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A Priori Estimates for the Existence of a Solution for a Multi-Point Boundary Value Problem

CHAITAN P. GUPTA[a,*] and SERGEI TROFIMCHUK[b]

[a] *Department of Mathematics, University of Nevada, Reno, Reno, NV 89557, USA;* [b] *Departamento de Matemática, Facultad de Ciencias, Universidade de Chile, Casilla 653, Santiago, Chile*

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Let $a_i \in R, \xi_i \in (0,1), i = 1,2,\ldots,m-2, 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, with $\alpha = \sum_{i=1}^{m-2} a_i \neq 1$ be given. Let $x(t) \in W^{2,1}(0,1)$ be such that $x'(0) = 0, x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ $(*)$ be given. This paper is concerned with the problem of obtaining Poincaré type *a priori* estimates of the form $\|x\|_\infty \leq C\|x''\|_1$. The study of such estimates is motivated by the problem of existence of a solution for the Caratheodory equation $x''(t) = f(t, x(t), x'(t)) + e(t),\ 0 < t < 1$, satisfying boundary conditions $(*)$. This problem was studied earlier by Gupta *et al.* (*Jour. Math. Anal. Appl.* **189** (1995), 575–584) when the a_i's, all had the same sign.

Keywords: *A priori* estimates; Multi-point boundary value problem; Leray–Schauder continuation theorem; Caratheodory's conditions

AMS(MOS) Subject Classification: 34B10, 34B15, 34G20

1 INTRODUCTION

Let $a_i \in R, \xi_i \in (0,1), i = 1,2,\ldots,m-2, 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, with $\alpha = \sum_{i=1}^{m-2} a_i \neq 1$ be given. Let $x(t) \in W^{2,1}(0,1)$ be such that $x'(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ be given. We are interested in obtaining Poincaré type *a priori* estimates of the form

$$\|x\|_\infty \leq C\|x''\|_1. \tag{1}$$

* Corresponding author.

The study of such estimates is motivated by the problem of existence of a solution for the multi-point boundary value problem

$$\begin{aligned} &x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \\ &x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{aligned} \tag{2}$$

where $f\colon [0,1] \times R^2 \rightarrow R$ is a function satisfying Caratheodory's conditions and $e: [0,1] \mapsto R$ be a function in $L^1[0,1]$. We also obtain some sharp Poincaré type estimates of the form $\|x\|_\infty \leq C\|x''\|_1$ when $m = 3$, i.e. for $x(t) \in W^{2,1}(0,1)$ with $x'(0) = 0$, $x(1) = \alpha x(\eta)$, where $\eta \in (0,1)$ and $\alpha \in R$ are given. We apply our estimates to the problem of existence of a solution for the multi-point boundary value problem (2) and for the three-point boundary value problem

$$\begin{aligned} &x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1, \\ &x'(0) = 0, \quad x(1) = \alpha x(\eta). \end{aligned} \tag{3}$$

We present the existence theorems for the boundary value problems (2) and (3) in Section 3.

The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev in [19,20] motivated by the work of Bitsadze and Samarski on non-local linear elliptic boundary problems, [1–3]. We refer the reader to [4–17] for some recent results on nonlinear multi-point boundary value problems.

2 *A PRIORI* ESTIMATES

In this section, we will establish some *a priori* estimates of the form (1). We recall that for $a \in R$, $a_+ = \max\{a, 0\}$, $a_- = \max\{-a, 0\}$ so that $a = a_+ - a_-$ and $|a| = a_+ + a_-$.

THEOREM 1 *Let $a_i \in R$, $\xi_i \in (0,1)$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, with $\alpha = \sum_{i=1}^{m-2} a_i \neq 1$ be given. Then for $x(t) \in W^{2,1}(0,1)$ with $x'(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ we have*

$$\|x\|_\infty \leq C\|x''\|_1, \tag{4}$$

where

$$C = \max \left\{ \begin{array}{l} \sum_{i=1}^{m-2} \left(\frac{a_i}{1-\alpha}\right)_- (1-\xi_i), \sum_{i=1}^{m-2} \left(\frac{a_i}{1-\alpha}\right)_- \xi_i + \left(\frac{1}{1-\alpha}\right)_+, \\ \sum_{i=1, i\neq j}^{m-2} \left(\frac{a_i}{1-\alpha}\right)_- |\xi_i - \xi_j| + \left(\frac{1}{1-\alpha}\right)_+ (1-\xi_j), \\ j = 1, 2, \ldots, m-2 \end{array} \right\}.$$

Proof Since $x(t) \in W^{2,1}(0,1)$ there exists a $c \in [0,1]$ such that $\|x\|_\infty = |x(c)|$. We may assume that $x(c) > 0$, by replacing $x(t)$ by $-x(t)$, if necessary. Now, two cases arise; either $c \in [0,1)$ or $c = 1$. In case, $c \in [0,1)$ we must have $x'(c) = 0$. First we set $a_{m-1} = -1$ and $\xi_{m-1} = 1$, to write the condition $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ in the symmetric form $\sum_{i=1}^{m-1} a_i x(\xi_i) = 0$ and then we apply the Taylor's formula with integral remainder after second term at each $\xi_i \in (0,1)$, $i = 1, 2, \ldots, m-1$, to get

$$\begin{aligned} &x(\xi_i) = x(c) + r_i, \\ &\quad \text{where } r_i = \int_c^{\xi_i} (\xi_i - s) x''(s)\, \mathrm{d}s \leq 0, \quad i = 1, 2, \ldots, m-1. \end{aligned} \tag{5}$$

Multiplying the equation in (5) by a_i, $i = 1, 2, \ldots, m-1$, and adding the resulting equations we obtain

$$0 = \sum_{i=1}^{m-1} a_i x(\xi_i) = \sum_{i=1}^{m-1} a_i x(c) + \sum_{i=1}^{m-1} a_i r_i = (\alpha - 1) x(c) + \sum_{i=1}^{m-1} a_i r_i. \tag{6}$$

Now, (6) implies that

$$\begin{aligned} 0 < x(c) &= \sum_{i=1}^{m-1} \frac{a_i}{1-\alpha} r_i = \sum_{i=1}^{m-1} \frac{a_i}{1-\alpha} \int_c^{\xi_i} (\xi_i - s) x''(s)\, \mathrm{d}s \qquad (7) \\ &\leq \sum_{i=1}^{m-1} \left(\frac{a_i}{1-\alpha}\right)_- \left| \int_c^{\xi_i} (\xi_i - s) x''(s)\, \mathrm{d}s \right|, \end{aligned}$$

since $r_i = \int_c^{\xi_i} (\xi_i - s)x''(s)\,\mathrm{d}s \leq 0$, $i = 1, 2, \ldots, m-1$. We, next, observe that $|\int_c^{\xi_i}(\xi_i - s)x''(s)\,\mathrm{d}s| \leq |\xi_i - c|\,|\int_c^{\xi_i}|x''(s)|\,\mathrm{d}s| \leq |\xi_i - c|\int_0^1 |x''(s)|\,\mathrm{d}s$, $i = 1, 2, \ldots, m-1$. We thus see from (7) that

$$\begin{aligned} \|x\|_\infty = x(c) &\leq \sum_{i=1}^{m-1}\left(\frac{a_i}{1-\alpha}\right)_- \left|\int_c^{\xi_i}(\xi_i - s)x''(s)\,\mathrm{d}s\right| \\ &\leq \left(\sum_{i=1}^{m-1}\left(\frac{a_i}{1-\alpha}\right)_- |\xi_i - c|\right)\int_0^1 |x''(s)|\,\mathrm{d}s \\ &\leq \max_{u\in[0,1]}\left(\sum_{i=1}^{m-1}\left(\frac{a_i}{1-\alpha}\right)_- |\xi_i - u|\right)\int_0^1 |x''(s)|\,\mathrm{d}s. \end{aligned} \tag{8}$$

Since, now, $\sum_{i=1}^{m-1}(a_i/(1-\alpha))_-|\xi_i - u|$ is a piecewise linear function, its maximum is attained at one of the points $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \xi_{m-1} = 1$. Accordingly we get

$$\begin{aligned} &\max_{u\in[0,1]}\left(\sum_{i=1}^{m-1}\left(\frac{a_i}{1-\alpha}\right)_- |\xi_i - u|\right) \\ &= \max\left\{ \begin{array}{l} \displaystyle\sum_{i=1}^{m-2}\left(\frac{a_i}{1-\alpha}\right)_-(1-\xi_i),\ \sum_{i=1}^{m-2}\left(\frac{a_i}{1-\alpha}\right)_-\xi_i + \left(\frac{1}{1-\alpha}\right)_+, \\ \displaystyle\sum_{i=1,\,i\neq j}^{m-2}\left(\frac{a_i}{1-\alpha}\right)_-|\xi_i - \xi_j| + \left(\frac{1}{1-\alpha}\right)_+(1-\xi_j), \\ j = 1, 2, \ldots, m-2 \end{array}\right\} = C, \end{aligned} \tag{9}$$

and estimate (4) holds in this case.

Finally, let us consider the case when $c = 1$, so that $\|x\|_\infty = x(1)$. Suppose, first, that $\alpha < 1$. Now we have

$$0 < (1-\alpha)x(1) = \sum_{i=1}^{m-2} a_i(x(\xi_i) - x(1)).$$

Now, for each $i = 1, 2, \ldots, m-2$ there exists a $\sigma_i \in (\xi_i, 1)$ such that $0 \leq x(1) - x(\xi_i) = (1-\xi_i)x'(\sigma_i) = (1-\xi_i)\int_0^{\sigma_i} x''(s)\,\mathrm{d}s$.

It follows that

$$\begin{aligned}(1-\alpha)\|x\|_\infty &= \sum_{i=1}^{m-2} a_i(\xi_i - 1)\int_0^{\sigma_i} x''(s)\,\mathrm{d}s \\ &\le \sum_{i=1}^{m-2}(a_i)_-(1-\xi_i)\int_0^{\sigma_i} x''(s)\,\mathrm{d}s \\ &\le \sum_{i=1}^{m-2}(a_i)_-(1-\xi_i)\|x''\|_1,\end{aligned}$$

and thus

$$\|x\|_\infty \le \sum_{i=1}^{m-2}\left(\frac{a_i}{1-\alpha}\right)_-(1-\xi_i)\|x''\|_1 \le C\|x''\|_1,$$

in view of (9). Similarly, if $\alpha > 1$, we have that

$$\begin{aligned}(\alpha-1)\|x\|_\infty &= \sum_{i=1}^{m-2} a_i(1-\xi_i)\int_0^{\sigma_i} x''(s)\,\mathrm{d}s \\ &\le \sum_{i=1}^{m-2}(a_i)_+(1-\xi_i)\int_0^{\sigma_i} x''(s)\,\mathrm{d}s \\ &\le \sum_{i=1}^{m-2}(a_i)_+(1-\xi_i)\|x''\|_1,\end{aligned}$$

and again we have

$$\begin{aligned}\|x\|_\infty &\le \sum_{i=1}^{m-2}\frac{(a_i)_+(1-\xi_i)}{\alpha-1}\|x''\|_1 \\ &= \sum_{i=1}^{m-2}\left(\frac{a_i}{1-\alpha}\right)_-(1-\xi_i)\|x''\|_1 \le C\|x''\|_1,\end{aligned}$$

in view of (9). Thus estimate (4) holds in this case too.

This completes the proof of the theorem.

Remark 1 Let $\alpha \in R$ and $\eta \in (0,1)$ be given. Then for $x(t) \in W^{2,1}(0,1)$ with $x'(0)=0$, $x(1)=\alpha x(\eta)$ we see from Theorem 1 that the estimate

$\|x\|_\infty \le C\|x''\|_1$ holds with

$$C = \begin{cases} \max\left\{\frac{|\alpha|(1-\eta)}{1+|\alpha|}, \frac{1+\eta|\alpha|}{1+|\alpha|}\right\} & \text{if } \alpha < -1, \\ \frac{1+\eta|\alpha|}{1+|\alpha|} & \text{if } -1 \le \alpha \le 0, \\ \frac{1}{1-\alpha} & \text{if } 0 < \alpha < 1, \\ \frac{\alpha}{\alpha-1}\max\{\eta, 1-\eta\} & \text{if } \alpha > 1. \end{cases}$$

The following theorem gives a better estimate than the one given by Theorem 1 in the case of a three-point boundary value problem.

THEOREM 2 *Let $\alpha \in R$ and $\eta \in (0,1)$ be given. Then for $x(t) \in W^{2,1}(0,1)$ with $x'(0)=0$, $x(1)=\alpha x(\eta)$ we have*

$$\|x\|_\infty \le M\|x''\|_1, \tag{10}$$

where

$$M = \max\left\{\frac{|\alpha|(1-\eta)}{1+|\alpha|}, \frac{1+\eta|\alpha|}{1+|\alpha|}\right\} \quad \textit{if } \alpha < -1.$$

$$M = \frac{1-\alpha\eta}{1-\alpha} \quad \textit{if } -1 \le \alpha < 1,$$

$$M = \max\left\{\frac{\eta}{2}, \frac{\alpha(1-\eta)}{\alpha-1}\right\} \quad \textit{if } \alpha > 1, \ \textit{and } \alpha\eta \le 1,$$

$$M = \max\left\{\frac{\eta}{2}, \frac{\alpha\eta-1}{\alpha-1}, \frac{\alpha(1-\eta)}{\alpha-1}\right\} \quad \textit{if } \alpha > 1, \ \textit{and } \alpha\eta > 1.$$

Proof For $\alpha \le 0$ we see from Theorem 1 that

$$M = \max\left\{\frac{|\alpha|(1-\eta)}{1+|\alpha|}, \frac{1+\eta|\alpha|}{1+|\alpha|}\right\}.$$

This implies, in particular, for $\alpha < -1$ that

$$M = \max\left\{\frac{|\alpha|(1-\eta)}{1+|\alpha|}, \frac{1+\eta|\alpha|}{1+|\alpha|}\right\}.$$

Now, we note that for $-1 \leq \alpha < 0$, that

$$\frac{1-\alpha\eta}{1-\alpha} = \frac{1+\eta|\alpha|}{1+|\alpha|} \geq \frac{|\alpha|(1+\eta)}{1+|\alpha|} > \frac{|\alpha|(1-\eta)}{1+|\alpha|}$$

and so we again see from Theorem 1 that $M = (1-\alpha\eta)/(1-\alpha)$ if $-1 \leq \alpha < 0$.

We, next, prove that $M = (1-\alpha\eta)/(1-\alpha)$ if $0 \leq \alpha < 1$. For this, we see using mean value theorem that there exists an $\xi \in (\eta, 1)$ such that $x(1) - x(\eta) = (1-\eta)x'(\xi)$, which implies using $x(1) = \alpha x(\eta)$ that $x(1) = \alpha(1-\eta)/(\alpha-1)x'(\xi)$. It then follows from the relation $x(t) = x(1) - \int_t^1 x'(s)\,ds$ that

$$\|x\|_\infty \leq \left(\frac{\alpha(1-\eta)}{1-\alpha} + 1\right)\|x'\|_\infty = \frac{1-\alpha\eta}{1-\alpha}\|x'\|_\infty \leq \frac{1-\alpha\eta}{1-\alpha}\|x''\|_1,$$

in view of the equation $x'(t) = \int_0^t x''(s)\,ds$, for $t \in [0,1]$ since $x'(0) = 0$. Thus $M = (1-\alpha\eta)/(1-\alpha)$, if $-1 \leq \alpha < 1$.

Finally, we consider the case $\alpha > 1$. Let $x(\eta) = z$ so that $x(1) = \alpha z$. We may assume without any loss of generality that $z \geq 0$, replacing $x(t)$ by $-x(t)$ if necessary. Suppose, now, $\|x\|_\infty = 1$ so that there exists a $c \in [0,1]$ such that either $x(c) = 1$ or $x(c) = -1$. We consider all possible cases of the location for c.

(i) Suppose that $c \in [0,\eta]$ and $x(c) = 1$. Then $x'(c) = 0$, $c \neq \eta$. Now, by mean value theorem there exist $\nu_1 \in [c,\eta]$, $\nu_2 \in [\eta,1]$ such that

$$x'(\nu_1) = \frac{x(\eta)-x(c)}{\eta-c} = -\frac{1-z}{\eta-c}, \quad x'(\nu_2) = \frac{x(1)-x(\eta)}{1-\eta} = \frac{\alpha z - z}{1-\eta}.$$

We note that $x'(\nu_1) \leq 0$, $x'(\nu_2) \geq 0$ since $0 \leq z \leq 1$ and $\alpha > 1$. It follows that

$$\begin{aligned}
\int_0^1 |x''(s)|\,ds &\geq \left|\int_c^{\nu_1} x''(s)\,ds\right| + \left|\int_{\nu_1}^{\nu_2} x''(s)\,ds\right| = 2|x'(\nu_1)| + x'(\nu_2) \\
&= 2\frac{1-z}{\eta-c} + \frac{\alpha z - z}{1-\eta} \geq \min_{c\in[0,\eta),\, z\in[0,1/\alpha]} \left\{2\frac{1-z}{\eta-c} + \frac{\alpha z - z}{1-\eta}\right\} \\
&\geq \min_{c\in[0,\eta)} \left\{\frac{2}{\eta-c}, \frac{2(\alpha-1)}{\alpha(\eta-c)} + \frac{\alpha-1}{\alpha(1-\eta)}\right\} \geq \min\left\{\frac{2}{\eta}, \frac{\alpha-1}{\alpha(1-\eta)}\right\}.
\end{aligned}$$

(ii) Let, now, $c \in [0, \eta]$, $x(c) = -1$. Then since $x'(c) = 0$, $c \neq \eta$, we again see from mean value theorem that there exist $\nu_3 \in [c, \eta]$, $\nu_4 \in [\eta, 1]$ such that

$$x'(\nu_3) = \frac{x(\eta) - x(c)}{\eta - c} = \frac{z+1}{\eta - c}, \quad x'(\nu_4) = \frac{x(1) - x(\eta)}{1 - \eta} = \frac{\alpha z - z}{1 - \eta}.$$

Again we note that $x'(\nu_3) > 0$, $x'(\nu_4) \geq 0$ since $0 \leq z \leq 1$ and $\alpha > 1$ and we have

$$\begin{aligned} \int_0^1 |x''(s)|\,\mathrm{d}s &\geq \left|\int_c^{\nu_3} x''(s)\,\mathrm{d}s\right| + \left|\int_{\nu_3}^{\nu_4} x''(s)\,\mathrm{d}s\right| \\ &= x'(\nu_3) + |x'(\nu_4) - x'(\nu_3)| \\ &= \frac{1+z}{\eta - c} + \left|\frac{\alpha z - z}{1 - \eta} - \frac{1+z}{\eta - c}\right|. \end{aligned} \tag{11}$$

Let

$$F(z, c) = \frac{1+z}{\eta - c} + \left|\frac{\alpha z - z}{1 - \eta} - \frac{1+z}{\eta - c}\right|.$$

We need to estimate $\min_{c \in [0,\eta), z \in [0,1/\alpha]} F(z, c)$. We note that $F(0, c) = 2/(\eta - c) \geq 2/\eta$ for $c \in [0, \eta)$,

$$F\left(\frac{1}{\alpha}, c\right) = \frac{\alpha + 1}{\alpha(\eta - c)} + \left|\frac{\alpha - 1}{\alpha(1 - \eta)} - \frac{\alpha + 1}{\alpha(\eta - c)}\right| \geq \frac{\alpha - 1}{\alpha(1 - \eta)}$$

for $c \in [0, \eta)$. Let, now, z_0 such that $(\alpha z_0 - z_0)/(1 - \eta) - (1 + z_0)/(\eta - c) = 0$ so that $z_0 = (1 - \eta)/(\alpha\eta - 1 - c(\alpha - 1))$. It is easy to see that $z_0 \in [0, 1/\alpha]$ if $\eta > (\alpha + 1)/2\alpha$ and $c \in (0, (2\alpha\eta - \alpha - 1)/(\alpha - 1))$. In this case we get

$$F(z_0, c) = \frac{\alpha - 1}{\alpha\eta - 1 - c(\alpha - 1)} \geq \frac{\alpha - 1}{\alpha\eta - 1}.$$

Accordingly we see that $F(z, c) \geq \min\{2/\eta, (\alpha - 1)/\alpha(1 - \eta)\}$ if $\alpha\eta \leq 1$ and $F(z, c) \geq \min\{2/\eta, (\alpha - 1)/\alpha(1 - \eta), (\alpha - 1)/(\alpha\eta - 1)\}$ if $\alpha\eta > 1$.

We thus have from (11) that

$$\int_0^1 |x''(s)|\,\mathrm{d}s \geq \left|\int_c^{\nu_3} x''(s)\,\mathrm{d}s\right| + \left|\int_{\nu_3}^{\nu_4} x''(s)\,\mathrm{d}s\right|$$
$$= x'(\nu_3) + |x'(\nu_4) - x'(\nu_3)|$$
$$= \frac{1+z}{\eta - c} + \left|\frac{\alpha z - z}{1-\eta} - \frac{1+z}{\eta - c}\right|$$
$$\geq \begin{cases} \min\left\{\dfrac{2}{\eta}, \dfrac{\alpha - 1}{\alpha(1-\eta)}\right\}, & \text{if } \alpha\eta \leq 1, \\ \min\left\{\dfrac{2}{\eta}, \dfrac{\alpha - 1}{\alpha(1-\eta)}, \dfrac{\alpha - 1}{\alpha\eta - 1}\right\}, & \text{if } \alpha\eta > 1. \end{cases}$$

(iii) Next, suppose that $c \in (\eta, 1), x(c) = 1$. Again, $x'(c) = 0$ and we have from mean value theorem that there exist $\nu_5 \in [\eta, c]$, $\nu_6 \in [c, 1]$ such that

$$x'(\nu_5) = \frac{x(c) - x(\eta)}{c - \eta} = \frac{1 - z}{c - \eta}, \quad x'(\nu_6) = \frac{x(1) - x(c)}{1 - c} = \frac{\alpha z - 1}{1 - c}.$$

Note that $x'(\nu_5) \geq 0$, $x'(\nu_6) \leq 0$ since $x(1) = \alpha z \leq 1$. Accordingly, we obtain

$$\int_0^1 |x''(s)|\,\mathrm{d}s \geq \left|\int_0^{\nu_5} x''(s)\,\mathrm{d}s\right| + \left|\int_{\nu_5}^{\nu_6} x''(s)\,\mathrm{d}s\right|$$
$$= x'(\nu_5) + |x'(\nu_6) - x'(\nu_5)| = 2x'(\nu_5) + |x'(\nu_6)|$$
$$= 2\frac{1-z}{c-\eta} + \frac{1-\alpha z}{1-c} \geq \frac{2(\alpha - 1)}{\alpha(1-\eta)}, \quad \text{since } 0 \leq z \leq \frac{1}{\alpha}. \tag{12}$$

(iv) Next, suppose that $c \in (\eta, 1)$, $x(c) = -1$. Again $x'(c) = 0$ and we have from mean value theorem that there exist $\nu_7 \in [\eta, c]$, $\nu_8 \in [c, 1]$ such that

$$x'(\nu_7) = \frac{x(c) - x(\eta)}{c - \eta} = \frac{-1 - z}{c - \eta}, \quad x'(\nu_8) = \frac{x(1) - x(c)}{1 - c} = \frac{\alpha z + 1}{1 - c}.$$

Note that $x'(\nu_7) \le 0$, $x'(\nu_8) \ge 0$. Accordingly, we obtain

$$\begin{aligned}\int_0^1 |x''(s)|\,\mathrm{d}s &\ge \left|\int_0^{\nu_7} x''(s)\,\mathrm{d}s\right| + \left|\int_{\nu_7}^{\nu_8} x''(s)\,\mathrm{d}s\right| \\ &= |x'(\nu_7)| + |x'(\nu_8) - x'(\nu_7)| \\ &= 2|x'(\nu_7)| + x'(\nu_8) \\ &= 2\frac{1+z}{c-\eta} + \frac{1+\alpha z}{1-c} \ge \frac{2}{c-\eta} + \frac{1}{1-c} \\ &\ge \frac{2}{1-\eta} \ge \frac{2(\alpha-1)}{\alpha(1-\eta)}.\end{aligned}$$

(v) Finally suppose that $c = 1$, so that $x(1) = 1 = \alpha z$. We then have that there exists a $\nu_9 \in (\eta, 1)$ such that $x'(\nu_9) = (\alpha-1)(\alpha(1-\eta))^{-1}$. Thus

$$\int_0^1 |x''(s)|\,\mathrm{d}s \ge \left|\int_0^{\nu_9} x''(s)\,\mathrm{d}s\right| = x'(\nu_9) = \frac{1-(1/\alpha)}{1-\eta} = \frac{\alpha-1}{\alpha(1-\eta)}.$$

We thus see from (i)–(v) that, for $\alpha > 1$, (10) holds with

$$M = \begin{cases} \max\left\{\dfrac{\eta}{2}, \dfrac{\alpha(1-\eta)}{\alpha-1}\right\} & \text{if } \alpha\eta \le 1, \\ \max\left\{\dfrac{\eta}{2}, \dfrac{\alpha\eta-1}{\alpha-1}, \dfrac{\alpha(1-\eta)}{\alpha-1}\right\} & \text{if } \alpha\eta > 1. \end{cases}$$

This completes the proof of the theorem.

The following theorem shows that for $-1 \le \alpha < 1$, $M = (1-\alpha\eta)/(1-\alpha)$ is the best constant in (10).

THEOREM 3 *Let* $-1 \le \alpha < 1$, $\eta \in (0,1)$ *and set*

$$\begin{aligned}\inf\{\|x''\|_1\colon x(t) \in W^{2,1}(0,1),\ x'(0) = 0,\\ x(1) = \alpha x(\eta),\ \|x\|_\infty = 1\} = M_1.\end{aligned}$$

Then $M_1 = (1-\alpha)/(1-\alpha\eta)$.

Proof We first see from Theorem 2 that $M_1 \ge (1-\alpha)/(1-\alpha\eta)$. We, next, note that

$$\frac{1-\eta}{1-\alpha\eta} - 1 = \frac{1-\eta-1+\alpha\eta}{1-\alpha\eta} = \frac{(\alpha-1)\eta}{1-\alpha\eta} < 0.$$

Let, now, $z_0 = (1-\eta)/(1-\alpha\eta)$ and $\varepsilon > 0$ be such that $0 \leq z_0 + \varepsilon < 1$ and let $k_\varepsilon = (z_0+\varepsilon)(\alpha-1)/(1-\eta) < 0$. We, next, consider the function $\varphi(t)$ defined by

$$\varphi(t) = 1 + \gamma t^\beta, \quad \text{if } t \in [0,\eta]$$

and

$$\varphi(t) = k_\varepsilon(t-\eta) + (z_0+\varepsilon), \quad \text{if } t \in [\eta, 1],$$

where

$$\beta = -\frac{k_\varepsilon \eta}{1 - z_0 - \varepsilon} > 1, \quad \gamma = \frac{k_\varepsilon}{\eta^{\beta-1}\beta}.$$

It is easy to check that

$$\varphi(t) \in W^{2,1}(0,1), \quad \varphi'(0) = 0, \quad \|\varphi(t)\|_0 = \varphi(0) = 1, \quad \alpha\varphi(\eta) = \varphi(1)$$

and

$$\int_0^1 |\varphi''(t)|\,\mathrm{d}t = \int_0^\eta |\gamma|\beta(\beta-1)t^{\beta-2}\,\mathrm{d}t = \frac{1-\alpha}{1-\eta\alpha} + \varepsilon\frac{1-\alpha}{1-\eta}.$$

This gives for every $\varepsilon > 0$, sufficiently small so that $z_0 + \varepsilon < 1$, that

$$\begin{aligned}\|\varphi''(t)\|_1 &= \frac{1-\alpha}{1-\eta\alpha}\left(1 + \varepsilon\frac{1-\eta\alpha}{1-\eta}\right)\|\varphi(t)\|_0 \\ &\leq \frac{1}{M_1}\left[\frac{1-\alpha}{1-\eta\alpha}\left(1+\varepsilon\frac{1-\eta\alpha}{1-\eta}\right)\right]\|\varphi''(t)\|_1\end{aligned}$$

and therefore

$$\frac{1}{M_1} \geq \frac{1-\eta\alpha}{1-\alpha}\left(1+\varepsilon\frac{1-\eta\alpha}{1-\eta}\right)^{-1}$$

for every $\varepsilon > 0$ sufficiently small. Thus $M_1 = (1-\alpha)/(1-\alpha\eta)$. This completes the proof of the theorem.

Remark 2 The following example shows that for $\eta = 0.5$; $\alpha = 4$. Theorem 2 gives the best possible constant $M = 2/3$ in estimate (10). Indeed, consider the function

$$\phi(t) = 2t^3 \quad \text{for } t \in [0, 1/2], \quad \phi(t) = (3t-1)/2 \quad \text{for } t \in [1/2, 1],$$

we gave $\phi(t) \in W^{2,1}(0,1)$, $\|\phi''(t)\|_1 = 3/2$ and

$$\phi'(0) = 0, \quad 4\phi(1/2) = \phi(1), \quad 1 = \phi(1) = \|\phi(t)\|_0 = 2/3\|\phi''(t)\|_1.$$

This shows that for this function the inequality in estimate (10) is indeed an equality, proving the assertion.

Moreover, it is possible to construct functions for every $\alpha \in R$ and $\eta \in (0,1)$ for which the estimate (10) holds with an equality for the corresponding M indicated in Theorem 2. We omit the details as it becomes technical. However, we should point out that the ideas for constructing such functions are generated by developing proofs for estimate (10) similar to the case $\alpha > 1$ given here.

3 EXISTENCE THEOREMS

DEFINITION 4 *A function $f: [0,1] \times R^2 \mapsto R$ satisfies Caratheodory's conditions if* (i) *for each $(x,y) \in R^2$, the function $t \in [0,1] \mapsto f(t,x,y) \in R$ is measurable on* $[0,1]$, (ii) *for a.e. $t \in [0,1]$, the function $(x,y) \in R^2 \mapsto f(t,x,y) \in R$ is continuous on R^2, and* (iii) *for each $r > 0$, there exists $\alpha_r(t) \in L^1[0,1]$ such that $|f(t,x,y)| \le \alpha_r(t)$ for a.e. $t \in [0,1]$ and all $(x,y) \in R^2$ with $\sqrt{x^2+y^2} \le r$.*

THEOREM 5 *Let $f: [0,1] \times R^2 \mapsto R$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t), q(t), r(t)$ such that the functions $p(t), q(t), r(t)$ are in $L^1(0,1)$ and*

$$|f(t,x_1,x_2)| \le p(t)|x_1| + q(t)|x_2| + r(t) \tag{13}$$

for a.e. $t \in [0,1]$ and all $(x_1,x_2) \in R^2$. Let $\alpha \in R$, $\alpha \ne 1$, and $\eta \in (0,1)$ be given. Then, the three-point boundary value problem (3) *has at least one solution in $C^1[0,1]$ provided*

$$M\|p(t)\|_1 + \|q(t)\|_1 < 1, \tag{14}$$

where M is as given in Theorem 2.

Proof Let X denote the Banach space $C^1[0,1]$ and Y denote the Banach space $L^1(0,1)$ with their usual norms. We define a linear mapping $L: D(L) \subset X \mapsto Y$ by setting

$$D(L) = \{x \in W^{2,1}(0,1) \mid x'(0) = 0, \ x(1) = \alpha x(\eta)\},$$

and for $x \in D(L)$,

$$Lx = x''.$$

We also define a nonlinear mapping $N: X \mapsto Y$ by setting

$$(Nx)(t) = f(t, x(t), x'(t)), \quad t \in [0, 1].$$

We note that N is a bounded mapping from X into Y. Next, it is easy to see that the linear mapping $L: D(L) \subset X \mapsto Y$, is a one-to-one mapping. Next, the linear mapping $K: Y \to X$, defined for $y \in Y$ by

$$(Ky)(t) = \int_0^t (t - s)y(s)\,\mathrm{d}s + At,$$

where A is given by

$$A(1 - \alpha\eta) = \alpha \int_0^\eta (\eta - s)y(s)\,\mathrm{d}s - \int_0^1 (1 - s)y(s)\,\mathrm{d}s$$

is such that for $y \in Y$, $Ky \in D(L)$ and $LKy = y$; and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela–Ascoli theorem that KN maps a bounded subset of X into a relatively compact subset of X. Hence $KN: X \mapsto X$ is a compact mapping.

We, next, note that $x \in C^1[0, 1]$ is a solution of the boundary value problem (3) if and only if x is a solution to the operator equation

$$Lx = Nx + e.$$

Now, the operator equation $Lx = Nx + e$ is equivalent to the equation

$$x = KNx + Ke.$$

We apply the Leray–Schauder continuation theorem (see, e.g. [18], Corollary IV.7) to obtain the existence of a solution for $x = KNx + Ke$ or equivalently to the boundary value problem (3).

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$\begin{gathered} x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \quad 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \alpha x(\eta), \end{gathered} \tag{15}$$

is, *a priori*, bounded in $C^1[0, 1]$ by a constant independent of $\lambda \in [0, 1]$.

This is straightforward to prove using the equation in (15), our assumptions that $f(t, x(t), x'(t)) \in L^1(0, 1)$, (13), estimate (10), the estimate $\|x'\|_\infty \leq \|x''\|_1$ for $x(t) \in W^{2,1}(0, 1)$, with $x'(0) = 0$ and the assumption (14).

This completes the proof of the theorem.

THEOREM 6 *Let f satisfy all conditions of Theorem* 1 *where the inequality* (14) *is replaced with* $C\|p(t)\|_1 + \|q(t)\|_1 < 1$, *where* C *is as given in Theorem* 5. *Let* $a_i \in R$, $\xi_i \in (0, 1)$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, *with* $\alpha = \sum_{i=1}^{m-2} a_i \neq 1$, *be given. Then the multi-point boundary value problem* (2) *has at least one solution in* $C^1[0, 1]$.

This proof is quite similar to the proof of Theorem 5 and we omit it accordingly.

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