\pi^2/4)-(r'\mathcal{K})^2}{r^2}\cdot\frac{r^2\log r}{(r'\mathcal{K})^2}\right\}$$
$$= \log 4,$$

and completes the proof of (1). Part (2) was proved in [18, Theorem 2.1(7)].

LEMMA 8 *Define the functions* f, g *on* $[0,1)$ *by*

$$f(r) = r\mathcal{K}(r)^2/\log((1+r)/(1-r)), f(0) = \pi^2/8,$$
$$g(r) = f(r)/\mathcal{K}(r).$$

Then f *is strictly increasing and* g *is strictly decreasing. In particular, for* $0<r<1$,

$$\left(\frac{\operatorname{arth} r}{r}\right)^{1/2} < \frac{2\mathcal{K}(r)}{\pi} < \frac{\operatorname{arth} r}{r}.$$

Both inequalities are sharp as $r\to 0$. *The second inequality is of the correct order as* $r\to 1$.

Proof This was proved in [3, Theorem 3.10].

3 PROOFS OF MAIN THEOREMS

In this section, we prove the main theorems stated in Section 1.

Proof of Theorem 1 For part (1), put $x = 2\sqrt{r}/(1+r)$. Then $r=(1-x')/(1+x')$ and

$$F(r) = 2\frac{(2/\pi)x\mathcal{K}(x)\mathcal{K}'(x) - x(\operatorname{arth} x')/x'}{\operatorname{arth} x} \equiv F_1(x), \tag{3.1}$$

by (2.2). It is sufficient to prove that F_1 is strictly increasing on $(0,1)$. For this purpose, let $F_2(x)\equiv(2/\pi)x\mathcal{K}\mathcal{K}' - x(\operatorname{arth} x')/x'$ and $F_3(x)\equiv\operatorname{arth} x$. Then $F_2(0+)=F_3(0)=0$ and

$$\frac{F_2'(x)}{F_3'(x)} = F_4(x) \equiv -\frac{2}{\pi}(x')^2\mathcal{K}\mathcal{K}' + \frac{4}{\pi}\mathcal{K}'\mathcal{E} - \frac{\operatorname{arth} x'}{x'}.$$

Put $t = x'$, and let $F_5(t) = F_4(t')$. Then

$$t(t')^2 F_5'(t) = \frac{4}{\pi}\mathcal{E}'(t)[\mathcal{E}(t) - (t')^2\mathcal{K}(t)] + \frac{(t')^2 \operatorname{arth} t}{t} - t^2 - 1,$$

which is negative for all $t \in (0, 1)$, by Lemma 5(4). Hence F_4 is strictly increasing on $(0, 1)$, and so is F_1 by Lemma 1.

The limiting values and the first two inequalities in (1.8) are clear, while the third inequality in (1.8) follows from Lemma 3(1).

For part (2), let $G_1(r) \equiv r[m(r) + \log r]/(r')^2$ and $G_2(r) \equiv \operatorname{arth} r$. Then $G_1(0+) = G_2(0) = 0$ and, by (2.3)

$$\begin{aligned} G_1'(r)/G_2'(r) &= \frac{1}{(r')^2}\left\{(1 + r^2)[m(r) + \log r] - \frac{4}{\pi}(r')^2\mathcal{K}'(\mathcal{K} - \mathcal{E})\right\} \\ &= \frac{(1 + r^2)}{(r')^2}[m(r) + \log r] - \frac{4}{\pi}\mathcal{K}\mathcal{E}' + 2, \end{aligned}$$

which is strictly decreasing on $(0, 1)$ by Theorem 5(4). Hence, the monotonicity of G follows from Lemma 1.

The limiting values $G(0+) = \log 4$, $G(1-) = 1$ and the first two inequalities in (1.9) are clear. The third inequality follows from Lemma 3(1).

Parts (3) and (4) follow from parts (1) and (2), respectively. Finally, part (5) was proved in [2, Lemma 4.2(2)].

COROLLARY 2 *Let* $f(r) \equiv ((r')^2 \operatorname{arth} r)/r$, $0 < r < 1$. *Then, for all* $r \in (0, 1)$,

$$\max\{f(r), f(\sqrt{r})\log 4\} < m(r) + \log r < \min\{2f(\sqrt{r}), f(r)\log 4\}. \tag{3.2}$$

Proof of Theorem 2 The monotoneities of f and g follow from Theorem 6 and Corollary 2.

The limiting values $f(1) = e^{a(r)}$, $\lim_{K\to\infty} f(K) = 1$ and $g(1) = e^{b(r)}$ are clear. By (1.2), (1.6), (1.8) and (1.9)

$$\mu(r) + \log r - b(r) > m(r) + \log r - b(r) > 0, \tag{3.3}$$

and, hence, if we put $u = \varphi_{1/K}(r)$, then

$$\begin{aligned} \lim_{K\to\infty} g(K) &= \lim_{u\to 0}\left\{u(e^{b(r)}/r)^{\mu(u)/\mu(r)}\right\} \\ &= (e^{b(r)}/r)^{(\log 4)/\mu(r)} \lim_{u\to 0} \exp\left\{(\mu(r))^{-1}[\mu(r) + \log r - b(r)]\log u\right\} \\ &= 0. \end{aligned} \tag{3.4}$$

The first inequality in (1.11), the first and third inequalities in (1.12) are clear. Since $a(r) \leq 4^{c(r)}$ and $b(r) \geq c(\sqrt{r}) \log 4$, the second inequality in (1.11) and the third inequality in (1.12) follow from Lemma 3(1) and (2), respectively.

COROLLARY 3 *For* $r \in (0,1)$ *and* $K \in (1,\infty)$,

$$\varphi_K(r)\varphi_K(r') < c^{1-1/K}(rr')^{1/K},$$

where $c = 4^{\sqrt{2}\log(\sqrt{2}+1)} = 5.629087\ldots$

Proof It follows from (1.11) that

$$\varphi_K(r)\varphi_K(r') < 4^{f(r)(1-1/K)}(rr')^{1/K}, \tag{3.5}$$

where

$$f(r) = (r')^2 \frac{\operatorname{arth} r}{r} + r^2 \frac{\operatorname{arth} r'}{r'}. \tag{3.6}$$

It is easy to verify that f is increasing on $(0, 1/\sqrt{2}]$ and decreasing on $[1/\sqrt{2}, 1)$, so that $1 < f(r) \leq f(1/\sqrt{2}) = \sqrt{2}\log(\sqrt{2}+1)$.

The following result is a direct consequence of Theorem 2(1) and the well-known quasiconformal Schwarz lemma [12, Theorem 3.1, p. 64] (see also (1.1)). This consequence is a significant improvement upon the existing explicit quasiconformal Schwarz lemma.

COROLLARY 4 *Suppose that* f *is a* K*-quasiconformal mapping of the unit disk* B *into itself with* $f(0) = 0$. *Then for each* $z \in B$,

$$|f(z)| \leq 4^{(1-|z|^2)^{2/3}(1-1/K)}|z|^{1/K}. \tag{3.7}$$

COROLLARY 5 *For* $r \in (0,1)$ *and* $K \in (1,\infty)$,

$$\varphi_K(r) > 4^{(1-1/K)(1-\varphi_K(r))}r^{1/K} > 4^{(1-1/K)a(r,K)}r^{1/K}, \tag{3.8}$$

$$\varphi_{1/K}(r) > 4^{(1-K)\varphi_K(r')^{4/3}}r^K > 4^{(1-K)b(r,K)}r^K, \tag{3.9}$$

where

$$a(r,K) = (r')^{2K}16^{b(r',K)(1-K)}/2, \qquad b(r,K) = \left\{(r')^{1/K}4^{r^{4/3}(1-1/K)}\right\}^{4/3}.$$

Proof It follows from (1.11) that

$$\varphi_{1/K}(r) > 4^{\varphi_K(r')^{4/3}(1-K)} r^K > 4^{(1-K)b(r,K)} r^K.$$

From (1.12), it follows that

$$\varphi_K(r) > 4^{(1-1/K)(1-\varphi_K(r))} r^{1/K}. \tag{3.10}$$

The second inequality in (3.8) follows from (3.9).

Remark 2 Corollary 5 improves the well-known lower bounds of $\varphi_K(r)$ and $\varphi_{1/K}(r)$ [9, Lemma]:

$$\varphi_K(r) > r^{1/K} \quad \text{and} \quad \varphi_{1/K}(r) > 4^{1-K} r^K, \tag{3.11}$$

for $r \in (0,1)$ and $K \in (1,\infty)$.

Proof of Theorem 3 For $r \in (0,1)$ and $K \in [1,\infty)$, set $s = s(K) \equiv \varphi_K(r^K)$.
(1) From (2.26), we get

$$Kf'(K)/f(K) = f_1(K) \equiv \frac{2}{\pi}\frac{s\mathcal{K}(s)\mathcal{K}'(s)}{\text{arth } s}\left\{1 + \frac{\log(r^K)}{m(r^K)}\right\} - 1. \tag{3.12}$$

By Lemmas 6 and 8, as a function of $K, s\mathcal{K}(s)\mathcal{K}'(s)/\text{arth}\, s$ is strictly decreasing on $[1,\infty)$. By Theorem 1(5), as a function of K, $(\log(r^K))/m(r^K)$ is strictly decreasing from $[1,\infty)$ onto $(-1,(\log r)/m(r)]$. Therefore, f_1 is strictly decreasing on $[1,\infty)$, with

$$f_1(1) = r[m(r) + \log r]/((r')^2 \text{arth } r) - 1,$$

which is positive on $(0,1)$ by Theorem 1(2).
On the other hand, since

$$\varphi_2(r) = 2\sqrt{r}/(1+r), \quad 0 \le r \le 1, \tag{3.13}$$

we have by (2.2),

$$2f_1(2) = [r(m(r^2) + \log(r^2))/((r')^2 \text{arth}\, r)] - 2,$$

which is negative on $(0,1)$ by Theorem 1(1).

Therefore, f_1 has a unique zero $K_1 \in (1,2)$, depending on r, such that $f_1(K) > 0$ for $K \in [1,K_1)$ and $f_1(K) < 0$ for $K \in (K_1,\infty)$, so that the monotoneities of $f|[1,K_1)$ and $f|(K_1,\infty)$ follow from (3.12).

Next, by (3.13), $f(2) = \operatorname{arth} r = f(1)$, while $\lim_{K\to\infty} f(K) = 0$ by Lemma 6.

(2) Logarithmic differentiation gives

$$K^2 F'(K)/F(K)$$

$$= F_1(K) \equiv \frac{s\mathcal{K}(s)^2}{\operatorname{arth} s}\left\{\frac{4}{\pi^2}\mu(r^K) + \frac{\log(r^K)}{(1-r^{2K})\mathcal{K}(r^K)^2}\right\} - \log 4, \qquad (3.14)$$

by (2.26). It follows from Lemmas 6–8 that F_1 is strictly increasing on $[1,\infty)$, with

$$F_1(1) = r[m(r) + \log r]/((r')^2 \operatorname{arth} r) - \log 4,$$

which is negative for all $r \in (0,1)$ by Theorem 1(4).

By (3.13) and (2.2), we have

$$F_1(2) = r[m(r^2) + \log r^2]/((r')^2 \operatorname{arth} r) - \log 4,$$

which is a positive and increasing function of r on $(0,1)$ by Theorem 1(3). Hence, F_1 has a unique zero $K_2 \in (1,2)$, depending on r, such that $F_1(K) < 0$ for $K \in [1, K_2)$ and $F_1(K) > 0$ for $K \in (K_2, \infty)$. This yields the piecewise monotonicity of F by (3.14).

By (3.13) and Lemma 6, $F(2) = (\operatorname{arth} \varphi_2(r^2))/2 = \operatorname{arth} r = F(1)$, and $\lim_{K\to\infty} F(K) = [\operatorname{arth}(\mu^{-1}(\log(1/r)))]/4$.

(3) It follows from part (1) that, for all $r \in (0,1)$,

$$\varphi_K(r)\begin{cases} \geq \operatorname{th}(K \operatorname{arth}(r^{1/K})), & \text{if } 1 \leq K \leq 2, \\ \leq \operatorname{th}(K \operatorname{arth}(r^{1/K})), & \text{if } 2 \leq K < \infty, \end{cases} \qquad (3.15)$$

with equality iff $K = 1$ or 2.

By part (2), for all $r \in (0,1)$,

$$\varphi_K(r)\begin{cases} \leq \operatorname{th}(4^{1-1/K} \operatorname{arth}(r^{1/K})), & \text{if } 1 \leq K \leq 2, \\ \geq \operatorname{th}(4^{1-1/K} \operatorname{arth}(r^{1/K})), & \text{if } 2 \leq K < \infty, \end{cases} \qquad (3.16)$$

with equality iff $K = 1$ or 2.

On the other hand, it is easy to verify that

$$c(K) = \begin{cases} 4^{1-1/K}, & \text{for } 1 \leq K \leq 2, \\ K, & \text{for } 2 < K < \infty, \end{cases} \qquad (3.17)$$

$$d(K) = \begin{cases} K, & \text{for } 1 \le K \le 2, \\ 4^{1-1/K}, & \text{for } 2 < K < \infty, \end{cases} \tag{3.18}$$

Now the assertion (3) follows from (3.15)–(3.18), as desired.

Remark 3 In [22, Theorem 2.23], it was proved that, for $K=2^p$, $p=1,2,\dots$ and $r \in [0,1]$,

$$\varphi_K(r) \le \text{th}(K\text{arth}(r^{1/K})). \tag{3.19}$$

Theorem 3 shows that (3.19) holds for all $K \ge 2$ and $r \in [0,1]$, and that it is reversed if $K \in [1,2)$.

COROLLARY 6 *For $r \in (0,1)$ and $K \in [1,\infty)$,*

$$(\text{th}(d_1(K)\text{arth } r))^K \le \varphi_{1/K}(r) \le (\text{th}(c_1(K)\text{arth } r))^K, \tag{3.20}$$

where

$$d_1(K) = \min\{1/K, 4^{(1/K)-1}\}, \quad c_1(K) = \max\{1/K, 4^{(1/K)-1}\}. \tag{3.21}$$

Equality holds iff $K=1$ or $K=2$.

Proof Let $u = \varphi_{1/K}(r)$. Then, by Theorem 3(3),

$$\text{th}(d(K)\text{arth}(u^{1/K})) \le r \le \text{th}(c(K)\text{arth}(u^{1/K})), \tag{3.22}$$

with equality iff $K=1$ or $K=2$, from which the result follows.

COROLLARY 7 *Suppose that f is a K-quasiconformal mapping of the unit disk B onto itself with $f(0)=0$. Then for each $z \in B$,*

$$\varphi_{1/K}(|z|) \le |f(z)| \le \varphi_K(|z|), \tag{3.23}$$

$$\{\text{th}(d_1(K)\text{arth } |z|)\}^K \le |f(z)| \le \text{th}(c(K)\text{arth}(|z|^{1/K})), \tag{3.24}$$

where $c(K)$ and $d_1(K)$ are as in Theorem 3(3) *and Corollary* 6, *respectively.*

Proof These follow from Theorem 3(3), Corollary 6 and the well-known quasiconformal Schwarz lemma [12, Theorem 3.1, p. 64].

Proof of Theorem 4 Since

$$\varphi_{1/2}(r) = (1-r')/(1+r') = A(r)^2, \quad 0 \le r \le 1, \tag{3.25}$$

$\varphi_K(r) = \varphi_{2K}(\varphi_{1/2}(r)) = \varphi_{2K}(A(r)^2)$ and the assertion (1) follows from (3.15) and (3.16).

For (2), set $r = A(t)^2$ and $u = \varphi_{1/K}(t)$. Then $t = \varphi_2(r) = 2\sqrt{r}/(1+r)$, and

$$\varphi_{1/K}(r) = \varphi_{1/K}(A(t)^2) = A(\varphi_{1/K}(t))^2 = A(u)^2.$$

Hence, by part (1) and (3.3),

$$\begin{aligned}\varphi_{1/K}(r) = A(u)^2 &\le \left\{\mathrm{th}\left(2^{(1/K)-2}\mathrm{arth}\ \varphi_K(u)\right)\right\}^{2K}\\ &= \left\{\mathrm{th}\left(2^{(1/K)-2}\mathrm{arth}\ t\right)\right\}^{2K}\\ &= \left\{\mathrm{th}\left(2^{(1/K)-1}\mathrm{arth}(\sqrt{r})\right)\right\}^{2K},\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\varphi_{1/K}(r) = A(u)^2 &\ge \left\{\mathrm{th}\left(\frac{1}{2K}\mathrm{arth}\varphi_K(u)\right)\right\}^{2K}\\ &= \left\{\mathrm{th}\left(\frac{1}{2K}\mathrm{arth}\ t\right)\right\}^{2K}\\ &= \left\{\mathrm{th}\left(\frac{1}{K}\mathrm{arth}(\sqrt{r})\right)\right\}^{2K},\end{aligned} \tag{3.27}$$

so that the assertion (2) follows.

3.1 Generalized Modular Equations

For a real number $\sigma \in (0,1)$ define the *generalized μ-function* or the *μ-function with signature σ* by

$$\mu_\sigma(r) = \frac{\pi}{2\sin(\pi\sigma)}\frac{F(\sigma, 1-\sigma; 1; 1-r^2)}{F(\sigma, 1-\sigma; 1; r^2)}, \quad r \in (0,1), \tag{3.28}$$

where F denotes the classical hypergeometric series [25, 2.38, p. 24]. Then $\mu_\sigma : (0,1) \to (0,\infty)$ is a homeomorphism. Since $\mu_\sigma = \mu_{1-\sigma}$ we may assume that $\sigma \in \left(0, \frac{1}{2}\right]$; note that $\mu_{1/2}(r) = \mu(r)$. The *generalized modular equation* or *modular equation of signature* σ and *degree* p is the equation

$$\mu_\sigma(s) = p\mu_\sigma(r). \tag{3.29}$$

For the particular case $\sigma = \frac{1}{2}$ we get the classical modular equations considered in [5,7]. The general case was first considered by Ramanujan in his notebooks. Obviously, the solution of (3.25) is given by

$$s = \mu_\sigma^{-1}(p\mu_\sigma(r)) \equiv \varphi_{1/p}^\sigma(r). \tag{3.30}$$

As far as we know there are no estimates for the function $\varphi_{1/p}^\sigma(r)$ defined in (3.30). It is a very interesting open problem to extend the inequalities for $\varphi_K(r)$ in this paper to the case of $\varphi_{1/p}^\sigma(r)$, $\sigma \in \left(0, \frac{1}{2}\right)$.

3.2 Ramanujan's α, β-Notation

Ramanujan has derived dozens of algebraic identities satisfied by the solutions of modular equations of prime degree for several small prime numbers p (and $\sigma = \frac{1}{2}$) He uses the notation $\alpha = r^2$, $\beta = \varphi_{1/p}(r)^2$ and proves for the solution of the classical degree 7 modular equations, e.g.

$$(\alpha\beta)^{1/8} + ((1-\alpha)(1-\beta))^{1/8} = 1, \quad \alpha = r^2, \ \beta = \varphi_{1/7}(r)^2$$

for $r \in (0,1)$ [5, p. 314, Entry 19(i)]. Berndt *et al.* [6, Theorem 7.1(i)] recently proved for the solution of the generalized modular equation of degree 2 with $\sigma = \frac{1}{3}$ that

$$(\alpha\beta)^{1/3} + ((1-\alpha)(1-\beta))^{1/3} = 1, \quad \alpha = r^2, \ \beta = (\varphi_{1/2}^{1/3}(r))^2$$

for $r \in (0,1)$ along with many similar results, stated originally by Ramanujan without proofs in his notebooks.

OPEN PROBLEM *For $K > 1$ and $r \in (0,1)$ let*

$$g(K,r) = \frac{\operatorname{arth} \varphi_K(r)}{\operatorname{arth}(r^{1/K})}.$$

Is it true that $g(K, r)$, *as a function of* r, *is decreasing onto* $(K, 4^{1-1/K})$ *when* $1 < K < 2$ *and increasing onto* $(4^{1-1/K}, K)$ *when* $K > 2$?

An affirmative answer to this problem would provide another proof of Theorem 3(3).

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