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Some Generalized Poincaré Inequalities and Applications to Problems Arising in Electromagnetism

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Two Poincaré type theorems for sufficiently regular fields are obtained. In particular, we prove that their $L^2(\Omega)$-norm can be controlled by the $L^2(\Omega)$-norms of their curl and divergence and the $L^2(\partial\Omega)$-norm of their tangential (or normal) component on the boundary. Finally, some applications of these results are given in the context of the electromagnetic theory.

Keywords: Modified Poincaré inequalities; Boundary value problems; Maxwell's equations

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1 INTRODUCTION

In this paper the Poincaré theorem for solenoidal fields is generalized to the case of general boundary conditions. It is, in fact, well known that if $\Omega \subset R^3$ is a sufficiently regular domain, then the following theorem [3,8] holds:

THEOREM 1.1 *Let $\mathbf{v} \in H^1(\Omega)$ such that $\nabla \cdot \mathbf{v} = 0$ and $\mathbf{v} \times \mathbf{n}|_{\partial\Omega} = 0$, then it satisfies the following inequality:*

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq c_1 \|\nabla \times \mathbf{v}\|_{L^2(\Omega)}, \tag{1}$$

where c_1 is a positive constant.

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Generally, Poincaré type theorems require the vanishing of at least one component of the field on the boundary. In fact, besides Theorem 1.1, it exists an analogous version for solenoidal fields with a null normal component on the boundary [3,8]:

THEOREM 1.2 *Let* $\mathbf{v} \in H^1(\Omega)$ *such that* $\nabla \cdot \mathbf{v} = 0$ *and* $\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$, *then it satisfies the following inequality:*

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq c_2 \|\nabla \times \mathbf{v}\|_{L^2(\Omega)}, \tag{2}$$

where c_2 *is a positive constant.*

The previous two theorems are strictly related to questions arising in mathematical physics. For example, if we study the behavior of a linear electromagnetic conductor, occupying a (regular) domain $\Omega \subset R^3$ with a boundary realized by a perfect conductor (that is on the boundary the tangential component of the electric field and the normal component of the magnetic field vanish), they are very important tools in order to prove existence, uniqueness and stability theorems.

In these last years, many works [6,9] in the context of the electromagnetic theory have been dedicated to the case of domains whose boundaries are realized by "good" but not perfect conductors. This physical situation can be well described by linking the tangential component of the magnetic field to the same component of the electric field. Since the tangential component of the electric (or magnetic) field does not vanish on the boundary and nothing is known about the normal components, Theorem 1.1 or 1.2 cannot be more used.

As said before, the aim of this paper is to obtain an extension of Theorems 1.1 and 1.2, applicable to sufficiently regular fields with a non-vanishing tangential (and normal) component on the boundary. More precisely, by making use of the orthogonal Hodge decompositions for the space $L^2(\Omega)$, we will prove that the $L^2(\Omega)$-norm of a field can be controlled by the $L^2(\Omega)$-norms of its curl and divergence and by the $L^2(\partial\Omega)$-norm of its tangential (or normal) component on the boundary.

These generalizations are the subjects, respectively, of Section 2 and Section 3. Finally, in Section 4, some applications of the previous results to problems arising in the electromagnetic theory are presented. More precisely, we shall study the quasi-static evolution of a dielectric when the boundary of the domain is dissipative and the time-harmonic evolution of a linear conductor with a conservative boundary. In both cases, we obtain an existence and uniqueness theorem as a consequence of the new theorems.

2 FIRST THEOREM

From now on with a "sufficiently regular" domain Ω, we intend a bounded domain of R^3 which satisfies the following conditions:

(1) Ω is simply connected, i.e. such that every continuous closed curve (entirely contained in Ω) can be deformed continuously until it has shrunk to a point;
(2) its complementar $\Omega' = R^3 \setminus \bar{\Omega}$ is connected;
(3) it admits a bounded C^2-boundary $\partial\Omega$ and Ω is situated, locally, on one side of $\partial\Omega$.

In order to obtain the above mentioned generalization of Theorems 1.1 and 1.2, we introduce the functional spaces

$$\begin{aligned}
\mathcal{D}(\Omega) &= \{\mathbf{v} \in L^2(\Omega),\ \nabla \cdot \mathbf{v} \in L^2(\Omega)\},\\
\mathcal{R}(\Omega) &= \{\mathbf{v} \in L^2(\Omega), \nabla \times \mathbf{v} \in L^2(\Omega)\},\\
\mathcal{D}^0(\Omega) &= \{\mathbf{v} \in L^2(\Omega),\ \nabla \cdot \mathbf{v} = 0\},\\
\mathcal{D}_{n0}(\Omega) &= \{\mathbf{v} \in \mathcal{D}^0(\Omega),\ \mathbf{v} \cdot \mathbf{n} = 0\},\\
\mathcal{D}_{t0}(\Omega) &= \{\mathbf{v} \in \mathcal{D}^0(\Omega),\ \mathbf{v} \times \mathbf{n} = \mathbf{0}\},\\
\mathcal{H}(\Omega) &= \{\mathbf{v} \in \mathcal{R}(\Omega) \cap \mathcal{D}(\Omega),\ \mathbf{v} \times \mathbf{n} \in L^2(\partial\Omega)\},\\
\mathcal{H}^0(\Omega) &= \{\mathbf{v} \in \mathcal{R}(\Omega) \cap \mathcal{D}^0(\Omega),\ \mathbf{v} \times \mathbf{n} \in L^2(\partial\Omega)\},\\
\mathcal{K}(\Omega) &= \{\mathbf{v} \in \mathcal{R}(\Omega) \cap \mathcal{D}(\Omega),\ \mathbf{v} \cdot \mathbf{n} \in L^2(\partial\Omega)\},\\
\mathcal{K}^0(\Omega) &= \{\mathbf{v} \in \mathcal{R}(\Omega) \cap \mathcal{D}^0(\Omega),\ \mathbf{v} \cdot \mathbf{n} \in L^2(\partial\Omega)\}
\end{aligned}$$

and recall the following [5]:

LEMMA 2.1 *Let Ω be a sufficiently regular domain. The space $L^2(\Omega)$ admits the following orthogonal decomposition:*

$$L^2(\Omega) = \mathrm{curl}\big(H^1(\Omega) \cap \mathcal{D}_{n0}(\Omega)\big) \oplus \mathrm{grad}\big(H_0^1(\Omega)\big).$$

The theorem we are looking for can be expressed as follows:

THEOREM 2.1 *Let Ω be a "sufficiently regular" domain and* $\mathbf{u} \in \mathcal{H}(\Omega)$. *Then it exists a positive constant k such that*

$$\|\mathbf{u}\|_{L^2(\Omega)} \leq k\Big[\|\nabla \times \mathbf{u}\|_{L^2(\Omega)} + \|\nabla \cdot \mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u} \times \mathbf{n}\|_{L^2(\partial\Omega)}\Big]. \tag{3}$$

Proof Since $\mathbf{u} \in \mathcal{H}(\Omega)$, Lemma 2.1 guarantees that there exist a vector $\mathbf{w} \in H^1(\Omega) \cap \mathcal{D}_{n0}(\Omega)$ and $\phi \in H_0^1(\Omega)$, such that

$$\mathbf{u} = \nabla \times \mathbf{w} + \nabla\phi. \tag{4}$$

From Green's formulas, it follows immediately that

$$\begin{aligned}\int_\Omega |\mathbf{u}(\mathbf{x})|^2 \mathrm{d}\mathbf{x} &= \int_\Omega [\nabla \times \mathbf{w}(\mathbf{x}) + \nabla\phi(\mathbf{x})] \cdot \mathbf{u}(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= \int_\Omega [\nabla \times \mathbf{u}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) - \phi(\mathbf{x})\nabla \cdot \mathbf{u}(\mathbf{x})]\, \mathrm{d}\mathbf{x} \\ &\quad + \int_{\partial\Omega} [\mathbf{w}(\sigma) \times \mathbf{u}(\sigma) + \phi(\sigma)\mathbf{u}(\sigma)] \cdot \mathbf{n}(\sigma)\, \mathrm{d}\sigma. \end{aligned} \tag{5}$$

Let us put

$$I_\Omega = \int_\Omega [\nabla \times \mathbf{u}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) - \phi(\mathbf{x})\nabla \cdot \mathbf{u}(\mathbf{x})]\, \mathrm{d}\mathbf{x}$$

and

$$I_{\partial\Omega} = \int_{\partial\Omega} [\mathbf{w}(\sigma) \times \mathbf{u}(\sigma) + \phi(\sigma)\mathbf{u}(\sigma)] \cdot \mathbf{n}(\sigma)\, \mathrm{d}\sigma.$$

An application of the Schwarz inequality, due to the fact that $\mathbf{u} \in \mathcal{H}(\Omega)$ and therefore $\nabla \times \mathbf{u}$ and $\nabla \cdot \mathbf{u}$ belong to $L^2(\Omega)$, yields to

$$\begin{aligned} I_\Omega \leq& \left[\int_\Omega |\mathbf{w}(\mathbf{x})|^2 \,\mathrm{d}x\right]^{1/2} \left[\int_\Omega |\nabla \times \mathbf{u}(\mathbf{x})|^2 \,\mathrm{d}x\right]^{1/2} \\ &+ \left[\int_\Omega |\phi(\mathbf{x})|^2 \,\mathrm{d}x\right]^{1/2} \left[\int_\Omega |\nabla \cdot \mathbf{u}(\mathbf{x})|^2 \,\mathrm{d}x\right]^{1/2}. \end{aligned} \tag{6}$$

Now, since $\mathbf{w} \in H^1(\Omega) \cap \mathcal{D}_{n0}$, from Theorem 1.2 we have

$$\|\mathbf{w}\|_{L^2(\Omega)} \leq \|\mathbf{w}\|_{H^1(\Omega)} \leq c_2 \|\nabla \times \mathbf{w}\|_{L^2(\Omega)}, \tag{7}$$

while $\phi \in H_0^1(\Omega)$ and the classical Poincaré inequality guarantees that

$$\|\phi\|_{L^2(\Omega)} \leq c_3 \|\nabla \phi\|_{L^2(\Omega)}. \tag{8}$$

So, we can rewrite (6) as follows:

$$I_\Omega \leq c_2 \|\nabla \times \mathbf{w}\|_{L^2(\Omega)} \|\nabla \times \mathbf{u}\|_{L^2(\Omega)} + c_3 \|\nabla \phi\|_{L^2(\Omega)} \|\nabla \cdot \mathbf{u}\|_{L^2(\Omega)}. \tag{9}$$

Turning now our attention to the boundary integral $I_{\partial\Omega}$ appearing in (5), we first observe that the occurence of the unit outward normal $\mathbf{n}$ makes $\mathbf{w}$ and $\mathbf{u}$ contribute only through their tangential parts $\mathbf{w}_\tau$ and $\mathbf{u}_\tau$ and then recall that $\phi \in H_0^1(\Omega)$. Hence we can write

$$I_{\partial\Omega} = \int_{\partial\Omega} \mathbf{w}_\tau(\sigma) \times \mathbf{u}_\tau(\sigma) \cdot \mathbf{n}(\sigma) \, \mathrm{d}\sigma.$$

Besides, $\mathbf{w}$ is in $H^1(\Omega)$; therefore its tangential component to the boundary $\partial\Omega$ belongs to $H^{1/2}(\partial\Omega)$ and satisfies the "trace inequality" [1,4]:

$$\|\mathbf{w}_\tau\|_{H^{1/2}(\partial\Omega)} \leq c_4 \|\mathbf{w}\|_{H^1(\Omega)}. \tag{10}$$

A further application of the Schwarz inequality, together with (10), yields to the following estimate:

$$I_{\partial\Omega} \leq c_4 \|\mathbf{w}\|_{H^1(\Omega)} \|\mathbf{u} \times \mathbf{n}\|_{L^2(\partial\Omega)}, \tag{11}$$

while relation (7) allows us to write

$$I_{\partial\Omega} \leq c_5 \|\nabla \times \mathbf{w}\|_{L^2(\Omega)} \|\mathbf{u} \times \mathbf{n}\|_{L^2(\partial\Omega)}. \tag{12}$$

Finally, we are able to conclude our proof; in fact, if we put (9) and (12) in (5), then we have

$$\begin{aligned}
\|\mathbf{u}\|^2_{L^2(\Omega)} &\leq c_6\|\nabla\times\mathbf{w}\|_{L^2(\Omega)}\Big(\|\nabla\times\mathbf{u}\|_{L^2(\Omega)}+\|\mathbf{u}\times\mathbf{n}\|_{L^2(\partial\Omega)}\Big)\\
&\quad + c_3\|\nabla\phi\|_{L^2(\Omega)}\|\nabla\cdot\mathbf{u}\|_{L^2(\Omega)}\\
&\leq \frac{1}{2}\Big(\|\nabla\times\mathbf{w}\|^2_{L^2(\Omega)}+\|\nabla\phi\|^2_{L^2(\Omega)}\Big)\\
&\quad + k_1\Big(\|\nabla\times\mathbf{u}\|^2_{L^2(\Omega)}+\|\mathbf{u}\times\mathbf{n}\|^2_{L^2(\partial\Omega)}+\|\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)}\Big)\\
&= \frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Omega)}+k_1\Big(\|\nabla\times\mathbf{u}\|^2_{L^2(\Omega)}+\|\mathbf{u}\times\mathbf{n}\|^2_{L^2(\partial\Omega)}+\|\nabla\cdot\mathbf{u}\|^2_{L^2(\Omega)}\Big),
\end{aligned} \tag{13}$$

that is our thesis.

Two direct consequences of the previous result are the following:

PROPOSITION 2.1 *Let Ω be a sufficiently regular domain and $\mathbf{u}\in\mathcal{H}^0(\Omega)$. Then we have*

$$\|\mathbf{u}\|_{L^2(\Omega)}\leq h_1\Big(\|\nabla\times\mathbf{u}\|_{L^2(\Omega)}+\|\mathbf{u}\times\mathbf{n}\|_{L^2(\partial\Omega)}\Big) \tag{14}$$

with h_1 a positive constant.

PROPOSITION 2.2 *Let Ω be a sufficiently regular domain and $\mathbf{u}\in H^1(\Omega)$. Then we have*

$$\|\mathbf{u}\|_{H^1(\Omega)}\leq h_2\Big(\|\nabla\times\mathbf{u}\|_{L^2(\Omega)}+\|\nabla\cdot\mathbf{u}\|_{L^2(\Omega)}+\|\mathbf{u}\times\mathbf{n}\|_{H^{1/2}(\partial\Omega)}\Big) \tag{15}$$

with h_2 a positive constant.

Proof Inequality (15) follows from the fact that the hypothesis of regularity on the domain Ω allows us to identify the Sobolev space $H^1(\Omega)$ with the space

$$\{\mathbf{u}\in\mathcal{D}(\Omega)\cap\mathcal{R}(\Omega);\mathbf{u}\times\mathbf{n}_{|\partial\Omega}\in H^{1/2}(\partial\Omega)\}$$

and that it exists a positive constant α_1 such that

$$\|\mathbf{u}\|_{H^1(\Omega)}\leq\alpha_1\Big(\|\mathbf{u}\|_{L^2(\Omega)}+\|\nabla\times\mathbf{u}\|_{L^2(\Omega)}+\|\nabla\cdot\mathbf{u}\|_{L^2(\Omega)}+\|\mathbf{u}\times\mathbf{n}\|_{H^{1/2}(\partial\Omega)}\Big) \tag{16}$$

for any $\mathbf{u}\in H^1(\Omega)$ (see [5], Cor. 1, p. 212).

Now, if the vector $\mathbf{u}$ belongs to $H^1(\Omega)$, by Theorem 2.1 its $L^2(\Omega)$-norm can be controlled by the $L^2(\Omega)$-norms of its curl and divergence and by the $L^2(\partial\Omega)$-norm of its tangential component on the boundary, so that the inequality (16) assumes the desired form (15).

3 SECOND THEOREM

In this section we are interested in finding an analogous of Theorem 2.1 which involves the normal component on the boundary. The first step consists in considering another Hodge decomposition of the space $L^2(\Omega)$:

LEMMA 3.1 *Let Ω be a "sufficiently regular" domain. The space $L^2(\Omega)$ admits the following orthogonal decomposition:*

$$L^2(\Omega) = \operatorname{curl}\big(H^1(\Omega) \cap \mathcal{D}_{t0}(\Omega)\big) \oplus \operatorname{grad}\big(H^1(\Omega)\big).$$

If we choose $\mathbf{u} \in \mathcal{K}(\Omega)$, then from the above decomposition there exist $p \in H^1(\Omega)$, unique to within an additive constant, and $\mathbf{w} \in H^1(\Omega) \cap \mathcal{D}_{t0}(\Omega)$, such that

$$\mathbf{u} = \nabla \times \mathbf{w} + \nabla p.$$

Moreover, $p \in H^1(\Omega)$ is a solution of the problem†

$$\begin{cases} \Delta p = \nabla \cdot \mathbf{u}, \\ \nabla p \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} \end{cases} \tag{17}$$

and this implies [10,12] the existence of a positive constant β such that

$$\|\nabla p\|_{L^2(\Omega)} \leq \beta\Big(\|\nabla \cdot \mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u} \cdot \mathbf{n}\|_{L^2(\partial\Omega)}\Big). \tag{18}$$

On the other hand, $\nabla \times \mathbf{w} \in \mathcal{R}(\Omega) \cap \mathcal{D}_{n0}(\Omega)$ and it satisfies the identity:

$$\nabla \times (\nabla \times \mathbf{w}) = \nabla \times \mathbf{u}.$$

† The boundary condition is a consequence of the fact [8] that the curl operator establishes a one-to-one correspondence between the spaces $H^1(\Omega) \cap \mathcal{D}_{t0}(\Omega)$ and $\mathcal{D}_{n0}(\Omega)$ and therefore $\mathbf{w} \in \mathcal{D}_{t0}(\Omega)$ implies that $\nabla \times \mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0$.

It follows that $\nabla \times \mathbf{w}$ fulfills the hypotheses of Theorem 1.2 and thanks to (2) the next inequality holds:

$$\|\nabla \times \mathbf{w}\|_{L^2(\Omega)} \leq \|\nabla \times \mathbf{w}\|_{H^1(\Omega)} \leq c_2 \|\nabla \times \mathbf{u}\|_{L^2(\Omega)}. \tag{19}$$

As a direct consequence of the inequalities (18) and (19), we are now able to state the theorem:

THEOREM 3.1 *Let Ω be a "sufficiently regular" domain, then, for any vector $\mathbf{u} \in \mathcal{K}(\Omega)$ it exists a positive constant γ such that*

$$\|\mathbf{u}\|_{L^2(\Omega)} \leq \gamma\Big(\|\nabla \cdot \mathbf{u}\|_{L^2(\Omega)} + \|\nabla \times \mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u} \cdot \mathbf{n}\|_{L^2(\partial\Omega)}\Big). \tag{20}$$

We can also deduce from Theorem 3.1 the following two corollaries.

PROPOSITION 3.1 *Let Ω be a sufficiently regular domain and $\mathbf{u} \in \mathcal{K}^0(\Omega)$. Then we have*

$$\|\mathbf{u}\|_{L^2(\Omega)} \leq h_3\Big(\|\nabla \times \mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u} \cdot \mathbf{n}\|_{L^2(\partial\Omega)}\Big) \tag{21}$$

with h_3 a positive constant.

PROPOSITION 3.2 *Let Ω be a sufficiently regular domain. Then the inequality*

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq h_4\Big(\|\nabla \times \mathbf{u}\|_{L^2(\Omega)} + \|\nabla \cdot \mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u} \cdot \mathbf{n}\|_{H^{1/2}(\partial\Omega)}\Big) \tag{22}$$

holds for any $\mathbf{u} \in H^1(\Omega)$, where h_4 is a positive constant.

Proof The proof is very similar to the one of Proposition 2.2. In this case we identify (see [5], Cor. 1, p. 212) the space $H^1(\Omega)$ with the space

$$\{\mathbf{u} \in \mathcal{D}(\Omega) \cap \mathcal{R}(\Omega); \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} \in H^{1/2}(\partial\Omega)\}.$$

Moreover, it exists a positive constant α_2 such that

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq \alpha_2\Big(\|\mathbf{u}\|_{L^2(\Omega)} + \|\nabla \times \mathbf{u}\|_{L^2(\Omega)} + \|\nabla \cdot \mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u} \cdot \mathbf{n}\|_{H^{1/2}(\partial\Omega)}\Big) \tag{23}$$

for any $\mathbf{u} \in H^1(\Omega)$.

Now, if we take into account Theorem 3.1, we get the thesis.

4 APPLICATIONS

As already observed in the introduction, Poincaré type theorems are strictly related to the theory of electromagnetism. It seems therefore natural to apply our new results to some problems arising in this context.

Let us consider a "sufficiently regular" domain Ω; the evolution of the electromagnetic field in $\Omega \times (0, T)$ is governed by the well-known Maxwell equations:

$$\frac{\partial}{\partial t}\mathbf{D}(\mathbf{x},t) - \nabla \times \mathbf{H}(\mathbf{x},t) = -\mathbf{J}(\mathbf{x},t) \quad \nabla \cdot \mathbf{D}(\mathbf{x},t) = \rho(\mathbf{x},t), \tag{24}$$

$$\frac{\partial}{\partial t}\mathbf{B}(\mathbf{x},t) + \nabla \times \mathbf{E}(\mathbf{x},t) = \mathbf{G}_f(\mathbf{x},t) \quad \nabla \cdot \mathbf{B}(\mathbf{x},t) = 0. \tag{25}$$

If we ask the material to be linear, isotropic and homogeneous, then the electric and magnetic fields $\mathbf{E}$, $\mathbf{H}$ are linked to the electric and magnetic inductions $\mathbf{D}$, $\mathbf{B}$ by the constitutive equations:

$$\mathbf{D}(\mathbf{x},t) = \epsilon\mathbf{E}(\mathbf{x},t), \quad \mathbf{B}(\mathbf{x},t) = \mu\mathbf{H}(\mathbf{x},t), \tag{26}$$

where ϵ, μ are positive constants, while we assume the charge density ρ equal to zero and the current density $\mathbf{J}$ as the sum of an unknown part $\mathbf{J}_c$ and an impressed one $\mathbf{J}_f$. The vector $\mathbf{G}_f$ is usually set equal to zero, which means that magnetic currents do not occur in nature; however, we let for a non zero $\mathbf{G}_f$, because it might represent a forced electric displacement current.

In this case the Maxwell equations assume the following form:

$$\epsilon\frac{\partial}{\partial t}\mathbf{E}(\mathbf{x},t) - \nabla \times \mathbf{H}(\mathbf{x},t) + \mathbf{J}_c(\mathbf{x},t) = -\mathbf{J}_f(\mathbf{x},t) \quad \nabla \cdot \mathbf{E}(\mathbf{x},t) = 0, \tag{27}$$

$$\mu\frac{\partial}{\partial t}\mathbf{H}(\mathbf{x},t) + \nabla \times \mathbf{E}(\mathbf{x},t) = \mathbf{G}_f(\mathbf{x},t) \quad \nabla \cdot \mathbf{H}(\mathbf{x},t) = 0. \tag{28}$$

4.1 A Quasi-Static Problem

We are interested in the quasi-static approximation of the system (27) and (28) on $\Omega \times (-\infty, +\infty)$ when the material is a dielectric, i.e. when $\mathbf{J}_c = \mathbf{0}$; therefore, since $\mathbf{E}$ and $\mathbf{H}$ are slowly-varying, we can omit their time-derivatives. The problem we obtain in this way consists in finding a pair $(\mathbf{E}, \mathbf{H})$ of solenoidal fields such that

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{x}, t) = \mathbf{G}_f(\mathbf{x}, t) \\ \nabla \times \mathbf{H}(\mathbf{x}, t) = \mathbf{J}_f(\mathbf{x}, t), \end{cases} \tag{29}$$

where the density currents $\mathbf{J}_f$, $\mathbf{G}_f$ belong to $L^2(-\infty, +\infty; \mathcal{D}^0(\Omega))$. Furthermore we assume the boundary of the domain Ω dissipative, so that the boundary condition has the form [7]:

$$\mathbf{E}_\tau(\mathbf{x}, t) = \lambda \mathbf{H}(\mathbf{x}, t) \times \mathbf{n}(\mathbf{x}) \quad \text{on } \partial\Omega, \tag{30}$$

where, as usual, the subscript τ denotes the tangential component to $\partial\Omega$ and λ is a positive constant.

Definition 4.1 *A pair* $(\mathbf{E}, \mathbf{H}) \in (L^2(-\infty, +\infty; \mathcal{D}^0(\Omega)))^2$ *is a weak solution for problem* (29) *and* (30) *with sources* $\mathbf{J}_f$, $\mathbf{G}_f \in L^2(-\infty, +\infty; \mathcal{D}^0(\Omega))$ *if the identity*

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_\Omega \{\mathbf{E}(\mathbf{x}, t) \cdot \nabla \times \mathbf{h}(\mathbf{x}, t) + \mathbf{H}(\mathbf{x}, t) \cdot \nabla \times \mathbf{e}(\mathbf{x}, t)\} \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = \int_{-\infty}^{\infty} \int_\Omega \{\mathbf{J}_f(\mathbf{x}, t) \cdot \mathbf{e}(\mathbf{x}, t) + \mathbf{G}_f(\mathbf{x}, t) \cdot \mathbf{h}(\mathbf{x}, t)\} \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{31}$$

holds for any pair $(\mathbf{e}, \mathbf{h}) \in H(Q) = \{(\mathbf{e}, \mathbf{h}) \in (L^2(-\infty, \infty; \mathcal{H}^0(\Omega)))^2;\ \mathbf{e}_\tau = \lambda \mathbf{h} \times \mathbf{n}$ *on* $\partial\Omega\}$.

Definition 4.2 *A pair* $(\mathbf{E}, \mathbf{H}) \in H(Q)$ *is a strong solution for problem* (29) *and* (30) *with sources* $\mathbf{J}_f$, $\mathbf{G}_f \in L^2(-\infty, +\infty; \mathcal{D}^0(\Omega))$ *if it satisfies* (29) *almost everywhere.*

It is easy to show that a weak solution for problem (29) and (30) is also a strong one. In fact, let $(\mathbf{E}, \mathbf{H}) \in (L^2(-\infty, +\infty; \mathcal{D}^0(\Omega)))^2$ be a weak solution and take the pairs $(\mathbf{e}_1, \mathbf{h}_1)$ with $\mathbf{e}_1 \in L^2(-\infty, \infty; H_0^1(\Omega))$, $\mathbf{h}_1 = \mathbf{0}$ and $(\mathbf{e}_2, \mathbf{h}_2)$ with $\mathbf{e}_2 = 0$, $h_2 \in L^2(-\infty, \infty; H_0^1(\Omega))$. Then $(\mathbf{e}_i, \mathbf{h}_i)$ belongs to $H(Q)$ for $i = 1, 2$ and applying (31) we have

$$\begin{aligned} \int_{-\infty}^{\infty} \int_\Omega \mathbf{H}(\mathbf{x}, t) \cdot \nabla \times \mathbf{e}_1(\mathbf{x}, t) \, \mathrm{d}x \, \mathrm{d}t &= \int_{-\infty}^{\infty} \int_\Omega \mathbf{J}_f(\mathbf{x}, t) \cdot \mathbf{e}_1(\mathbf{x}, t) \, \mathrm{d}x \, \mathrm{d}t, \\ \int_{-\infty}^{\infty} \int_\Omega \mathbf{E}(\mathbf{x}, t) \cdot \nabla \times \mathbf{h}_2(\mathbf{x}, t) \, \mathrm{d}x \, \mathrm{d}t &= \int_{-\infty}^{\infty} \int_\Omega \mathbf{G}_f(\mathbf{x}, t) \cdot \mathbf{h}_2(\mathbf{x}, t) \, \mathrm{d}x \, \mathrm{d}t, \end{aligned}$$

that is $(\mathbf{E},\mathbf{H})$ belongs to $(L^2(-\infty,\infty;\mathcal{R}(\Omega)))^2$ and satisfies (29) almost everywhere on Ω. Moreover we observe that, thanks to (31) and (29), we have

$$\begin{aligned}0 &= \int_{-\infty}^{\infty}\int_{\Omega}\{\mathbf{H}(\mathbf{x},t)\cdot\nabla\times\mathbf{e}(\mathbf{x},t)+\mathbf{E}(\mathbf{x},t)\cdot\nabla\times\mathbf{h}(\mathbf{x},t)\\ &\quad -\mathbf{h}(\mathbf{x},t)\cdot\nabla\times\mathbf{E}(\mathbf{x},t)-\mathbf{e}(\mathbf{x},t)\cdot\nabla\times\mathbf{H}(\mathbf{x},t)\}\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{-\infty}^{\infty}\int_{\partial\Omega}\{\mathbf{h}(\sigma,t)\times\mathbf{E}(\sigma,t)+\mathbf{e}(\sigma,t)\times\mathbf{H}(\sigma,t)\}\cdot\mathbf{n}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}t\end{aligned}$$

for any pair $(\mathbf{e},\mathbf{h})\in H(Q)$. By recalling that $\mathbf{e}_\tau=\lambda\mathbf{h}\times\mathbf{n}$ in $L^2(-\infty,\infty;L^2(\partial\Omega))$, we can conclude that the tangential components of the electric and magnetic fields belong to $L^2(-\infty,\infty;L^2(\partial\Omega))$ and satisfy (30).

Besides, the following energy inequality holds:

THEOREM 4.1 *Let* $(\mathbf{E},\mathbf{H})$ *be a weak solution for problem* (29) *and* (30), *then it exists a positive constant* K *such that*

$$\begin{aligned}&\int_{-\infty}^{+\infty}\{\|\mathbf{E}(t)\|^2_{L^2(\Omega)}+\|\mathbf{H}(t)\|^2_{L^2(\Omega)}\}\,\mathrm{d}t\\ &\quad\le K\int_{-\infty}^{+\infty}\{\|\mathbf{J}_f(t)\|^2_{L^2(\Omega)}+\|\mathbf{G}_f(t)\|^2_{L^2(\Omega)}\}\,\mathrm{d}t.\end{aligned}\tag{32}$$

Proof The Poynting theorem, applied to problem (29) and (30), leads to the identity:

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{\Omega}\{\mathbf{J}_f(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)-\mathbf{G}_f(\mathbf{x},t)\cdot\mathbf{H}(\mathbf{x},t)\}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad=-\lambda\int_{-\infty}^{\infty}\int_{\partial\Omega}|\mathbf{H}(\sigma,t)\times\mathbf{n}(\sigma)|^2\,\mathrm{d}\sigma\,\mathrm{d}t.\end{aligned}\tag{33}$$

Proposition 2.1 can be applied to both electric and magnetic fields solutions of problem (29) and (30); in this way we obtain the inequality:

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{\Omega}\{|\mathbf{E}(\mathbf{x},t)|^2+|\mathbf{H}(\mathbf{x},t)|^2\}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\le k^2\int_{-\infty}^{\infty}\left(\int_{\Omega}\{|\mathbf{J}_f(\mathbf{x},t)|^2+|\mathbf{G}_f(\mathbf{x},t)|^2\}\,\mathrm{d}x\right.\\ &\qquad\left.+(1+\lambda^2)\int_{\partial\Omega}|\mathbf{H}(\sigma,t)\times\mathbf{n}(\sigma)|^2\,\mathrm{d}\sigma\right)\mathrm{d}t.\end{aligned}\tag{34}$$

On the other hand, (33) yields to

$$\lambda \int_{-\infty}^{\infty} \|\mathbf{H}(t) \times \mathbf{n}\|_{L^2(\partial\Omega)}^2 \,\mathrm{d}t$$
$$= \int_{-\infty}^{\infty} \int_{\Omega} \{\mathbf{G}_f(\mathbf{x},t) \cdot \mathbf{H}(\mathbf{x},t) - \mathbf{J}_f(\mathbf{x},t) \cdot \mathbf{E}(\mathbf{x},t)\} \,\mathrm{d}x \,\mathrm{d}t$$
$$\leq \int_{-\infty}^{\infty} \{\|\mathbf{J}_f(t)\|_{L^2(\Omega)} \|\mathbf{E}(t)\|_{L^2(\Omega)} + \|\mathbf{G}_f(t)\|_{L^2(\Omega)} \|\mathbf{H}(t)\|_{L^2(\Omega)}\} \,\mathrm{d}t. \quad (35)$$

The use of (35) in (34) leads to the energy inequality (32), where the constant K depends on the constant of the Poincaré type theorem Proposition 2.1 and on λ.

An immediate consequence of the foregoing result is the following uniqueness theorem:

THEOREM 4.2 *Problem* (29) *and* (30) *has at most one weak solution corresponding to data* $\mathbf{J}_f$, $\mathbf{G}_f \in L^2(-\infty, +\infty; \mathcal{D}^0(\Omega))$.

Proof Let us consider two weak solutions of problem (29) and (30), then their difference satisfies the same problem with vanishing data and must be zero just because of (32).

Furthermore, Theorem 4.1 plays a very important role also in the proof of the existence theorem:

THEOREM 4.3 *Problem* (29) *and* (30) *with sources* $\mathbf{J}_f$, $\mathbf{G}_f \in L^2(-\infty, +\infty; \mathcal{D}^0(\Omega))$ *has a* (*unique*) *weak solution.*

Proof Let us introduce the operator Λ, defined on $H(Q)$ by $\Lambda(\mathbf{E},\mathbf{H}) = (\nabla \times \mathbf{H}, \nabla \times \mathbf{E})$ and rewrite problem (29) as $\Lambda(\mathbf{E},\mathbf{H}) = (\mathbf{J}_f, \mathbf{G}_f)$.

First of all we want to prove that $R(\Lambda)$, the range of Λ, is dense in $W = (L^2(-\infty, \infty; L^2(\Omega)))^2$. To this end, let us consider an element $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ orthogonal to $\overline{R(\Lambda)}$, then it satisfies the identity

$$0 = \int_{-\infty}^{\infty} \int_{\Omega} \{\mathbf{J}_f(\mathbf{x},t) \cdot \tilde{\mathbf{E}}(\mathbf{x},t) + \mathbf{G}_f(\mathbf{x},t) \cdot \tilde{\mathbf{H}}(\mathbf{x},t)\} \,\mathrm{d}x \,\mathrm{d}t$$
$$= \int_{-\infty}^{\infty} \int_{\Omega} \big(\nabla \times \mathbf{H}(\mathbf{x},t) \cdot \tilde{\mathbf{E}}(\mathbf{x},t) + \nabla \times \mathbf{E}(\mathbf{x},t) \cdot \tilde{\mathbf{H}}(\mathbf{x},t)\big) \,\mathrm{d}x \,\mathrm{d}t$$

for any $(\mathbf{E},\mathbf{H}) \in H(Q)$. This relation assures that $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ is a weak solution for problem (29) and (30) with vanishing data; therefore, as a consequence of Theorem 4.2, it must be $\tilde{\mathbf{E}} = \tilde{\mathbf{H}} = \mathbf{0}$.

Take now $(\mathbf{J}_f, \mathbf{G}_f) \in W$. Then, thanks to the property already shown, it exists a sequence $(\mathbf{E}_n, \mathbf{H}_n) \in H(Q)$ whose source field $(\mathbf{J}_n, \mathbf{G}_n)$ converges to $(\mathbf{J}_f, \mathbf{G}_f)$ in W. Besides, from the energy inequality (32) we have

$$\int_{-\infty}^{\infty}\left(\|\mathbf{E}_n(.,t)-\mathbf{E}_m(.,t)\|^2_{L^2(\Omega)}+\|\mathbf{H}_n(.,t)-\mathbf{H}_m(.,t)\|^2_{L^2(\Omega)}\right)\mathrm{d}t$$
$$\leq K\int_{-\infty}^{\infty}\left(\|\mathbf{J}_n(.,t)-\mathbf{J}_m(.,t)\|^2_{L^2(\Omega)}+\|\mathbf{G}_n(.,t)-\mathbf{G}_m(.,t)\|^2_{L^2(\Omega)}\right)\mathrm{d}t. \tag{36}$$

It follows that $(\mathbf{E}_n, \mathbf{H}_n)$ is a Cauchy sequence in $(L^2(-\infty,\infty;L^2(\Omega)))^2$, which is complete. Hence, the limit field $(\mathbf{E},\mathbf{H})$ exists and is a weak solution (then a strong one) having the prescribed source $(\mathbf{J}_f, \mathbf{G}_f)$, since $(\mathbf{E}_n, \mathbf{H}_n)$ is a weak solution with data $(\mathbf{J}_n, \mathbf{G}_n)$.

Remark 4.1 It is possible [9] to extend the previous result, by means of the same technique, to more general situations such as problem (29) with a boundary condition with memory of the type

$$\mathbf{E}_\tau(t) = \lambda_0\mathbf{H}(t)\times\mathbf{n} + \int_0^\infty \lambda(s)\mathbf{H}(t-s)\times\mathbf{n}\,\mathrm{d}s, \tag{37}$$

when the thermodynamic restriction

$$\lambda_0 + \int_0^\infty \lambda(s)\cos(\omega s)\,\mathrm{d}s > 0$$

is satisfied.

4.2 A Time-Harmonic Problem

Let us now suppose the domain Ω occupied by a linear conductor, i.e. $\mathbf{J}_c = \sigma\mathbf{E}$ with $\sigma > 0$ and put $\mathbf{G}_f = \mathbf{0}$. The time-harmonic evolution of the electromagnetic field is described by vectors of the type $\mathbf{A}(\mathbf{x},t) = \Re\{\mathrm{e}^{i\omega t}\mathbf{A}(\mathbf{x})\}$, where $\omega \neq 0$ is the frequency and $\mathbf{A}$ a complex vector independent on t. In this case the Maxwell equations (27) and (28) can be rewritten as

$$\begin{aligned}\nabla\times\mathbf{H} - (\mathrm{i}\omega\epsilon+\sigma)\mathbf{E} &= \mathbf{J}_f,\\ \nabla\times\mathbf{E} + \mathrm{i}\omega\mu\mathbf{H} &= \mathbf{0},\end{aligned} \tag{38}$$

while we assume the boundary condition of the type

$$\mathbf{n} \times \mathbf{E} = \mathrm{i}\gamma \mathbf{H}_\tau \quad \text{on } \partial\Omega, \tag{39}$$

with $\gamma > 0$. Condition (39) represents a conservative boundary, in the sense that it satisfies the relation

$$\Re\{\mathbf{E}^*(\sigma) \times \mathbf{H}(\sigma) \cdot \mathbf{n}(\sigma)\} = 0, \quad \sigma \in \partial\Omega$$

and a practical example where it applies, is the junction of a cavity and a tuning stub [2].

DEFINITION 4.3 *A pair* $(\mathbf{E}, \mathbf{H}) \in \Gamma(\Omega) = \{(\mathbf{e}, \mathbf{h}) \in \mathcal{H}^0(\Omega) \times \mathcal{H}^0(\Omega);\ \mathbf{n} \times \mathbf{e} = \mathrm{i}\gamma \mathbf{h}_\tau$ *on* $\partial\Omega\}$ *is a solution for problem* (38) *and* (39) *with source* $\mathbf{J}_f \in \mathcal{D}^0(\Omega)$ *if* (38) *holds almost everywhere.*

We can now establish the following:

THEOREM 4.4 *If* $(\mathbf{E}, \mathbf{H})$ *is a solution to problem* (38) *and* (39), *then the following inequality holds:*

$$\|\mathbf{E}\|^2_{L^2(\Omega)} + \|\mathbf{H}\|^2_{L^2(\Omega)} \leq C\|\mathbf{J}_f\|^2_{L^2(\Omega)}, \tag{40}$$

where C is a positive constant.

Proof Let the pair $(\mathbf{E}, \mathbf{H})$ be a solution to problem (38) and (39). Then, since both electric and magnetic fields are solenoidal, Proposition 2.1 assures that

$$\begin{aligned}
\|\mathbf{E}\|_{L^2(\Omega)} &\leq k(\|\nabla \times \mathbf{E}\|_{L^2(\Omega)} + \|\mathbf{E} \times \mathbf{n}\|_{L^2(\partial\Omega)}),\\
\|\mathbf{H}\|_{L^2(\Omega)} &\leq k(\|\nabla \times \mathbf{H}\|_{L^2(\Omega)} + \|\mathbf{H} \times \mathbf{n}\|_{L^2(\partial\Omega)}).
\end{aligned}$$

But if we recall (39), it results that the $L^2(\partial\Omega)$-norm of the tangential component of the electric field can be controlled by the $L^2(\partial\Omega)$-norm of $\mathbf{H} \times \mathbf{n}$. Therefore

$$\begin{aligned}
&\|\mathbf{E}\|^2_{L^2(\Omega)} + \|\mathbf{H}\|^2_{L^2(\Omega)}\\
&\quad \leq h_1\left(\|\nabla \times \mathbf{E}\|^2_{L^2(\Omega)} + \|\nabla \times \mathbf{H}\|^2_{L^2(\Omega)} + \|\mathbf{H} \times \mathbf{n}\|^2_{L^2(\partial\Omega)}\right)
\end{aligned} \tag{41}$$

and in order to obtain the inequality (40), we must estimate the right-hand side of (41). Let us now put

$$a(\omega) = (\mathrm{i}\omega\epsilon + \sigma), \quad b(\omega) = \mathrm{i}\omega\mu,$$

and consider the bilinear form defined by

$$I(\mathbf{E}, \mathbf{H}) = \int_\Omega \frac{1}{a^*(\omega)} \nabla \times \mathbf{H}(\mathbf{x}) \cdot \mathbf{J}_f^*(\mathbf{x})\, \mathrm{d}x. \tag{42}$$

If we substitute (38) in (42), we get

$$\begin{aligned} I(\mathbf{E}, \mathbf{H}) = &\int_\Omega \{\mathbf{H}(\mathbf{x}) \cdot \nabla \times \mathbf{E}^*(\mathbf{x}) - \mathbf{E}^*(\mathbf{x}) \cdot \nabla \times \mathbf{H}(\mathbf{x})\}\, \mathrm{d}x \\ &+ \int_\Omega \left\{ \frac{1}{b(\omega)} |\nabla \times \mathbf{E}(\mathbf{x})|^2 + \frac{1}{a^*(\omega)} |\nabla \times \mathbf{H}(\mathbf{x})|^2 \right\} \mathrm{d}x; \end{aligned} \tag{43}$$

while, thanks to (39), we have

$$\int_{\partial\Omega} \mathbf{E}_\tau^*(\sigma) \cdot \mathbf{H}(\sigma) \times \mathbf{n}(\sigma)\, \mathrm{d}\sigma = -\mathrm{i}\gamma \int_{\partial\Omega} |\mathbf{H}(\sigma) \times \mathbf{n}(\sigma)|^2\, \mathrm{d}\sigma. \tag{44}$$

Since

$$\begin{aligned} &\int_{\partial\Omega} \mathbf{E}_\tau^*(\sigma) \cdot \mathbf{H}(\sigma) \times \mathbf{n}(\sigma)\, \mathrm{d}\sigma \\ &= \int_\Omega \{\mathbf{H}(\mathbf{x}) \cdot \nabla \times \mathbf{E}^*(\mathbf{x}) - \mathbf{E}^*(\mathbf{x}) \cdot \nabla \times \mathbf{H}(\mathbf{x})\}\, \mathrm{d}x, \end{aligned} \tag{45}$$

if we put (44) and (45) in (43), we obtain

$$\begin{aligned} &\int_\Omega \left\{ \frac{1}{b(\omega)} |\nabla \times \mathbf{E}(\mathbf{x})|^2 + \frac{1}{a^*(\omega)} |\nabla \times \mathbf{H}(\mathbf{x})|^2 \right\} \mathrm{d}x \\ &\quad - \mathrm{i}\gamma \int_{\partial\Omega} |\mathbf{H}(\sigma) \times \mathbf{n}(\sigma)|^2\, \mathrm{d}\sigma \\ &\quad = \int_\Omega \frac{1}{a^*(\omega)} \nabla \times \mathbf{H}(\mathbf{x}) \cdot \mathbf{J}_f^*(\mathbf{x})\, \mathrm{d}x. \end{aligned} \tag{46}$$

By considering the real part of (46), we get

$$\frac{\sigma}{\sigma^2+\omega^2\epsilon^2}\int_\Omega |\nabla\times\mathbf{H}(\mathbf{x})|^2\,\mathrm{d}x = \Re\bigg(\int_\Omega \frac{1}{a^*(\omega)}\mathbf{J}_f^*(\mathbf{x})\cdot\nabla\times\mathbf{H}(\mathbf{x})\,\mathrm{d}x\bigg). \tag{47}$$

An application of the Schwarz inequality to the right-hand side of (47), leads to

$$\|\nabla\times\mathbf{H}\|_{L^2(\Omega)} \le k_1\|\mathbf{J}_f\|_{L^2(\Omega)}. \tag{48}$$

It should be underlined that the inequality (48) is strictly related to the dissipative character of the system, represented by the term $\sigma\mathbf{E}$ in (38).

On the other hand, if we take account of the imaginary part of (46), we find that

$$\begin{aligned}\gamma\int_{\partial\Omega}&|\mathbf{H}(\sigma)\times\mathbf{n}|^2\,\mathrm{d}\sigma+\frac{1}{\omega\mu}\int_\Omega|\nabla\times\mathbf{E}(\mathbf{x})|^2\,\mathrm{d}x\\ &=\int_\Omega\bigg(\frac{\omega\epsilon}{\omega^2\epsilon^2+\sigma^2}|\nabla\times\mathbf{H}(\mathbf{x})|^2\bigg)\,\mathrm{d}x\\ &\quad+\Im\bigg(-\int_\Omega\frac{1}{a^*(\omega)}\mathbf{J}_f^*(\mathbf{x})\cdot\nabla\times\mathbf{H}(\mathbf{x})\,\mathrm{d}x\bigg)\end{aligned} \tag{49}$$

and a further application of the Schwarz inequality together with (48) yield to

$$\|\mathbf{H}\times\mathbf{n}\|^2_{L^2(\partial\Omega)}+\|\nabla\times\mathbf{E}\|^2_{L^2(\Omega)}\le k_2\|\mathbf{J}_f\|^2_{L^2(\Omega)}. \tag{50}$$

From the last two inequalities we get the thesis.

Reasoning in the same way as done in the previous subsection for the quasi-static problem, it is possible to use the inequality (40) to prove an existence and uniqueness theorem for problem (38) and (39).

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