

A New Approach to the Extragradient Method for Nonlinear Variational Inequalities

RAM U. VERMA

*International Publications, USA, Mathematical Sciences Division,
12046 Coed Drive, Suite A-29, Orlando, FL 32826, USA*

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The approximation-solvability of the following nonlinear variational inequality (NVI) problem is presented:

Determine an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geq 0 \quad \text{for all } x \in K,$$

where $T: K \rightarrow H$ is a mapping from a nonempty closed convex subset K of a real Hilbert space H into H . The iterative procedure is characterized as a nonlinear variational inequality (for any arbitrarily chosen initial point $x^0 \in K$)

$$\begin{aligned} \langle \rho T(P_K[x^k - \rho T(x^k)]) + x^{k+1} - x^k, x - x^{k+1} \rangle \geq 0 \\ \text{for all } x \in K \text{ and for } k \geq 0, \end{aligned}$$

which is equivalent to a double projection formula

$$x^{k+1} = P_K[x^k - \rho T(P_K[x^k - \rho T(x^k)])],$$

where P_K denotes the projection of H onto K .

Keywords: Extragradient method; g - α -cocoercive mapping; Double projection equation; Nonlinear quasivariational inequalities; Iterative algorithms; Expanding mappings

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1. INTRODUCTION

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $T: K \rightarrow H$ be a mapping and K a closed convex subset of H . We present the convergence of a sequence $\{x^k\}$ generated by a double projection formula to a solution of the nonlinear variational inequality (NVI) problem: find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geq 0 \quad \text{for all } x \in K, \quad (1.1)$$

which is equivalent to a projection equation

$$x^* = P_K[x^* - \rho T(x^*)] \quad \text{for } \rho > 0, \quad (1.2)$$

where P_K is the projection of H onto K .

Next, we consider an auxiliary nonlinear variational inequality (ANVI) problem: find an element $x^* \in K$ such that

$$\langle T(P_K[x^* - \rho T(x^*)]), x - x^* \rangle \geq 0 \quad \text{for all } x \in K, \quad (1.3)$$

which is equivalent to a double projection formula

$$x^* = P_K[x^* - \rho T(P_K[x^* - \rho T(x^*)])]. \quad (1.4)$$

ALGORITHM 1.1 For an arbitrarily chosen initial point $x^0 \in K$, we consider an iterative algorithm generated by the following variational inequality (for $k \geq 0$):

$$\begin{aligned} \langle \rho T(P_K[x^0 - \rho T(x^0)]) + x^1 - x^0, x - x^1 \rangle &\geq 0 \quad \text{for all } x \in K, \\ &\vdots \\ \langle \rho T(P_K[x^k - \rho T(x^k)]) + x^{k+1} - x^k, x - x^{k+1} \rangle &\geq 0 \quad \text{for all } x \in K. \end{aligned} \quad (1.5)$$

The iterative procedure (1.5) is equivalent to a double projection equation

$$x^{k+1} = P_K[x^k - \rho T(P_K[x^k - \rho T(x^k)])] \quad \text{for } k \geq 0. \quad (1.6)$$

The extragradient method, introduced by Korpelevich [13], is applicable to the solvability of the monotone variational inequalities using the iterative algorithm (1.6) with T Lipschitz continuous. Among several other existing methods in use to the solvability of the NVI problem (1.1), the projection method, where the iterative scheme is constructed based on the projection Eq. (1.2), is the simplest, but it restricts T or T^{-1} to be strongly monotone for convergence. The extragradient method overcomes this problem by updating x in the double projection formula (1.4), where ρ is the positive stepsize. Since it uses only function evaluations and projection onto K , it is easy to implement, requires a little storage, and can readily exploit any sparsity problem or separable structure in T or K , as has been the case in others. On the top of that, its convergence, in contrast to other methods, requires only a solution to exist. We intend, unlike the case of Korpelevich [13], in which the convergence is achieved as usual using the iterative algorithm generated by the double projection equation, to establish the convergence of the sequence constructed by an iterative procedure characterized as a variational inequality instead to a solution of the NVI problem (1.1). This variational inequality iterative scheme is an extension to that of Marcotte and Wu [14]. Recently Marcotte and Wu [14] established the convergence of the projection method by using an iterative algorithm characterized as a variational inequality for the monotone variational inequality problem involving cocoercive mappings in R^n . Similar estimates for convergence using the projection equation type iterations are studied by He [8–11], Korpelevich [13], Chan and Pang [2], and others. For more details on the solvability of the nonlinear variational inequalities and related materials, we refer to [1–21].

As far as the approximation-solvability of the NVI problem (1.1) based on iterative algorithms is concerned, we mention the following criteria, most commonly used in the literature:

LEMMA 1.1 *An element $u \in K$ is a solution of the NVI problem (1.1) if and only if*

$$u = P_K[u - \rho T(u)] \quad \text{for } \rho > 0,$$

where $T: K \rightarrow H$ is a mapping from a closed convex subset K of a real Hilbert space H into H and P_K is the projection of H onto K .

LEMMA 1.2 *An element $u \in K$ is a solution of the NVI problem (1.1) if and only if*

$$R(u) := u - P_K[u - \rho T(u)] = 0,$$

where $R(u)$ denotes the residue function.

LEMMA 1.3 *An element $u \in K$ is a solution of the NVI problem (1.1) if*

$$\langle T(u), x - u \rangle \geq 0 \quad \text{for all } x \in K.$$

2. CONVERGENCE AND SOLVABILITY

In this section, we consider the approximation-solvability of the NVI problem (1.1) and ANVI problem (1.3) involving α -cocoercive mappings along with a discussion of an alternative to the existing notion of α -cocoercivity [14], which is also referred as the Dunn property [5,6]. The author [20] introduced an alternative to the existing notion of the cocoercivity [14] – new, yet compatible with the existing notion of the cocoercivity [14] – in the following manner:

A mapping $T: H \rightarrow H$ is said to be α -cocoercive if for all $x, y \in H$, we have

$$\|x - y\|^2 \geq \alpha^2 \|T(x) - T(y)\|^2 + \|\alpha(T(x) - T(y)) - (x - y)\|^2,$$

where $\alpha > 0$ is a constant.

A mapping $T: H \rightarrow H$ is called α -cocoercive [14] if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq \alpha \|T(x) - T(y)\|^2 \quad \text{for all } x, y \in H.$$

A mapping $T: H \rightarrow H$ is said to be g - α -cocoercive if there exist a mapping $g: H \rightarrow H$ and a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), g(x) - g(y) \rangle \geq \alpha \|T(x) - T(y)\|^2 \quad \text{for all } x, y \in H.$$

This implies that

$$\|T(x) - T(y)\| \leq (1/\alpha)\|g(x) - g(y)\|,$$

that is T is g - $(1/\alpha)$ -Lipschitz continuous. When $g \equiv I$ (identity), T is referred as $(1/\alpha)$ -Lipschitz continuous.

A mapping $T: H \rightarrow H$ is called r -strongly monotone if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq r\|x - y\|^2 \quad \text{for all } x, y \in H.$$

This implies that

$$\|T(x) - T(y)\| \geq r\|x - y\|,$$

that is, T is r -expanding. When $r = 1$, T is called an expanding mapping.

We note that if T is α -cocoercive and expanding, then T is α -strongly monotone. On the top of that if T is α -strongly monotone and β -Lipschitz continuous, then T is (α/β^2) -cocoercive for $\beta > 0$. Clearly every α -cocoercive mapping T is $(1/\alpha)$ -Lipschitz continuous.

PROPOSITION 2.1 *Let $T: H \rightarrow H$ be a mapping from a Hilbert space H into itself. Then the following statements are equivalent:*

(1) *For each $x, y \in H$ and for a constant $\alpha > 0$, we have*

$$\|x - y\|^2 \geq \alpha^2\|T(x) - T(y)\|^2 + \|\alpha(T(x) - T(y)) - (x - y)\|^2.$$

(2) *For each $x, y \in H$, we have*

$$\langle T(x) - T(y), x - y \rangle \geq \alpha\|T(x) - T(y)\|^2,$$

where $\alpha > 0$ is a constant.

LEMMA 2.1 *For all $v, w \in H$, we have*

$$\|v\|^2 + \langle v, w \rangle \geq -(1/4)\|w\|^2.$$

LEMMA 2.2 *Let $v, w \in H$. Then we have*

$$\langle v, w \rangle = (1/2)[\|v + w\|^2 - \|v\|^2 - \|w\|^2].$$

LEMMA 2.3 [12] *Let an element $z \in H$. Then $u = P_K(z)$ if and only if $u \in K$ and*

$$\langle u - z, y - u \rangle \geq 0 \quad \text{for all } y \in K.$$

LEMMA 2.4 *An element $u \in K$ is a solution of the NVI problem (1.1) if and only if u is a fixed point of the projection $P_K[u - \rho T(u)]$.*

LEMMA 2.5 *Let $T: K \rightarrow H$ be an α -cocoercive mapping. Then an element $u \in K$ is a solution of the NVI problem (1.1) if and only if $u \in K$ is a solution of the ANVI problem (1.3).*

THEOREM 2.1 *Let H be a real Hilbert space and $T: K \rightarrow H$ an α -cocoercive mapping from a nonempty closed convex subset K of H into H . Let $x^* \in K$ be a solution of the NVI problem (1.1) and $\{x^k\}$ the sequence generated by iteration (1.5). Then we have:*

(i) *The estimate*

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - [1 - (\rho/2\alpha)]\|x^k - x^{k+1}\|^2.$$

(ii) *The sequence $\{x^k\}$ converges to x^* for $\rho < 2\alpha$.*

Proof Since by Lemma 2.5, a solution of the NVI problem (1.1) is also a solution of the ANVI problem (1.3), it suffices to show that the sequences $\{x^k\}$ generated by the iterative algorithm (1.5) converges to x^* , a solution of the NVI problem (1.3). Since x^{k+1} satisfies the iterative algorithm (1.5), we have for a constant $\rho > 0$ that

$$\langle \rho T(P_K[x^k - \rho T(x^k)]) + x^{k+1} - x^k, x - x^{k+1} \rangle \geq 0 \quad \text{for all } x \in K. \quad (2.1)$$

Thus, for a given solution x^* of the NVI problem (1.3), we have for a constant $\rho > 0$ that

$$\langle \rho T(P_K[x^* - \rho T(x^*)]), x - x^* \rangle \geq 0. \quad (2.2)$$

Replacing x by x^* in (2.1) and x by x^{k+1} in (2.2), and adding, we obtain

$$\begin{aligned} 0 &\leq \langle \rho\{T(P_K[x^k - \rho T(x^k)]) - T(P_K[x^* - \rho T(x^*)])\}, x^* - x^{k+1} \rangle \\ &\quad + \langle x^{k+1} - x^k, x^* - x^{k+1} \rangle \\ &= -\rho \langle T(P_K[x^k - \rho T(x^k)]) - T(P_K[x^* - \rho T(x^*)]), x^k - x^* \rangle \\ &\quad - \rho \langle T(P_K[x^k - \rho T(x^k)]) - T(P_K[x^* - \rho T(x^*)]), x^{k+1} - x^k \rangle \\ &\quad + \langle x^{k+1} - x^k, x^* - x^{k+1} \rangle. \end{aligned}$$

Since by (1.2), $x^k = P_K[x^k - \rho T(x^k)]$ and by Lemma 1.1, $x^* = P_K[x^* - \rho T(x^*)]$, and since T is α -cocoercive, we have

$$\begin{aligned} 0 &\leq -\sigma\alpha \|T(P_K[x^k - \rho T(x^k)]) - T(P_K[x^* - \rho T(x^*)])\|^2 \\ &\quad - \rho \langle T(P_K[x^k - \rho T(x^k)]) - T(P_K[x^* - \rho T(x^*)]), x^{k+1} - x^k \rangle \\ &\quad + \langle x^{k+1} - x^k, x^* - x^{k+1} \rangle \\ &= -\rho\alpha \{ \|T(P_K[x^k - \rho T(x^k)]) - T(P_K[x^* - \rho T(x^*)])\|^2 \\ &\quad + (1/\alpha) \langle T(P_K[x^k - \rho T(x^k)]) - T(P_K[x^* - \rho T(x^*)]), x^{k+1} - x^k \rangle \} \\ &\quad + \langle x^{k+1} - x^k, x^* - x^{k+1} \rangle. \tag{2.3} \end{aligned}$$

Setting $v = T(P_K[x^k - \rho T(x^k)]) - T(P_K[x^* - \rho T(x^*)])$ and $w = (1/\alpha) \times [x^{k+1} - x^k]$ in Lemma 2.1, we obtain

$$\begin{aligned} &-\{ \|T(P_K[x^k - \rho T(x^k)]) - T(P_K[x^* - \rho T(x^*)])\|^2 \\ &\quad + (1/\alpha) \langle T(P_K[x^k - \rho T(x^k)]) - T(P_K[x^* - \rho T(x^*)]), x^{k+1} - x^k \rangle \} \\ &\leq (1/4\alpha^2) \|x^{k+1} - x^k\|^2. \tag{2.4} \end{aligned}$$

Applying (2.4) to (2.3), we get

$$0 \leq (\rho/4\alpha) \|x^{k+1} - x^k\|^2 + \langle x^{k+1} - x^k, x^* - x^{k+1} \rangle \tag{2.5}$$

Taking $v = x^{k+1} - x^k$ and $w = x^* - x^{k+1}$ in Lemma 2.2, and applying to (2.5), we have

$$0 \leq (\rho/4\alpha)\|x^{k+1} - x^k\|^2 + (1/2)[\|x^* - x^k\|^2 - \|x^{k+1} - x^k\|^2 - \|x^* - x^{k+1}\|^2].$$

This implies that

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - [1 - (\rho/2\alpha)]\|x^{k+1} - x^k\|^2. \quad (2.6)$$

Since $\rho < 2\alpha$, it follows from (2.6) that

$$\text{either } \lim_{k \rightarrow \infty} \|x^k - x^*\| = 0, \quad \text{or} \quad \lim_{k \rightarrow \infty} \|x^k - x^{k+1}\| = 0.$$

Assume that the first alternative holds. Then the sequence $\{x^k\}$ converges to x^* and

$$\lim_{k \rightarrow \infty} \|x^k - x^{k+1}\| = 0.$$

as well.

Next, assume that the second alternative holds, that is,

$$\lim_{k \rightarrow \infty} \|x^k - x^{k+1}\| = 0.$$

Let \bar{x} be a cluster point of the sequence $\{x^k\}$. Then there exists a subsequence $\{x^{k_i}\}$ such that $\{x^{k_i}\}$ converges to \bar{x} since the left hand term of (2.6) is bounded. Finally, the continuity of the projection (1.2), in light of the $(1/\alpha)$ -Lipschitz continuity of T , implies that \bar{x} is a fixed point of the projection (1.2) and, as a result, \bar{x} is a solution of the NVI problem (1.1) by Lemma 2.5. This completes the proof.

3. QUASIVARIATIONAL INEQUALITIES

Let $T, g : H \rightarrow H$ be single-valued mappings on H and K a closed convex subset of H . We consider a nonlinear quasivariational inequality (NQVI) problem: find an element $u \in H$ such that $g(u) \in K$ and

$$\langle T(u), g(x) - g(u) \rangle \geq 0 \quad \text{for all } g(x) \in K \text{ and for } x \in H, \quad (3.1)$$

which is equivalent to a projection equation

$$g(u) = P_K[g(u) - \rho T(u)] \quad \text{for } \rho > 0, \tag{3.2}$$

where P_K is the projection of H onto K .

Next, we consider an auxiliary nonlinear quasivariational inequality (ANQVI) problem: find an element $u \in H$ such that $g(u) \in K$ and

$$\langle T(P_K[g(u) - \rho T(u)]), g(x) - g(u) \rangle \geq 0 \quad \text{for all } g(x) \in K, \tag{3.3}$$

which is equivalent to a double projection formula

$$g(u) = P_K[g(u) - \rho T(P_K[g(u) - \rho T(u)])]. \tag{3.4}$$

ALGORITHM 3.1 For an arbitrarily chosen initial point $x^0 \in H$, we consider an iterative algorithm generated by the following variational inequality (for $k \geq 0$):

$$\langle \rho T(P_K[g(x^0) - \rho T(x^0)]) + g(x^1) - g(x^0), g(x) - g(x^1) \rangle \geq 0$$

for all $g(x) \in K$,

⋮

$$\langle \rho T(P_K[g(x^k) - \rho T(x^k)]) + g(x^{k+1}) - g(x^k), g(x) - g(x^{k+1}) \rangle \geq 0$$

for all $g(x) \in K$. (3.5)

The iterative procedure (3.5) is equivalent to a double projection equation

$$g(x^{k+1}) = P_K[g(x^k) - \rho T(P_K[g(x^k) - \rho T(x^k)])] \quad \text{for } k \geq 0. \tag{3.6}$$

LEMMA 3.1 An element $u \in H$ is a solution of the NQVI Problem (3.1) if and only if u is a solution of the ANQVI problem (3.3).

THEOREM 3.1 Let H be a real Hilbert space, $T, g : H \rightarrow H$ any mappings on H , and $x^* \in H$ a solution of the NQVI problem (3.1). Suppose that the following assumptions hold:

- (i) T is g - α -cocoercive.
- (ii) g is an expanding mapping.
- (iii) The sequence $\{x^k\}$ is generated by the iterative algorithm (3.5).

Then we have the following conclusions:

(a) *The estimate*

$$\begin{aligned} & \|g(x^{k+1}) - g(x^*)\|^2 \\ & \leq \|g(x^k) - g(x^*)\|^2 - [1 - (\rho/2\alpha)]\|g(x^k) - g(x^{k+1})\|^2. \end{aligned}$$

(b) *The sequence $\{x^k\}$ converges to x^* for $\rho < 2\alpha$.*

Proof Since x^* is a solution of the NQVI problem (3.1) and hence, $g(x^*) = P_K[g(x^*) - \rho T(x^*)]$, it satisfies (3.3), that is,

$$\rho \langle T(P_K[g(x^*) - \rho T(x^*)]), g(x) - g(x^*) \rangle \geq 0. \quad (3.7)$$

In light of algorithm (3.5), we can write

$$\begin{aligned} & \langle \rho T(P_K[g(x^k) - \rho T(x^k)]) + g(x^{k+1}) - g(x^k), g(x) - g(x^{k+1}) \rangle \geq 0 \\ & \text{for all } g(x) \in K. \end{aligned} \quad (3.8)$$

Replacing x by x^{k+1} in (3.7) and by x^* in (3.8), and adding, we obtain

$$\begin{aligned} 0 & \leq \langle \rho \{ T(P_K[g(x^k) - \rho T(x^k)]) - T(P_K[g(x^*) - \rho T(x^*)]) \} \\ & \quad + g(x^{k+1}) - g(x^k), g(x^*) - g(x^{k+1}) \rangle \\ & = -\rho \langle T(P_K[g(x^k) - \rho T(x^k)]) \\ & \quad - T(P_K[g(x^*) - \rho T(x^*)]), g(x^k) - g(x^*) \rangle \\ & \quad - \rho \langle T(P_K[g(x^k) - \rho T(x^k)]) \\ & \quad - T(P_K[g(x^*) - \rho T(x^*)]), g(x^{k+1}) - g(x^k) \rangle \\ & \quad + \langle g(x^{k+1}) - g(x^k), g(x^*) - g(x^{k+1}) \rangle. \end{aligned}$$

Since, by (3.2), $g(x^k) = P_K[g(x^k) - \rho T(x^k)]$ and $g(x^*) = P_K[g(x^*) - \rho T(x^*)]$, and since T is g - α -cocoercive, we have

$$\begin{aligned} 0 & \leq -\rho\alpha \{ \| T(P_K[g(x^k) - \rho T(x^k)]) - T(P_K[g(x^*) - \rho T(x^*)]) \|^2 \\ & \quad + (1/\alpha) \langle T(P_K[g(x^k) - \rho T(x^k)]) \\ & \quad - T(P_K[g(x^*) - \rho T(x^*)]), g(x^{k+1}) - g(x^k) \rangle \} \\ & \quad + \langle g(x^{k+1}) - g(x^k), g(x^*) - g(x^{k+1}) \rangle. \end{aligned} \quad (3.9)$$

Applying Lemma 2.1 by taking $v = T(P_K[g(x^k) - \rho T(x^k)]) - T(P_K[g(x^*) - \rho T(x^*)])$ and $w = [g(x^{k+1}) - g(x^k)]/\alpha$, it follows from (3.9) that

$$0 \leq (\rho/4\alpha)\|g(x^{k+1}) - g(x^k)\|^2 + \langle g(x^{k+1}) - g(x^k), g(x^*) - g(x^{k+1}) \rangle. \quad (3.10)$$

Next applying Lemma 2.2 to (3.10) taking $v = g(x^{k+1}) - g(x^k)$ and $w = g(x^*) - g(x^{k+1})$, we arrive at

$$0 \leq (\rho/4\alpha)\|g(x^{k+1}) - g(x^k)\|^2 + (1/2)[\|g(x^*) - g(x^k)\|^2 - \|g(x^{k+1}) - g(x^k)\|^2 - \|g(x^*) - g(x^{k+1})\|^2].$$

This implies that

$$\begin{aligned} & \|g(x^{k+1}) - g(x^*)\|^2 \\ & \leq \|g(x^k) - g(x^*)\|^2 - [1 - (\rho/2\alpha)]\|g(x^{k+1}) - g(x^k)\|^2. \end{aligned}$$

Since the convergence of the sequence $\{g(x^k)\}$ to $g(x^*)$ is similar to that of Theorem 2.1, all we need is to show that the sequence $\{x^k\}$ converges to x^* . As given g is expanding, we have

$$\|x^k - x^*\| \leq \|g(x^k) - g(x^*)\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

This concludes the proof.

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