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Generalized Bi-quasi-variational Inequalities for Quasi-semi-monotone and Bi-quasi-semi-monotone Operators with Applications in Non-compact Settings and Minimization Problems

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Results are obtained on existence theorems of generalized bi-quasi-variational inequalities for quasi-semi-monotone and bi-quasi-semi-monotone operators in both compact and non-compact settings. We shall use the concept of escaping sequences introduced by Border (Fixed Point Theorem with Applications to Economics and Game Theory, Cambridge University Press, Cambridge, 1985) to obtain results in non-compact settings. Existence theorems on non-compact generalized bi-complementarity problems for quasi-semi-monotone and bi-quasi-semi-monotone operators are also obtained. Moreover, as applications of some results of this paper on generalized bi-quasi-variational inequalities, we shall obtain existence of solutions for some kind of minimization problems with quasi-semi-monotone and bi-quasi-semi-monotone operators.

Keywords: Bilinear functional; Generalized bi-quasi-variational inequality; Locally convex space; Lower semicontinuous; Upper semicontinuous; Upper hemicontinuous; h-bi-quasi-semi-monotone; Bi-quasi-semi-monotone; h-quasi-semi-monotone; Quasi-semi-monotone operators; Minimization problems

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1. INTRODUCTION

If X is a non-empty set, we shall denote by 2^X the family of all non-empty subsets of X. If X and Y are topological spaces and $T: X \to 2^Y$, then the graph of T is the set $G(T) := \{(x,y) \in X \times Y: y \in T(x)\}$. Throughout this paper, Φ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

Let E be a topological vector space over Φ, F be a vector space over Φ and $\langle\, , \rangle : F \times E \to \Phi$ be a bilinear functional. If X is a non-empty subset of E, then a map $T: X \to 2^F$ is called (i) F-monotone (i.e., monotone with respect to the bilinear functional $\langle\, , \rangle$) if for each $x, y \in X$, each $u \in T(x)$ and each $w \in T(y)$, $\operatorname{Re}\langle w-u, y-x\rangle \geq 0$ and (ii) F-semi-monotone (i.e., semi-monotone with respect to the bilinear functional $\langle\, , \rangle$) if for each $x, y \in X$, $\inf_{u \in T(x)} \operatorname{Re}\langle u, y-x\rangle \leq \inf_{w \in T(y)} \operatorname{Re}\langle w, y-x\rangle$. Note that when $F = E^*$, the vector space of all continuous linear functionals on E and $\langle\, , \rangle$ is the usual pairing between E^* and E, the F-monotonicity and F-semi-monotonicity notions coincide with the usual definitions of monotonicity and semi-monotonicity (see, e.g., Browder [4, p. 79] and Bae *et al.* [2, p. 237] respectively). But for simplicity of notions we shall use the terms monotone and semi-monotone instead of F-monotone and F-semi-monotone. Note also that $T: X \to 2^F$ is monotone if and only if its graph $G(T)$ is a monotone subset of $X \times F$; i.e., for all $(x_1, y_1), (x_2, y_2) \in G(T)$, $\operatorname{Re}\langle y_2 - y_1, x_2 - x_1\rangle \geq 0$.

For each $x_0 \in E$, each non-empty subset A of E and each $\epsilon > 0$, let $W(x_0;\epsilon) := \{y \in F: |\langle y, x_0\rangle| < \epsilon\}$ and $U(A;\epsilon) := \{y \in F: \sup_{x \in A} |\langle y, x\rangle| < \epsilon\}$. Let $\sigma\langle F, E\rangle$ be the (weak) topology on F generated by the family $\{W(x;\epsilon): x \in E \text{ and } \epsilon > 0\}$ as a subbase for the neighborhood system at 0 and $\delta\langle F, E\rangle$ be the (strong) topology on F generated by the family $\{U(A;\epsilon): A \text{ is a non-empty bounded subset of } E \text{ and } \epsilon > 0\}$ as a base for the neighborhood system at 0. We note then that F, when equipped with the (weak) topology $\sigma\langle F, E\rangle$ or the (strong) topology $\delta\langle F, E\rangle$, becomes a locally convex topological vector space which is not necessarily Hausdorff. But if the bilinear functional $\langle\, , \rangle : F \times E \to \Phi$ separates points in F, i.e., for each $y \in F$ with $y \neq 0$, there exists $x \in E$ such that $\langle y, x\rangle \neq 0$, then F also becomes Hausdorff. Furthermore, for a net $\{y_\alpha\}_{\alpha \in \Gamma}$ in F and for $y \in F$, (i) $y_\alpha \to y$ in $\sigma\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ for each $x \in E$ and (ii) $y_\alpha \to y$ in $\delta\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ uniformly for $x \in A$ for each non-empty bounded subset A of E.

Let X be a non-empty subset of E. Then X is a cone in E if X is convex and $\lambda X \subset X$ for all $\lambda \geq 0$. If X is a cone in E and $\langle\,,\rangle : F \times E \to \Phi$ is a bilinear functional, then $\hat{X} = \{w \in F\colon \operatorname{Re}\langle w, x\rangle \geq 0 \text{ for all } x \in X\}$ is also a cone in F, called the dual cone of X (with respect to the bilinear functional $\langle\,,\rangle$).

The following result is Lemma 1 of Shih and Tan in [13, pp. 334–335]:

LEMMA A *Let X be a non-empty subset of a Hausdorff topological vector space E and $S: X \to 2^E$ be an upper semicontinuous map such that $S(x)$ is a bounded subset of E for each $x \in X$. The for each continuous linear functional p on E, the map $f_p: X \to \mathbb{R}$ defined by $f_p(y) = \sup_{x \in S(y)} \operatorname{Re}\langle p, x\rangle$ is upper semicontinuous; i.e., for each $\lambda \in \mathbb{R}$, the set $\{y \in X\colon f_p(y) = \sup_{x \in S(y)} \operatorname{Re}\langle p, x\rangle < \lambda\}$ is open in X.*

The following result is Lemma 3 of Takahashi in [15, p. 177] (see also Lemma 3 in [14, pp. 71–72]:

LEMMA B *Let X and Y be topological spaces, $f: X \to \mathbb{R}$ be non-negative and continuous and $g: Y \to \mathbb{R}$ be lower semicontinuous. Then the map $F: X \times Y \to \mathbb{R}$, defined by $F(x, y) = f(x)g(y)$ for all $(x, y) \in X \times Y$, is lower semicontinuous.*

We now state the following result which follows from Theorem 3.1 of Chowdhury and Tan in [9] (see also Lemma 2.1 of Tarafdar and Yuan [17] and Theorem 2.2 of Tarafdar in [16]) and is a generalization of the celebrated 1972 Ky Fan's minimax inequality in [11, Theorem 1]:

THEOREM A *Let E be a topological vector space, and X be a non-empty compact convex subset of E. Suppose that $f, g: X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ are two mappings satisfying the following conditions:*

(i) *for each $x \in X$, $g(x, x) \leq 0$ and for each $x, y \in X$, $f(x, y) > 0$ implies $g(x, y) > 0$;*
(ii) *for each fixed $x \in X$, the map $y \mapsto f(x, y)$ is lower semicontinuous on X;*
(iii) *for each fixed $y \in X$, the set $\{x \in X\colon g(x, y) > 0\}$ is convex;*

Then there exists a point $\hat{y} \in X$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

We shall need the following Kneser's minimax theorem in [12, pp. 2418–2420] (see also Aubin [1, pp. 40–41]):

THEOREM B *Let X be a non-empty convex subset of a vector space and Y be a non-empty compact convex subset of a Hausdorff topological* *vector space. Suppose that f is a real-valued function on $X \times Y$ such that for each fixed $x \in X$, the map $y \mapsto f(x,y)$ is lower semicontinuous and convex on Y and for each fixed $y \in Y$, the map $x \mapsto f(x,y)$ is concave on X. Then*

$$\min_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\min_{y\in Y} f(x,y).$$

2. GENERALIZED BI-QUASI-VARIATIONAL INEQUALITIES FOR QUASI-SEMI-MONOTONE AND BI-QUASI-SEMI-MONOTONE OPERATORS

In this section we shall obtain some existence theorems of generalized bi-quasi-variational inequalities for quasi-semi-monotone and bi-quasi-semi-monotone operators. Our results will extend and or generalize the corresponding results in [6] and [14].

Let E and F be Hausdorff topological vector spaces over the field Φ, let $\langle\, ,\rangle : F \times E \to \Phi$ be a bilinear functional, and let X be a non-empty subset of E. Given a set-valued map $S: X \to 2^X$ and two set-valued maps $M, T: X \to 2^F$, the *generalized bi-quasi-variational inequality* (GBQVI) problem is to find a point $\hat{y} \in X$ and a point $\hat{w} \in T(\hat{y})$ such that $\hat{y} \in S(\hat{y})$ and $\operatorname{Re}\langle f - \hat{w}, \hat{y} - x\rangle \le 0$ for all $x \in S(\hat{y})$ and for all $f \in M(\hat{y})$ or to find a point $\hat{y} \in X$, a point $\hat{w} \in T(\hat{y})$ and a point $\hat{f} \in M(\hat{y})$ such that $\hat{y} \in S(\hat{y})$ and $\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x\rangle \le 0$ for all $x \in S(\hat{y})$. The above definition of GBQVI problem was given in [6, p. 139] which is a slight modification of the original definition of GBQVI problem of Shih and Tan in [14].

The following definition is Definition 4.4.2 in [6] and generalizes Definition 2.1(b) in [9]:

DEFINITION 1 *Let E be a topological vector space over Φ, F be a vector space over Φ and X be a non-empty subset of E. Let $\langle\, ,\rangle : F \times E \to \Phi$ be a bilinear functional and $M: X \to 2^F$ be a map. Then M is said to be upper hemicontinuous on X if and only if for each $p \in E$, the function $f_p: X \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f_p(z) = \sup_{u\in M(z)} \operatorname{Re}\langle u, p\rangle \quad \textit{for each } z \in X,$$

is upper semicontinuous on X (if and only if for each $p \in E$, the function $g_p : X \to \mathbb{R} \cup \{-\infty\}$ defined by

$$g_p(z) = \inf_{u \in M(z)} \operatorname{Re}\langle u, p\rangle \quad \textit{for each } z \in X,$$

is lower semicontinuous on X).

The following result is Proposition 4.4.3 in [6] and generalizes Proposition 2.4 in [9]:

PROPOSITION 1 *Let E be a topological vector space over Φ, F be a vector space over Φ and X be a non-empty subset of E. Let $\langle\,,\rangle : F \times E \to \Phi$ be a bilinear functional such that for each $p \in E$, $u \mapsto \langle u, p\rangle$ is $\sigma\langle F, E\rangle$-continuous on F when F is equipped with the $\sigma\langle F, E\rangle$-topology. Let $M : X \to 2^F$ be upper semicontinuous from the relative topology on X to the weak topology $\sigma\langle F, E\rangle$ on F. Then M is upper hemicontinuous on X.*

Note that the converse of Proposition 1 is not true as can be seen in Example 2.5 of [9] which is Example 2.3 in [18, p. 392]:

The following definition is a generalization of (3) and (4) of the Definition 2.6 in [9, p. 31].

DEFINITION 2 *Let E be a topological vector space and X be a non-empty subset of E. Let F be a vector space over Φ and $\langle\,,\rangle : F \times E \to \Phi$ be a bilinear functional. Let $M : X \to 2^F$ be a map. Suppose $h : X \to \mathbb{R}$. Then M is said to be h-quasi-semi-monotone if for each $x, y \in X$,*

$$\inf_{w \in M(y)} \operatorname{Re}\langle w, y - x\rangle + h(y) - h(x) > 0$$

whenever

$$\inf_{u \in M(x)} \operatorname{Re}\langle u, y - x\rangle + h(y) - h(x) > 0.$$

M is said to be quasi-semi-monotone if M is h-quasi-semi-monotone with $h \equiv 0$.

We shall now introduce the following definition:

DEFINITION 3 *Let E be a topological vector space and X be a non-empty subset of E. Let F be a vector space over Φ and $\langle\,,\rangle : F \times E \to \Phi$ be a* *bilinear functional. Let $M, T: X \to 2^F$ be two maps. Suppose $h: X \to \mathbb{R}$. Then M is said to be h-bi-quasi-semi-monotone if for each $x, y \in X$ and each finite set $\{\beta_j: j = 0, 1, \ldots, n\}$ of non-negative real-valued functions on X,*

$$\beta_0(y)\left[\inf_{g\in M(y)}\inf_{w\in T(y)}\mathrm{Re}\langle g-w, y-x\rangle + h(y) - h(x)\right] + \sum_{k=1}^{n}\beta_k(y)\mathrm{Re}\langle p_k, y-x\rangle > 0$$

whenever

$$\beta_0(y)\left[\inf_{f\in M(x)}\inf_{w\in T(y)}\mathrm{Re}\langle f-w, y-x\rangle + h(y) - h(x)\right] + \sum_{k=1}^{n}\beta_k(y)\mathrm{Re}\langle p_k, y-x\rangle > 0,$$

where $p_k \in E^*$ for $k = 1, \ldots, n$.

M is said to be bi-quasi-semi-monotone if M is h-bi-quasi-semi-monotone with $h \equiv 0$. If "each finite set $\{\beta_j: j = 0, 1, \ldots, n\}$ of non-negative real-valued functions on X" is replaced by "each family $\{\beta_0, \beta_p: p \in E^*\}$ of non-negative real-valued functions on X" and $T \equiv 0$, M becomes a generalized h-quasi-semi-monotone operator as defined in [7, Definition 4, p. 296].

Clearly, a semi-monotone operator is also an h-bi-quasi-semi-monotone (respectively, a generalized h-quasi-semi-monotone) operator. But the converse is not true; because if $T \equiv 0$, $\beta_0 \equiv 1$ and $\beta_k \equiv 0$ for each $k = 1, 2, \ldots, n$ (respectively, if $\beta_0 \equiv 1$ and for each $p \in E^*$, $\beta_p \equiv 0$), then an h-bi-quasi-semi-monotone (respectively, a generalized h-quasi-semi-monotone) operator is an h-quasi-semi-monotone operator which is not necessarily a semi-monotone operator. The following example, which is Example 2.8 in [9] shows that an h-bi-quasi-semi-monotone operator T need not be a semi-monotone operator.

Example 1 Define $T: \mathbb{R} \to 2^{\mathbb{R}}$ by

$$T(x) = \begin{cases} [x, 1/x], & \text{if } 0 < x < 1; \\ [1/x, x], & \text{if } x \geq 1. \end{cases}$$

It is shown in [9, pp. 31–32] that T is not semi-monotone although it is quasi-monotone.

The following example, which is Example 2.9 in [9] shows that an h-bi-quasi-monotone operator T need not be a quasi-monotone operator.

Example 2 Define $T: \mathbb{R} \to 2^{\mathbb{R}}$ by

$$T(x) = \begin{cases} [0, 2x], & \text{if } x \geq 0; \\ [2x, 0], & \text{if } x < 0. \end{cases}$$

It is shown in [9, p. 32] that T is not quasi-monotone although it is semi-monotone (as shown in [2, p. 241]) and therefore an h-bi-quasi-semi-monotone operator need not be a quasi-monotone operator.

These examples justify the validity of bi-quasi-monotone operators.

The following result is Lemma 4.4.4 in [6]:

LEMMA 1 *Let E be a topological vector space over Φ, X be a non-empty compact subset of E and F be a Hausdorff topological vector space over Φ. Let $\langle , \rangle : F \times E \to \Phi$ be a bilinear functional and $T: X \to 2^F$ be an upper semicontinuous map such that each $T(x)$ is compact. Let M be a non-empty compact subset of F, $x_0 \in X$ and $h: X \to \mathbb{R}$ be continuous. Define $g: X \to \mathbb{R}$ by $g(y) = [\inf_{f \in M} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x_0 \rangle] + h(y)$ for each $y \in X$. Suppose that $\langle , \rangle$ is continuous on the (compact) subset $[M - \bigcup_{y \in X} T(y)] \times X$ of $F \times E$. Then g is lower semicontinuous on X.*

When $h \equiv 0$ and $M = \{0\}$, replacing T by $-T$, Lemma 1 reduces to the Lemma 2 of Shih and Tan in [14, pp. 70–71].

The following result is Lemma 4.4.5 in [6] and generalizes Lemma 4.2 in [9]:

LEMMA 2 *Let E be a topological vector space over Φ, F be a vector space over Φ and X be a non-empty convex subset of E. Let $\langle , \rangle : F \times E \to \Phi$ be a bilinear functional. Equip F with the $\sigma\langle F, E \rangle$-topology. Let D be a non-empty $\sigma\langle F, E \rangle$-compact subset of F, $h: X \to \mathbb{R}$ be convex and $M: X \to 2^F$ be upper hemicontinuous along line segments in X. Suppose $\hat{y} \in X$ is such that $\inf_{f \in M(x)} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Then*

$$\inf_{f \in M(\hat{y})} \inf_{g \in D} \operatorname{Re}\langle f - g, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \quad \textit{for all } x \in X.$$

We shall now establish the following result:

THEOREM 1 *Let E be a locally convex Hausdorff topological vector space over Φ, X be a non-empty compact convex subset of E and F be a Hausdorff topological vector space over Φ. Let $\langle\,,\rangle : F \times E \to \Phi$ be a bilinear functional which is continuous on compact subsets of $F \times X$. Suppose that*

(a) $S: X \to 2^X$ *is an upper semicontinuous map such that each $S(x)$ is closed convex;*
(b) $T: X \to 2^F$ *is upper semicontinuous such that each $T(x)$ is compact convex;*
(c) $h: X \to \mathbb{R}$ *is convex and continuous;*
(d) $M: X \to 2^F$ *is upper hemicontinuous along line segments in X and h-bi-quasi-semi-monotone (with respect to $\langle\,,\rangle$) such that each $M(x)$ is compact convex and*
(e) *the set*

$$\Sigma = \left\{ y \in X: \sup_{x \in S(y)} \inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x) > 0 \right\}$$

is open in X. Then there exists a point $\hat{y} \in X$ such that
(i) $\hat{y} \in S(\hat{y})$ *and*
(ii) *there exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex and if $T \equiv 0$, the continuity assumption on $\langle\,,\rangle$ can be weakened to the assumption that for each $f \in F$, the map $x \mapsto \langle f, x \rangle$ is continuous on X.

Proof We divide the proof into three steps:
Step 1 There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} \left[\inf_{f \in M(x)} \inf_{w \in T(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{f \in M(x)} \inf_{w \in T(y)} \operatorname{Re}\langle f - w, y - x \rangle + h(y) - h(x) > 0$; that is, for each $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by a separation theorem, there exists $p \in E^*$ such that

$$\mathrm{Re}\langle p, y\rangle - \sup_{x \in S(y)} \mathrm{Re}\langle p, x\rangle > 0.$$

For each $p \in E^*$, let

$$V(p) = \left\{ y \in X \colon \mathrm{Re}\langle p, y\rangle - \sup_{x \in S(y)} \mathrm{Re}\langle p, x\rangle > 0 \right\}.$$

Then $V(p)$ is open by Lemma A. Since $X = \Sigma \cup \bigcup_{p \in E^*} V(p)$, by compactness of X, there exist $p_1, p_2, \ldots, p_n \in E^*$ such that $X = \Sigma \cup \bigcup_{i=1}^{n} V(p_i)$. For simplicity of notations, let $V_0 := \Sigma$ and $V_i = V(p_i)$ for $i = 1, 2, \ldots, n$. Let $\{\beta_0, \beta_1, \ldots, \beta_n\}$ be a continuous partition of unity on X subordinated to the covering $\{V_0, V_1, \ldots, V_n\}$. Then $\beta_0, \beta_1, \ldots, \beta_n$ are continuous non-negative real-valued functions on X such that β_i vanishes on $X \backslash V_i$, for each $i = 0, 1, \ldots, n$ and $\sum_{i=0}^{n} \beta_i(x) = 1$ for all $x \in X$. Define $\phi, \psi : X \times X \to \mathbb{R}$ by

$$\begin{aligned}\phi(x, y) = \beta_0(y) &\left[\inf_{f \in M(x)} \inf_{w \in T(y)} \mathrm{Re}\langle f - w, y - x\rangle + h(y) - h(x) \right] \\ &+ \sum_{i=1}^{n} \beta_i(y) \mathrm{Re}\langle p_i, y - x\rangle,\end{aligned}$$

and

$$\begin{aligned}\psi(x, y) = \beta_0(y) &\left[\inf_{g \in M(y)} \inf_{w \in T(y)} \mathrm{Re}\langle g - w, y - x\rangle + h(y) - h(x) \right] \\ &+ \sum_{i=1}^{n} \beta_i(y) \mathrm{Re}\langle p_i, y - x\rangle,\end{aligned}$$

for each $x, y \in X$. Then we have the following.

(1) For each $x \in X$, $\psi(x, x) = 0$ and for each $x, y \in X$, since M is h-bi-quasi-semi-monotone, $\phi(x, y) > 0$ implies $\psi(x, y) > 0$.
(2) For each fixed $x \in X$, the map

$$y \mapsto \inf_{f \in M(x)} \inf_{w \in T(y)} \mathrm{Re}\langle f - w, y - x\rangle + h(y) - h(x)$$

is lower semicontinuous on X by Lemma 1; therefore the map

$$y \mapsto \beta_0(y)\left[\inf_{f\in M(x)} \inf_{w\in T(y)} \operatorname{Re}\langle f-w, y-x\rangle + h(y) - h(x)\right]$$

is lower semicontinuous on X by Lemma B. Hence for each fixed $x \in X$, the map $y \mapsto \phi(x, y)$ is lower semicontinuous on X.

(3) Clearly, for each fixed $y \in X$, the set $\{x \in X: \psi(x, y) > 0\}$ is convex.

Then ϕ and ψ satisfy all the hypotheses of Theorem A. Thus by Theorem A, there exists $\hat{y} \in X$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$, i.e.,

$$\beta_0(\hat{y})\left[\inf_{f\in M(x)} \inf_{w\in T(\hat{y})} \operatorname{Re}\langle f-w, \hat{y}-x\rangle + h(\hat{y}) - h(x)\right] + \sum_{i=1}^{n} \beta_i(\hat{y})\operatorname{Re}\langle p_i, \hat{y}-x\rangle \leq 0 \tag{2.1}$$

for all $x \in X$.

Choose $\hat{x} \in S(\hat{y})$ such that

$$\inf_{f\in M(\hat{x})} \inf_{w\in T(\hat{y})} \operatorname{Re}\langle f-w, \hat{y}-\hat{x}\rangle + h(\hat{y}) - h(\hat{x}) > 0 \quad \text{whenever } \beta_0(\hat{y}) > 0;$$

it follows that

$$\beta_0(\hat{y})\left[\inf_{f\in M(\hat{x})} \inf_{w\in T(\hat{y})} \operatorname{Re}\langle f-w, \hat{y}-\hat{x}\rangle + h(\hat{y}) - h(\hat{x})\right] > 0 \quad \text{whenever } \beta_0(\hat{y}) > 0.$$

If $i \in \{1, \ldots, n\}$ is such that $\beta_i(\hat{y}) > 0$, then $\hat{y} \in V(p_i)$ and hence

$$\operatorname{Re}\langle p_i, \hat{y}\rangle > \sup_{x\in S(\hat{y})} \operatorname{Re}\langle p_i, x\rangle \geq \operatorname{Re}\langle p_i, \hat{x}\rangle$$

so that $\operatorname{Re}\langle p_i, \hat{y}-\hat{x}\rangle > 0$. Then note that

$$\beta_i(\hat{y})\operatorname{Re}\langle p_i, \hat{y}-\hat{x}\rangle > 0 \quad \text{whenever } \beta_i(\hat{y}) > 0 \text{ for } i = 1, \ldots, n.$$

Since $\beta_i(\hat{y}) > 0$, for at least one $i \in \{0, 1, \ldots, n\}$, it follows that

$$\beta_0(\hat{y})\left[\inf_{f\in M(\hat{x})}\inf_{w\in T(\hat{y})}\mathrm{Re}\langle f-w, \hat{y}-\hat{x}\rangle + h(\hat{y}) - h(\hat{x})\right] + \sum_{i=1}^{n}\beta_i(\hat{y})\mathrm{Re}\langle p_i, \hat{y}-\hat{x}\rangle > 0,$$

which contradicts (2.1). This contradiction proves Step 1.

Step 2

$$\inf_{f\in M(\hat{y})}\inf_{w\in T(\hat{y})}\mathrm{Re}\langle f-w, \hat{y}-x\rangle \le h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}).$$

Indeed, from Step 1, $\hat{y} \in S(\hat{y})$ which is a convex subset of X, and

$$\inf_{f\in M(x)}\inf_{w\in T(\hat{y})}\mathrm{Re}\langle f-w, \hat{y}-x\rangle \le h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}). \tag{2.2}$$

Hence by Lemma 2, we have

$$\inf_{f\in M(\hat{y})}\inf_{w\in T(\hat{y})}\mathrm{Re}\langle f-w, \hat{y}-x\rangle \le h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}).$$

Step 3 There exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ with $\mathrm{Re}\langle \hat{f}-\hat{w}, \hat{y}-x\rangle \le h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

From Step 2 we have

$$\sup_{x\in S(\hat{y})}\left[\inf_{f\in M(\hat{y})}\inf_{w\in T(\hat{y})}\mathrm{Re}\langle f-w, \hat{y}-x\rangle + h(\hat{y}) - h(x)\right] \le 0;$$

i.e.,

$$\sup_{x\in S(\hat{y})}\inf_{(f,w)\in M(\hat{y})\times T(\hat{y})}\mathrm{Re}\langle f-w, \hat{y}-x\rangle + h(\hat{y}) - h(x) \le 0, \tag{2.3}$$

where $M(\hat{y}) \times T(\hat{y})$ is a compact convex subset of the Hausdorff topological vector space $F \times F$ and $S(\hat{y})$ is a convex subset of X.

Let $Q = M(\hat{y}) \times T(\hat{y})$ and the map $g : S(\hat{y}) \times Q \to \mathbb{R}$ be defined by $g(x, q) = g(x, (f, w)) = \mathrm{Re}\langle f-w, \hat{y}-x\rangle + h(\hat{y}) - h(x)$ for each

$x \in S(\hat{y})$ and each $q = (f, w) \in Q = M(\hat{y}) \times T(\hat{y})$. Note that for each fixed $x \in S(\hat{y})$, the map $(f, w) \mapsto g(x, (f, w))$ is lower semicontinuous from the relative product topology on Q to $\mathbb{R}$ and also convex on Q. Clearly, for each fixed $q = (f, w) \in Q$, the map $x \mapsto g(x, q) = g(x, (f, w))$ is concave on $S(\hat{y})$. Then by Theorem B we have

$$\min_{(f,w)\in Q} \sup_{x\in S(\hat{y})} g(x, (f, w)) = \sup_{x\in S(\hat{y})} \min_{(f,w)\in Q} g(x, (f, w)).$$

Thus

$$\min_{(f,w)\in Q} \sup_{x\in S(\hat{y})} \operatorname{Re}\langle f - w, \hat{y} - x\rangle + h(\hat{y}) - h(x) \leq 0, \quad \text{by (2.3)}.$$

Since $Q = M(\hat{y}) \times T(\hat{y})$ is compact, there exists $(\hat{f}, \hat{w}) \in M(\hat{y}) \times T(\hat{y})$ such that

$$\sup_{x\in S(\hat{y})} \operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x\rangle + h(\hat{y}) - h(x) \leq 0.$$

Therefore

$$\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}).$$

Hence there exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ with

$$\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}).$$

Next we note from the above proof that E is required to be locally convex when and only when the separation theorem is applied to the case $y \notin S(y)$. Thus if $S : X \to 2^X$ is the constant map $S(x) = X$ for all $x \in X$, E is not required to be locally convex.

Finally, if $T \equiv 0$, in order to show that for each $x \in X$, $y \mapsto \phi(x, y)$ is lower semi-continuous, Lemma 1 is no longer needed and the weaker continuity assumption on $\langle\,,\rangle$ that for each $f \in F$, the map $x \mapsto \langle f, x\rangle$ is continuous on X is sufficient. This completes the proof.

When M is h-quasi-semi-monotone instead of h-bi-quasi-semi-monotone, the result follows immediately from Theorem 1.

Note that if the map $S : X \to 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma$, M is upper semicontinuous at some point x in $S(y)$

with $\inf_{f\in M(x)}\inf_{w\in T(x)}\operatorname{Re}\langle f-w,y-x\rangle+h(y)-h(x)>0$, then the set Σ in Theorem 1 is always open in X.

THEOREM 2 *Let E be a locally convex Hausdorff topological vector space over Φ, X be a non-empty compact convex subset of E and F be a vector space over Φ. Let $\langle\,,\rangle : F\times E\to\Phi$ be a bilinear functional such that $\langle\,,\rangle$ separates points if F and for each $f\in F$, the map $x\mapsto\langle f,x\rangle$ is continuous on X. Equip F with the strong topology $\delta\langle F,E\rangle$. Suppose that*

(a) *$S:X\to 2^X$ is a continuous map such that each $S(x)$ is closed and convex;*
(b) *$T:X\to 2^F$ is upper semicontinuous such that each $T(x)$ is strongly compact and convex;*
(c) *$h:X\to\mathbb{R}$ is convex and continuous;*
(d) *$M:X\to 2^F$ is upper hemicontinuous along line segments in X and h-bi-quasi-semi-monotone (with respect to $\langle\,,\rangle$) such that each $M(x)$ is $\delta\langle F,E\rangle$-compact convex; also, for each $y\in\Sigma=\{y\in X$: $\sup_{x\in S(y)}[\inf_{f\in M(x)}\inf_{w\in T(y)}\operatorname{Re}\langle f-w,y-x\rangle+h(y)-h(x)]>0\}$, M is upper semicontinuous at some point x in $S(y)$ with $\inf_{f\in M(x)}\inf_{w\in T(y)}\operatorname{Re}\langle f-w,y-x\rangle+h(y)-h(x)>0$.*

Then there exists a point $\hat{y}\in X$ such that

(i) *$\hat{y}\in S(\hat{y})$ and*
(ii) *there exist a point $\hat{f}\in M(\hat{y})$ and a point $\hat{w}\in T(\hat{y})$ with $\operatorname{Re}\langle\hat{f}-\hat{w},\hat{y}-x\rangle\le h(x)-h(\hat{y})$ for all $x\in S(\hat{y})$.*

Moreover, if $S(x)=X$ for all $x\in X$, E is not required to be locally convex.

Proof As $\langle\,,\rangle : F\times E\to\Phi$ is a bilinear functional such that for each $f\in F$, the map $x\mapsto\langle f,x\rangle$ is continuous on X and as F is equipped with the strong topology $\delta\langle F,E\rangle$, it is easy to see that $\langle\,,\rangle$ is continuous on compact subsets of $F\times X$. Thus by Theorem 1, it suffices to show that the set

$$\Sigma=\left\{y\in X:\ \sup_{x\in S(y)}[\inf_{f\in M(x)}\inf_{w\in T(y)}\operatorname{Re}\langle f-w,y-x\rangle+h(y)-h(x)]>0\right\}$$

is open in X. Indeed, let $y_0\in\Sigma$; then by the last part of the hypothesis (*d*), M is upper semicontinuous at some point x_0 in $S(y_0)$ with

$\inf_{f\in M(x_0)} \inf_{w\in T(y_0)} \operatorname{Re}\langle f-w, y_0-x_0\rangle + h(y_0) - h(x_0) > 0$. Let

$$\alpha := \inf_{f\in M(x_0)} \inf_{w\in T(y_0)} \operatorname{Re}\langle f-w, y_0-x_0\rangle + h(y_0) - h(x_0).$$

Then $\alpha > 0$. Also let

$$W := \left\{ w \in F\colon \sup_{z_1,z_2\in X} |\langle w, z_1 - z_2\rangle| < \alpha/6 \right\}.$$

Then W is an open neighborhood of 0 in F so that $U_1 := T(y_0) + W$ is an open neighborhood of $T(y_0)$ in F. Since T is upper semicontinuous at y_0, there exists an open neighborhood N_1 of y_0 in X such that $T(y) \subset U_1$ for all $y \in N_1$.

Let $U_2 := M(x_0) + W$, the U_2 is an open neighborhood of $M(x_0)$ in F. Since M is upper semicontinuous at x_0, there exists an open neighborhood V_1 of x_0 in X such that $M(x) \subset U_2$ for all $x \in V_1$.

As the map $x \mapsto \inf_{f\in M(x_0)} \inf_{w\in T(y_0)} \operatorname{Re}\langle f-w, x_0-x\rangle + h(x_0) - h(x)$ is continuous at x_0, there exists an open neighborhood V_2 of x_0 in X such that

$$\left| \inf_{f\in M(x_0)} \inf_{w\in T(y_0)} \operatorname{Re}\langle f-w, x_0-x\rangle + h(x_0) - h(x) \right| < \alpha/6$$

for all $x \in V_2$.

Let $V_0 := V_1 \cap V_2$; then V_0 is an open neighborhood of x_0 in X. Since $x_0 \in V_0 \cap S(y_0) \neq \emptyset$ and S is lower semicontinuous at y_0, there exists an open neighborhood N_2 of y_0 in X such that $S(y) \cap V_0 \neq \emptyset$ for all $y \in N_2$.

Since the map $y \mapsto \inf_{f\in M(x_0)} \inf_{w\in T(y_0)} \operatorname{Re}\langle f-w, y-y_0\rangle + h(y) - h(y_0)$ is continuous at y_0, there exists an open neighborhood N_3 of y_0 in X such that

$$\left| \inf_{f\in M(x_0)} \inf_{w\in T(y_0)} \operatorname{Re}\langle f-w, y-y_0\rangle + h(y) - h(y_0) \right| < \alpha/6$$

for all $y \in N_3$.

Let $N_0 := N_1 \cap N_2 \cap N_3$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have

(i) $T(y_1) \subset U_1 = T(y_0) + W$ as $y_1 \in N_1$;
(ii) $S(y_1) \cap V_0 \neq \emptyset$ as $y_1 \in N_2$; so we can choose any $x_1 \in S(y_1) \cap V_0$;
(iii) $\left|\inf_{f\in M(x_0)} \inf_{w\in T(y_0)} \operatorname{Re}\langle f-w, y_1 - y_0\rangle + h(y_1) - h(y_0)\right| < \alpha/6$ as $y_1 \in N_3$;
(iv) $M(x_1) \subset U_2 = M(x_0) + W$ as $x_1 \in V_1$;
(v) $\left|\inf_{f\in M(x_0)} \inf_{w\in T(y_0)} \operatorname{Re}\langle f-w, x_0 - x_1\rangle + h(x_0) - h(x_1)\right| < \alpha/6$ as $x_1 \in V_2$;

it follows that

$$
\begin{aligned}
&\inf_{f\in M(x_1)} \inf_{w\in T(y_1)} \operatorname{Re}\langle f-w, y_1 - x_1\rangle + h(y_1) - h(x_1)\\
&\quad\geq \inf_{[f\in M(x_0)+W]} \inf_{[w\in T(y_0)+W]} \operatorname{Re}\langle f-w, y_1 - x_1\rangle + h(y_1) - h(x_1)\\
&\qquad\qquad \text{(by (i) and (iv))},\\
&\quad\geq \inf_{f\in M(x_0)} \inf_{w\in T(y_0)} \operatorname{Re}\langle f-w, y_1 - x_1\rangle + h(y_1) - h(x_1)\\
&\qquad + \inf_{f\in W} \inf_{w\in W} \operatorname{Re}\langle f-w, y_1 - x_1\rangle\\
&\quad\geq \inf_{f\in M(x_0)} \inf_{w\in T(y_0)} \operatorname{Re}\langle f-w, y_1 - y_0\rangle + h(y_1) - h(y_0)\\
&\qquad + \inf_{f\in M(x_0)} \inf_{w\in T(y_0)} \operatorname{Re}\langle f-w, y_0 - x_0\rangle + h(y_0) - h(x_0)\\
&\qquad + \inf_{f\in M(x_0)} \inf_{w\in T(y_0)} \operatorname{Re}\langle f-w, x_0 - x_1\rangle + h(x_0) - h(x_1)\\
&\qquad + \inf_{f\in W} \operatorname{Re}\langle f, y_1 - x_1\rangle + \inf_{w\in W} \operatorname{Re}\langle -w, y_1 - x_1\rangle\\
&\quad\geq -\alpha/6 + \alpha - \alpha/6 - \alpha/6 - \alpha/6 = \alpha/3 > 0\\
&\qquad\qquad \text{(by (iii) and (v))};
\end{aligned}
$$

therefore

$$
\sup_{x\in S(y_1)} \left[\inf_{f\in M(x)} \inf_{w\in T(y_1)} \operatorname{Re}\langle f-w, y_1 - x\rangle + h(y_1) - h(x)\right] > 0
$$

as $x_1 \in S(y_1)$. This shows that $y_1 \in \Sigma$ for all $y_1 \in N_0$, so that Σ is open in X. This proves the theorem.

When M is h-quasi-semi-monotone instead of h-bi-quasi-semi-monotone, the result follows immediately from Theorem 2.

Since a semi-monotone operator is also an h-quasi-semi-monotone operator and an h-bi-quasi-semi-monotone operator, Theorems 1 and 2 are extensions of Theorems 4.4.6 and 4.4.7 respectively in [6]. The proof of Theorem 1 here is obtained by modifying the proof of Theorem 4.4.6 in [6]. Although M is h-quasi-semi-monotone or h-bi-quasi-semi-monotone instead of semi-monotone, there is no difference between the proof of Theorem 2 here and the proof of Theorem 4.4.7 in [6]. But for completeness we have included the proof of Theorem 2 here.

In Sections 3 and 4, we shall present out main non-compact contribution of this paper.

3. NON-COMPACT GENERALIZED BI-QUASI-VARIATIONAL INEQUALITIES FOR QUASI-SEMI-MONOTONE AND BI-QUASI-SEMI-MONOTONE OPERATORS

Let X be a topological space such that $X = \bigcup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact subsets of X. Then a sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be escaping from X relative to $\{C_n\}_{n=1}^{\infty}$ [3, p. 34] if for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $x_k \notin C_n$ for all $k \geq m$.

In this section, we shall apply Theorem 2 together with the concept of escaping sequences to obtain existence theorems on non-compact generalized bi-quasi-variational inequalities for quasi-semi-monotone and bi-quasi-semi-monotone operators.

We shall now establish the following result:

THEOREM 3 *Let E be a locally convex Hausdorff topological vector space over Φ, X be a non-empty (convex) subset of E such that $X = \bigcup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of X and F be a vector space over Φ. Let $\langle\, , \rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle\, , \rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x\rangle$ is continuous on X. Equip F with the strong topology $\delta\langle F, E\rangle$. Suppose that*

(1) *$S : X \to 2^X$ is a continuous map such that*
  (a) *for each $x \in X$, $S(x)$ is a closed convex subset of X and*
  (b) *for each $n \in \mathbb{N}$, $S(x) \subset C_n$ for all $x \in C_n$;*

(2) $T: X \to 2^F$ *is upper semicontinuous such that each* $T(x)$ *is* $\delta\langle F, E\rangle$-*compact convex*;
(3) $h: X \to \mathbb{R}$ *is convex and continuous*;
(4) $M: X \to 2^F$ *is upper hemicontinuous along line segments in* X *and* h-*bi-quasi-semi-monotone* (*with respect to* $\langle , \rangle$) *such that each* $M(x)$ *is* $\delta\langle F, E\rangle$-*compact convex*; *also, for each* $y \in \Sigma = \{y \in X: \sup_{x\in S(y)}[\inf_{f\in M(x)} \inf_{w\in T(y)} \mathrm{Re}\langle f - w, y - x\rangle + h(y) - h(x)] > 0\}$, M *is upper semicontinuous at some point* x *in* $S(y)$ *with* $\inf_{f\in M(x)} \inf_{w\in T(y)} \mathrm{Re}\langle f - w, y - x\rangle + h(y) - h(x) > 0$ *and* M *is upper semicontinuous on* C_n *for each* $n \in \mathbb{N}$;
(5) *for each sequence* $\{y_n\}_{n=1}^{\infty}$ *in* X, *with* $y_n \in C_n$ *for each* $n \in \mathbb{N}$, *which is escaping from* X *relative to* $\{C_n\}_{n=1}^{\infty}$, *either there exists* $n_0 \in \mathbb{N}$ *such that* $y_{n_0} \notin S(y_{n_0})$ *or there exist* $n_0 \in \mathbb{N}$ *and* $x_{n_0} \in S(y_{n_0})$ *such that* $\min_{f\in M(y_{n_0})} \min_{w\in T(y_{n_0})} \mathrm{Re}\langle f - w, y_{n_0} - x_{n_0}\rangle + h(y_{n_0}) - h(x_{n_0}) > 0$.

Then there exists a point $\hat{y} \in X$ *such that*

(i) $\hat{y} \in S(\hat{y})$ *and*
(ii) *there exist a point* $\hat{f} \in M(\hat{y})$ *and a point* $\hat{w} \in T(\hat{y})$ *with*

$$\mathrm{Re}\langle \hat{f} - \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y}) \quad \textit{for all } x \in S(\hat{y}).$$

Moreover, if $S(x) = X$ *for all* $x \in X$, E *is not required to be locally convex.*

Proof Fix an arbitrary $n \in \mathbb{N}$. Note that C_n is a non-empty compact convex subset of E. Define $S_n: C_n \to 2^{C_n}$, $h_n: C_n \to \mathbb{R}$ and M_n, $T_n: C_n \to 2^F$ by $S_n(x) = S(x)$, $h_n(x) = h(x)$, $M_n(x) = M(x)$ and $T_n(x) = T(x)$ respectively for each $x \in C_n$; i.e., $S_n = S|_{C_n}$, $h_n = h|_{C_n}$, $M_n = M|_{C_n}$ and $T_n = T|_{C_n}$ respectively. By Theorem 2, there exists a point $\hat{y}_n \in C_n$ such that

(i)′ $\hat{y}_n \in S_n(\hat{y}_n)$ and
(ii)′ there exists a point $\hat{f}_n \in M(\hat{y}_n) = M_n(\hat{y}_n)$ and a point $\hat{w}_n \in T(\hat{y}_n) = T_n(\hat{y}_n)$ with $\mathrm{Re}\langle \hat{f}_n - \hat{w}_n, \hat{y}_n - x\rangle \leq h(x) - h(\hat{y}_n)$ for all $x \in S_n(\hat{y}_n)$.

Note that $\{\hat{y}_n\}_{n=1}^{\infty}$ is a sequence in $X = \bigcup_{n=1}^{\infty} C_n$ with $\hat{y}_n \in C_n$ for each $n \in \mathbb{N}$.

Case 1 $\{\hat{y}_n\}_{n=1}^{\infty}$ is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$.

Then by hypothesis (5), there exists $n_0 \in \mathbb{N}$ such that $\hat{y}_{n_0} \notin S(\hat{y}_{n_0}) = S_{n_0}(\hat{y}_{n_0})$, which contradicts (i)′ or there exist $n_0 \in \mathbb{N}$ and

$x_{n_0} \in S(\hat{y}_{n_0}) = S_{n_0}(\hat{y}_{n_0})$ such that

$$\min_{f \in M(\hat{y}_{n_0})} \min_{w \in T(\hat{y}_{n_0})} \operatorname{Re}\langle f - w, \hat{y}_{n_0} - x_{n_0}\rangle + h(\hat{y}_{n_0}) - h(x_{n_0}) > 0,$$

which contradicts (ii)$'$.

Case 2 $\{\hat{y}_n\}_{n=1}^{\infty}$ is not escaping from X relative to $\{C_n\}_{n=1}^{\infty}$.

Then there exist $n_1 \in \mathbb{N}$ and a subsequence $\{\hat{y}_{n_j}\}_{j=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ such that $\hat{y}_{n_j} \in C_{n_1}$ for all $j = 1, 2, \ldots$ Since C_{n_1} is compact, there exist a subnet $\{\hat{z}_\alpha\}_{\alpha \in \Gamma}$ of $\{\hat{y}_{n_j}\}_{j=1}^{\infty}$ and $\hat{y} \in C_{n_1} \subset X$ such that $\hat{z}_\alpha \to \hat{y}$.

For each $\alpha \in \Gamma$, let $\hat{z}_\alpha = \hat{y}_{n_\alpha}$, where $n_\alpha \to \infty$. Then according to our choice of $\hat{y}_{n_\alpha}$ in C_{n_α}, we have

(i)$''$ $\hat{y}_{n\alpha} \in S_{n_\alpha}(\hat{y}_{n_\alpha}) = S(\hat{y}_{n_\alpha})$ and

(ii)$''$ there exist a point $\hat{f}_{n_\alpha} \in M_{n_\alpha}(\hat{y}_{n_\alpha}) = M(\hat{y}_{n_\alpha})$ and a point $\hat{w}_{n\alpha} \in T_{n_\alpha}(\hat{y}_{n_\alpha}) = T(\hat{y}_{n_\alpha})$ with $\operatorname{Re}\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x\rangle + h(\hat{y}_{n_\alpha}) - h(x) \le 0$ for all $x \in S_{n_\alpha}(\hat{y}_{n_\alpha}) = S(\hat{y}_{n_\alpha})$. Since $n_\alpha \to \infty$, there exists $\alpha_0 \in \Gamma$ such that $n_\alpha \ge n_1$ for all $\alpha \ge \alpha_0$. Thus $C_{n_1} \subset C_{n_\alpha}$, for all $\alpha \ge \alpha_0$. From (i)$''$ above we have $(\hat{y}_{n_\alpha}, \hat{y}_{n_\alpha}) \in G(S)$ for all $\alpha \in \Gamma$. Since S is upper semicontinuous with closed values, $G(S)$ is closed in $X \times X$; it follows that $\hat{y} \in S(\hat{y})$.

Moreover, since $\{\hat{f}_{n_\alpha}\}_{\alpha \ge \alpha_0}$ and $\{\hat{w}_{n_\alpha}\}_{\alpha \ge \alpha_0}$ are nets in the compact sets $\bigcup_{x \in C_{n_1}} M(x)$ and $\bigcup_{x \in C_{n_1}} T(x)$ respectively, without loss of generality, we may assume that the nets $\{\hat{f}_{n_\alpha}\}_{\alpha \in \Gamma}$ and $\{\hat{w}_{n_\alpha}\}_{\alpha \in \Gamma}$ converge to some $\hat{f} \in \bigcup_{x \in C_{n_1}} M(x)$ and some $\hat{w} \in \bigcup_{x \in C_{n_1}} T(x)$ respectively. Since M and T have closed graphs on C_{n_1}, $\hat{f} \in M(\hat{y})$ and $\hat{w} \in T(\hat{y})$.

Let $x \in S(\hat{y})$ be arbitrarily fixed. Let $n_2 \ge n_1$ be such that $x \in C_{n_2}$. Since S is lower semicontinuous at $\hat{y}$, without loss of generality we may assume that for each $\alpha \in \Gamma$, there is an $x_{n_\alpha} \in S(\hat{y}_{n_\alpha})$ such that $x_{n_\alpha} \to x$. By (ii)$''$ we have, $\operatorname{Re}\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle + h(\hat{y}_{n_\alpha}) - h(x_{n_\alpha}) \le 0$ for all $\alpha \in \Gamma$. Note that $\hat{f}_{n_\alpha} - \hat{w}_{n_\alpha} \to \hat{f} - \hat{w}$ in $\delta\langle F, E\rangle$ and $\{\hat{y}_{n_\alpha} - x_{n_\alpha}\}_{\alpha \in \Gamma}$ is a net in the compact (and hence bounded) set $C_{n_2} - \bigcup_{y \in C_{n_2}} S(y)$. Thus, we have for each $\epsilon > 0$, there exists $\alpha_1 \ge \alpha_0$ such that $|\operatorname{Re}\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha} - (\hat{f} - \hat{w}), \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle| < \epsilon/2$ for all $\alpha \ge \alpha_1$. Since $\langle \hat{f} - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle \to \langle \hat{f} - \hat{w}, \hat{y} - x\rangle$, there exists $\alpha_2 \ge \alpha_1$ such that $|\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle - \operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x\rangle| < \epsilon/2$ for all $\alpha \ge \alpha_2$. Thus for $\alpha \ge \alpha_2$,

$$\begin{aligned}
&|\operatorname{Re}\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle - \operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y} - x\rangle| \\
&\quad \le |\operatorname{Re}\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha} - (\hat{f} - \hat{w}), \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle| \\
&\qquad + |\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y}_{n_\alpha} - x_{n_\alpha} - (\hat{y} - x)\rangle| \\
&\quad < \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Thus

$$\lim_{\alpha} \mathrm{Re}\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle = \mathrm{Re}\langle \hat{f} - \hat{w}, \hat{y} - x\rangle.$$

By continuity of h, we have

$$\begin{aligned} &\mathrm{Re}\langle \hat{f} - \hat{w}, \hat{y} - x\rangle + h(\hat{y}) - h(x) \\ &\quad = \lim_{\alpha}[\mathrm{Re}\langle \hat{f}_{n_\alpha} - \hat{w}_{n_\alpha}, \hat{y}_{n_\alpha} - x_{n_\alpha}\rangle + h(\hat{y}_{n_\alpha}) - h(x_{n_\alpha})]. \\ &\quad \leq 0 \end{aligned}$$

COROLLARY 1 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a non-empty closed convex subset of E and F be a vector space over Φ. Let $\langle\, ,\rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle\, ,\rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x\rangle$ is continuous on X. Equip F with the strong topology $\delta\langle F, E\rangle$. Let $S: X \to 2^X$ be weakly continuous such that $S(x)$ is closed convex for each $x \in X$, $T: X \to 2^F$ be weakly upper semi-continuous such that each $T(x)$ is $\delta\langle F, E\rangle$-compact convex, $h: X \to \mathbb{R}$ be convex and (weakly) continuous and $M: X \to 2^F$ be (weakly) upper hemicontinuous along line segments in X and h-bi-quasi-semi-monotone (with respect to $\langle\, ,\rangle$) such that each $M(x)$ is $\delta\langle F, E\rangle$-compact convex. Also, for each $y \in \Sigma = \{y \in X: \sup_{x \in S(y)}[\inf_{f \in M(x)} \inf_{w \in T(y)} \mathrm{Re}\langle f - w, y - x\rangle + h(y) - h(x)] > 0\}$, M is weakly upper semicontinuous at some point x in $S(y)$ with $\inf_{f \in M(x)} \inf_{w \in T(y)} \mathrm{Re}\langle f - w, y - x\rangle + h(y) - h(x) > 0$ and M is weakly upper semicontinuous on C_n for each $n \in \mathbb{N}$. Suppose that*

(1) *there exists an increasing sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers with $r_n \to \infty$ such that $S(x) \subset C_n$ for each $x \in C_n$ and each $n \in \mathbb{N}$ where $C_n = \{x \in X: \|x\| \leq r_n\}$;*
(2) *for each sequence $\{y_n\}_{n=1}^{\infty}$ in X, with $\|y_n\| \to \infty$, either there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in S(y_{n_0})$ such that*

$$\min_{f \in M(y_{n_0})} \min_{w \in T(y_{n_0})} \mathrm{Re}\langle f - w, y_{n_0} - x_{n_0}\rangle + h(y_{n_0}) - h(x_{n_0}) > 0.$$

Then there exists $\hat{y} \in X$ such that

(a_1) *$\hat{y} \in S(\hat{y})$ and*
(b_1) *there exist a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ with*

$$\mathrm{Re}\langle \hat{f} - \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y}) \quad \textit{for all } x \in S(\hat{y}).$$

Proof Equip E with the weak topology. Then C_n is weakly compact convex for each $n \in \mathbb{N}$ such that $X = \bigcup_{n=1}^{\infty} C_n$. Now if $\{y_n\}_{n=1}^{\infty}$ is a sequence in X, with $y_n \in C_n$ for each $n = 1, 2, \ldots$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, then $\|y_n\| \to \infty$. By hypothesis (2), either there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in S(y_{n_0})$ such that $\min_{f \in M(y_{n_0})} \min_{w \in T(y_{n_0})} \operatorname{Re}\langle f - w, y_{n_0} - x_{n_0}\rangle + h(y_{n_0}) - h(x_{n_0}) > 0$. Thus all hypotheses of Theorem 3 are satisfied so that the conclusion follows.

When M is h-quasi-semi-monotone instead of h-bi-quasi-semi-monotone, the result follows immediately from Theorem 3.

By taking $M \equiv 0$ and replacing T by $-T$ in Theorem 3, we obtain the following result of Chowdhury and Tan in [8, Corollary 3]:

COROLLARY 2 *Let E be a locally convex Hausdorff topological vector space over Φ, X be a non-empty (convex) subset of E such that $X = \bigcup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of X and F be a vector space over Φ. Let $\langle\, ,\rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle\, ,\rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x\rangle$ is continuous on X. Equip F with the strong topology $\delta\langle F, E\rangle$. Suppose that*

(1) *$S : X \to 2^X$ is a continuous map such that*
(i)′ (b) *for each $x \in X$, $S(x)$ is a closed convex subset of X and*
(iii)′ (d) *for each $n \in \mathbb{N}$, $S(x) \subset C_n$ for all $x \in C_n$;*
(2) *$T : X \to 2^F$ is upper semicontinuous such that each $T(x)$ is $\delta\langle F, E\rangle$-compact convex;*
(3) *$h : X \to \mathbb{R}$ is convex and continuous;*
(4) *for each sequence $\{y_n\}_{n=1}^{\infty}$ in X, with $y_n \in C_n$ for each $n \in \mathbb{N}$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, either there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} \notin S(y_{n_0})$ or there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in S(y_{n_0})$ such that $\min_{w \in T(y_{n_0})} \operatorname{Re}\langle w, y_{n_0} - x_{n_0}\rangle + h(y_{n_0}) - h(x_{n_0}) > 0$. Then there exists a point $\hat{y} \in X$ such that*
   (i) *$\hat{y} \in S(\hat{y})$ and*
   (ii) *there exists a point $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x\rangle \le h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$. Moreover, if $S(x) = X$ for all $x \in X$, E is not required to be locally convex.*

4. NON-COMPACT GENERALIZED BI-COMPLEMENTARITY PROBLEMS FOR QUASI-SEMI-MONOTONE AND BI-QUASI-SEMI-MONOTONE OPERATORS

In this section, we shall obtain existence theorems on non-compact generalized bi-complementarity problems for quasi-semi-monotone and bi-quasi-semi-monotone operators.

By modifying the proof of the result observed by S.C. Fang (e.g. see [5, p. 213] and [10, p. 59]), the following result was obtained in [6, Lemma 4.4.10]:

LEMMA 3 *Let X be a cone in a topological vector space E over Φ and F be a vector space over Φ. Let $\langle\, ,\rangle : F\times E\to\Phi$ be a bilinear functional. Let $M, T: X\to 2^F$ be two maps. Then the following are equivalent*:

(a) *There exist $\hat{y}\in X, \hat{f}\in M(\hat{y})$ and $\hat{w}\in T(\hat{y})$ such that*

$$\operatorname{Re}\langle \hat{f}-\hat{w}, \hat{y}-x\rangle \le 0 \quad \textit{for all } x\in X.$$

(b) *There exist $\hat{y}\in X, \hat{f}\in M(\hat{y})$ and $\hat{w}\in T(\hat{y})$ such that*

$$\operatorname{Re}\langle \hat{f}-\hat{w}, \hat{y}\rangle = 0 \;\textit{ and }\; \hat{f}-\hat{w}\in\widehat{X}.$$

Proof (a) $\Rightarrow$ (b):

If $x=0$ by (a) we have $\operatorname{Re}\langle \hat{f}-\hat{w},\hat{y}\rangle\le 0$. Let $x=\lambda\hat{y}$, $\lambda>1$; then $\lambda\hat{y}\in X$. Substituting $x=\lambda\hat{y}$ in (a) we get $\operatorname{Re}\langle \hat{f}-\hat{w},\hat{y}-\lambda\hat{y}\rangle\le 0$. Thus $\operatorname{Re}\langle \hat{f}-\hat{w},(1-\lambda)\hat{y}\rangle\le 0$. Hence $(1-\lambda)\operatorname{Re}\langle \hat{f}-\hat{w},\hat{y}\rangle\le 0$ so that $\operatorname{Re}\langle \hat{f}-\hat{w},\hat{y}\rangle\ge 0$. Hence $\operatorname{Re}\langle \hat{f}-\hat{w},\hat{y}\rangle = 0$.

Now suppose that $\hat{f}-\hat{w}\notin\widehat{X}$. Then there exists $x\in X$ such that $\operatorname{Re}\langle \hat{f}-\hat{w},\hat{y}\rangle<0$. But then $\operatorname{Re}\langle \hat{f}-\hat{w},\hat{y}-x\rangle=\operatorname{Re}\langle \hat{f}-\hat{w},\hat{y}\rangle-\operatorname{Re}\langle \hat{f}-\hat{w},x\rangle=0-\operatorname{Re}\langle \hat{f}-\hat{w},x\rangle>0$, which contradicts (a). Therefore $\hat{f}-\hat{w}\in\widehat{X}$.

(b) $\Rightarrow$ (a):

We have $\operatorname{Re}\langle \hat{f}-\hat{w},\hat{y}-x\rangle=\operatorname{Re}\langle \hat{f}-\hat{w},\hat{y}\rangle-\operatorname{Re}\langle \hat{f}-\hat{w},x\rangle=0-\operatorname{Re}\langle \hat{f}-\hat{w},x\rangle\le 0$ for all $x\in X$.

When X is a cone in E, by applying Lemma 3 and Theorem 3 with $h\equiv 0$ and $S(x)=X$ for all $x\in X$, we have immediately the following existence theorem of a non-compact generalized bi-complementarity problem for bi-quasi-semi-monotone operator:

THEOREM 4 *Let E be a Hausdorff topological vector space over Φ, X be a cone in E such that $X = \bigcup_{n=1}^{\infty} C_n$, where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact convex subsets of X and F be a vector space over Φ. Let $\langle\, ,\rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle\, ,\rangle$ separates points in F and for each $f \in F$ the map $x \mapsto \langle f, x\rangle$ is continuous on X. Equip F with the strong topology $\delta\langle F, E\rangle$. Suppose that*

(1) *$T : X \to 2^F$ is upper semicontinuous such that each $T(x)$ is $\delta\langle F, E\rangle$-compact convex;*
(2) *$M : X \to 2^F$ is upper hemicontinuous along line segments in X and bi-quasi-semi-monotone (with respect to $\langle\, ,\rangle$) such that each $M(x)$ is $\delta\langle F, E\rangle$-compact convex; also, for each $y \in \Sigma = \{y \in X$: $\sup_{x\in S(y)}[\inf_{f\in M(x)} \inf_{w\in T(y)} \mathrm{Re}\langle f - w, y - x\rangle] > 0\}$, M is upper semicontinuous at some point x in $S(y)$ with $\inf_{f\in M(x)} \inf_{w\in T(y)} \mathrm{Re}\langle f - w, y - x\rangle > 0$ and M is upper semicontinuous on C_n for each $n \in \mathbb{N}$;*
(3) *for each sequence $\{y_n\}_{n=1}^{\infty}$ in X, with $y_n \in C_n$ for each $n \in \mathbb{N}$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in X$ such that*

$$\min_{f\in M(y_{n_0})} \min_{w\in T(y_{n_0})} \mathrm{Re}\langle f - w, y_{n_0} - x_{n_0}\rangle > 0.$$

Then there exist a point $\hat{y} \in X$, a point $\hat{f} \in M(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $\mathrm{Re}\langle \hat{f} - \hat{w}, \hat{y}\rangle = 0$ and $\hat{f} - \hat{w} \in \widehat{X}$.

COROLLARY 3 *Let $(E, \|\cdot\|)$ be a reflexive Banach space, X be a closed cone in E and F be a vector space over Φ. Let $\langle\, ,\rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle\, ,\rangle$ separates points in F and for each $f \in F$, the map $x \mapsto \langle f, x\rangle$ is continuous on X. Equip F with the strong topology $\delta\langle F, E\rangle$. Let $T : X \to 2^F$ be weakly upper semicontinuous such that each $T(x)$ is $\delta\langle F, E\rangle$-compact convex and $M : X \to 2^F$ be weakly upper hemicontinuous along line segments in X and bi-quasi-semi-monotone (with respect to $\langle\, ,\rangle$) such that each $M(x)$ is $\delta\langle F, E\rangle$-compact convex. Also, for each $y \in \Sigma = \{y \in X$: $\sup_{x\in S(y)}[\inf_{f\in M(x)} \inf_{w\in T(y)} \mathrm{Re}\langle f - w, y - x\rangle] > 0\}$, M is weakly upper semicontinuous at some point x in $S(y)$ with $\inf_{f\in M(x)} \inf_{w\in T(y)} \mathrm{Re}\langle f - w, y - x\rangle > 0$ and M is weakly upper semicontinuous on C_n for* *each* $n \in \mathbb{N}$. *Let* $\{r_n\}_{n=1}^{\infty}$ *be an increasing sequence of positive numbers with* $r_n \to \infty$ *and* $C_n = \{x \in X\colon \|x\| \le r_n\}$ *for each* $n \in \mathbb{N}$. *Suppose that for each sequence* $\{y_n\}_{n=1}^{\infty}$ *in* X, *with* $\|y_n\| \to \infty$, *there exist* $n_0 \in \mathbb{N}$ *and* $x_{n_0} \in X$ *such that* $\min_{f \in M(y_{n_0})} \min_{w \in T(y_{n_0})} \operatorname{Re}\langle f - w, y_{n_0} - x_{n_0}\rangle > 0$. *Then there exist* $\hat{y} \in X, \hat{f} \in M(\hat{y})$ *and* $\hat{w} \in T(\hat{y})$ *such that*

$$\operatorname{Re}\langle \hat{f} - \hat{w}, \hat{y}\rangle = 0 \quad \textit{and} \quad \hat{f} - \hat{w} \in \widehat{X}.$$

Proof Equip E with the weak topology. Then C_n is weakly compact convex for each $n \in \mathbb{N}$ such that $X = \bigcup_{n=1}^{\infty} C_n$. Now if $\{y_n\}_{n=1}^{\infty}$ is a sequence in X, with $y_n \in C_n$ for each $n = 1, 2, \ldots$, which is escaping from X relative to $\{C_n\}_{n=1}^{\infty}$, then $\|y_n\| \to \infty$. Hence by hypothesis, there exist $n_0 \in \mathbb{N}$ and $x_{n_0} \in X$ such that

$$\min_{f \in M(y_{n_0})} \min_{w \in T(y_{n_0})} \operatorname{Re}\langle f - w, y_{n_0} - x_{n_0}\rangle > 0.$$

Thus all hypotheses of Theorem 4 are satisfied so that the conclusion follows.

When M is a quasi-semi-monotone instead of bi-quasi-semi-monotone, the result follows immediately from Theorem 4.

5. APPLICATIONS TO MINIMIZATION PROBLEMS

In this section, as application of Theorem 2 on generalized bi-quasi-variational inequalities established in Section 2, we shall consider the existence of solutions for the following minimization problem:

$$\inf_{x \in E} \Gamma(x) \tag{5.1}$$

where Γ is the sum of two extended real-valued functions $g, h : E \to (-\infty, +\infty]$ and E is a topological vector space. We shall prove an existence theorem of solutions for (5.1). To this end we shall now introduce the following definition on subdifferential which is obtained by modifying the usual definition of subdifferential.

DEFINITION 4 *Let* E *be a topological vector space over* Φ, X *be a non-empty convex subset of* E *and* F *be a vector space over* Φ.

Let $\langle\,,\rangle : F \times E \to \Phi$ *be a bilinear functional. Suppose that* $\Gamma : X \to (-\infty, +\infty]$ *is a function with non-empty domain. Suppose* $\hat{y} \in D_\Gamma$, *the domain of* Γ. *Then we define the 'F-subdifferential* $F\text{-}\partial\Gamma(\hat{y})$ *of* Γ *at* $\hat{y}$' *as the subset of F defined by*

$$F\text{-}\partial\Gamma(\hat{y}) = \{p \in F:\ \Gamma(\hat{y}) - \Gamma(x) \leq \operatorname{Re}\langle p, \hat{y} - x\rangle \text{ for all } x \in X\}. \tag{5.2}$$

The mapping $F\text{-}\partial\Gamma : D_\Gamma \to F$ *is said to be the F-subdifferential map.* Γ *is said to be F-subdifferentiable on X if* $F\text{-}\partial\Gamma(x) \neq 0$ *for all* $x \in X$. *The elements* $p \in F\text{-}\partial\Gamma(\hat{y})$ *are said to be F-subgradients.*

PROPOSITION 2 *Let E be a topological vector space over* Φ, *X be a non-empty convex subset of E and F be a vector space over* Φ. *Let* $\langle\,,\rangle : F \times E \to \Phi$ *be a bilinear functional. Suppose that* $\Gamma : X \to (-\infty, +\infty]$ *is F-subdifferentiable. Let* $M : X \to 2^F$ *be defined by* $M(x) = F\text{-}\partial\Gamma(x)$. *Then M is F-monotone, i.e., monotone with respect to the bilinear functional* $\langle\,,\rangle$.

Proof Suppose $x, y \in X$, $p \in M(x)$ and $q \in M(y)$. Then we have

$$\langle p, x - y\rangle \geq \Gamma(x) - \Gamma(y)$$

and

$$\langle q, y - x\rangle \geq \Gamma(y) - \Gamma(x).$$

Thus $\langle p, x - y\rangle - \langle q, x - y\rangle = \langle p, x - y\rangle + \langle q, y - x\rangle \geq 0$. Hence M is F-monotone, i.e., monotone with respect to the bilinear functional $\langle\,,\rangle$.

We shall now give the following proposition which will show that the existence of solutions of generalized bi-quasi-variational inequalities guarantee the existence of the minimizers for the minimization problem (5.1).

PROPOSITION 3 *Let E be a Hausdorff topological vector space over* Φ, *X be a non-empty convex subset of E and F be a vector space over* Φ. *Let* $\langle\,,\rangle : F \times E \to \Phi$ *be a bilinear functional. Suppose that* $\Gamma = g + h$ *where* $g, h : X \to (-\infty, +\infty]$, *g is an F-subdifferential map and h is a convex function. Then a point* $\hat{y} \in X$ *minimizes* Γ *if there exists* $p \in F\text{-}\partial g(\hat{y})$ *such that*

$$\sup_{x \in X}[\operatorname{Re}\langle p, \hat{y} - x\rangle + h(\hat{y}) - h(x)] \leq 0. \tag{5.3}$$

Proof Suppose there exists a point $p \in F\text{-}\partial g(\hat{y})$ such that $\mathrm{Re}\langle p, \hat{y} - x\rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in X$. Then we have

$$\begin{aligned}\Gamma(\hat{y}) - \Gamma(x) &= (g+h)(\hat{y}) - (g+h)(x)\\ &= g(\hat{y}) - g(x) + h(\hat{y}) - h(x)\\ &\leq \mathrm{Re}\langle p, \hat{y} - x\rangle + h(\hat{y}) - h(x) \quad \text{for all } x \in X\\ &\leq 0 \quad \text{for all } x \in X.\end{aligned}$$

Thus $\Gamma(\hat{y}) \leq \Gamma(x)$ for all $x \in X$. Hence $\Gamma(\hat{y}) = \inf_{x\in X}\Gamma(x)$. Therefore $\hat{y}$ minimizes Γ.

THEOREM 5 *Let E be a Hausdorff topological vector space over Φ and F be a vector space over Φ. Let $\langle\,,\rangle : F \times E \to \Phi$ be a bilinear functional such that $\langle\,,\rangle$ separates points in F and for each $u \in F$, the map $x \mapsto \langle u, x\rangle$ is continuous on E. Equip F with the strong topology $\delta\langle F, E\rangle$. Let $\Gamma : E \to \mathbb{R}$ be a function. Suppose that*

(a) *$h : E \to \mathbb{R}$ is convex and continuous;*
(b) *$M : E \to 2^F$, defined by $M(x) = F\text{-}\partial\Gamma(x)$ for each $x \in E$ is, upper hemicontinuous along line segments in E such that each $M(x)$ is $\delta\langle F, E\rangle$-compact convex; also, for each $y \in \Sigma = \{y \in E: \sup_{x\in E}[\inf_{f\in M(x)} \mathrm{Re}\langle f, y - x\rangle + h(y) - h(x)] > 0\}$, M is upper semicontinuous at some point x in E with $\inf_{f\in M(x)} \mathrm{Re}\langle f, y - x\rangle + h(y) - h(x) > 0$.*

Then there exist a point $\hat{y} \in E$ and a point $p \in M(\hat{y}) = F - \partial\Gamma(\hat{y})$ such that

$$\sup_{x\in E}[\mathrm{Re}\langle p, \hat{y} - x\rangle + h(\hat{y}) - h(x)] \leq 0,$$

i.e., $\hat{y}$ minimizes Γ on E.

Proof Let $S : E \to 2^E$ be defined by $S(x) = E$ for all $x \in E$ and $T \equiv 0$ in Theorem 2. Since M is F-subdifferentiable, by Proposition 2, M is F-monotone, i.e., monotone with respect to the bilinear functional $\langle\,,\rangle$. Thus M is h-bi-quasi-semi-monotone (respectively, h-quasi-semi-monotone) with respect to $\langle\,,\rangle$. Then from Theorem 2 with $X = E$ and $T \equiv 0$, it follows that there exist a point $\hat{y} \in E$ and a point $p \in M(\hat{y}) = F\text{-}\partial\Gamma(\hat{y})$ such that

$$\operatorname{Re}\langle p, \hat{y} - x\rangle + h(\hat{y}) - h(x) \leq 0$$

for all $x \in X$. Thus by Proposition 3 it follows that $\hat{y}$ minimizes Γ on E.

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