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Large Time Behavior of Solutions to the Cauchy Problem for One-Dimensional Thermoelastic System with Dissipation

KENJI NISHIHARA [a,*] and SHINYA NISHIBATA [b]

[a] *School of Political Science and Economics, Waseda University, Tokyo 169-8050, Japan;* [b] *Department of Mathematics, Fukuoka Institute of Technology, Fukuoka 811-0295, Japan*

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In this paper we investigate the large time behavior of solutions to the Cauchy problem on **R** for a one-dimensional thermoelastic system with dissipation. When the initial data is suitably small, (S. Zheng, *Chin. Ann. Math.* **8B** (1987), 142–155) established the global existence and the decay properties of the solution. Our aim is to improve the results and to obtain the sharper decay properties, which seems to be optimal. The proof is given by the energy method and the Green function method.

Keywords: Thermoelastic system; Dissipation; Decay rate; Green function

AMS Subject Classifications: 35B40, 35L60, 35L70, 76R50

1 INTRODUCTION

In this paper we investigate the large time behavior of solutions to the Cauchy problem for a one-dimensional thermoelastic system with dissipation on $\mathbf{R} \times (0, \infty)$:

$$\begin{gathered} w_{tt} - a(w_x, \theta)w_{xx} + b(w_x, \theta)\theta_x + \alpha w_t = 0, \\ c(w_x, \theta)\theta_t + b(w_x, \theta)w_{xt} - d(\theta, \theta_x)\theta_{xx} = 0, \\ w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad \theta(x, 0) = \theta_0(x), \end{gathered} \tag{1.1}$$

* Corresponding author.

where α is a positive constant, and smooth functions a, b, c and d satisfy

$$b \neq 0, \quad \text{and} \quad a, c, d \geq \delta_0 > 0 \text{ under considerations.} \tag{1.2}$$

For the derivation of this system refer to [1,9]. In [9] Slemrod also showed the global existence theorem for the system (1.1) with $\alpha = 0$ on the interval $[0, 1]$. Damping mechanism was discussed in [1]. Nevertheless, for lack of the Poincaré type inequality our problem (1.1) is not necessarily clear. Instead of this system, by introducing new unknown functions

$$w_x = v, \quad w_t = u, \quad \theta = \theta, \tag{1.3}$$

Zheng [10] considered the corresponding system

$$\begin{aligned} v_t - u_x &= 0, \\ u_t - a(v, \theta)v_x + b(v, \theta)\theta_x + \alpha u &= 0, \\ c(v, \theta)\theta_t + b(v, \theta)u_x - d(\theta, \theta_x)\theta_{xx} &= 0 \end{aligned} \tag{1.4}$$

with

$$(v, u, \theta)|_{t=0} = (v_0, u_0, \theta_0)(x). \tag{1.5}$$

In [10] he established the global existence of the solution of (1.4) and (1.5) together with its decay order, when the initial data (v_0, u_0, θ_0) in $H^3(\mathbf{R})$ are suitably small.

Our main purpose is to observe the large time behavior of the solution of (1.1). However, instead of treating (1.1) directly, we first consider (1.4) and (1.5) using L^2-energy method, which improves the result in [10]. Faster decay estimates of $\partial_{x,t}^k u_t$ obtained here play an important role in the next process. That is, regarding u_t in (1.4) as an inhomogeneous term, we have a parabolic system of (v, θ) and hence the "explicit" formula of (v, θ) using the Green functions $G_1(x, t)$, $G_2(x, t)$, which will give sharper estimates of (v, u, θ) if $(v_0, u_0, \theta_0) \in L^1(\mathbf{R})$. This method has been developed by the first author [5,6]. See also [7]. Finally, define a solution $(w, \theta)(x, t)$ of (1.1) by $w(x, t) = \int_{-\infty}^x v(y, t)\, dy$, where (v, u, θ) is a solution of (1.4) with its initial data

$$v_0 = w_{0x}, \quad u_0 = w_1, \quad \theta_0 = \theta_0. \tag{1.6}$$

Thus we obtain a solution to the original Cauchy problem (1.1). Below, we sketch this procedure and state theorems.

First, linearize (1.2) around $(v, u, \theta) = (0, 0, 0)$:

$$\begin{aligned} v_t - u_x &= 0, \\ u_t - v_x + b_0\theta_x + u &= g_2, \\ \theta_t + b_0 u_x - \theta_{xx} &= g_3, \end{aligned} \tag{1.7}$$

where we have normalized as

$$\alpha = 1, \quad a(0,0) = d(0,0) = 1, \quad b(0,0) = b_0 \tag{1.8}$$

and set

$$\begin{aligned} g_2 &= (a(v,\theta) - 1)v_x - (b(v,\theta) - b_0)\theta_x, \\ g_3 &= \frac{1}{c(v,\theta)}((b_0 - b(v,\theta))u_x + (d(\theta,\theta_x) - 1)\theta_{xx}). \end{aligned} \tag{1.9}$$

By denoting the Lebesgue space (resp. the Sobolev space) by $L^p = L^p(\mathbf{R})$ with its norm $\|\cdot\|_{L^p}$ (resp. $H^m = H^m(\mathbf{R})$ with its norm $\|\cdot\|_m$), especially $\|\cdot\|_{L^2} = \|\cdot\|_0 := \|\cdot\|$, our first theorem based on the L^2-energy method is the following:

THEOREM 1 *Suppose that* $(v_0, u_0, \theta_0) \in H^4(\mathbf{R})$ *is suitably small. Then, the Cauchy problem* (1.4) *and* (1.5) *has a unique global solution* $(v, u, \theta) \in C([0, \infty]; H^4(\mathbf{R}))$, *which satisfies*

$$\begin{aligned} &E(t; v, u, \theta) \\ &\quad := E_1(t; v, u, \theta) + \int_0^t E_2(\tau; v, u, \theta)\,\mathrm{d}\tau \\ &\quad = \|(v,\theta)(t)\|^2 + (1+t)\|(v_x, u, \theta_x)(t)\|^2 \\ &\qquad + (1+t)^2\|\partial_x(v_x, u, \theta_x), \partial_t(v,\theta)(t)\|^2 \\ &\qquad + (1+t)^3\|\partial_x^2(v_x, u, \theta_x), \partial_t(v_x, u, \theta_x)(t)\|^2 \\ &\qquad + (1+t)^4\|\partial_x^3(v_x, u, \theta_x), \partial_x^4 u, \partial_t\partial_x(v_x, u, \theta_x), \partial_t u_{xx}, \partial_t^2(v, u, \theta)(t)\|^2 \\ &\qquad + \int_0^t \{\|(v_x, u, \theta_x)(\tau)\|^2 + (1+\tau)\|\partial_x(v_x, u, \theta_x), \partial_t(v,\theta)(\tau)\|^2 \\ &\qquad + (1+\tau)^2\|\partial_x^2(v_x, u, \theta_x), \partial_t(v_x, u, \theta_x)(\tau)\|^2 \\ &\qquad + (1+\tau)^3\|\partial_x^3(v_x, u, \theta_x), \partial_x\partial_t(v_x, u, \theta_x), \partial_t^2(v,\theta)(\tau)\|^2 \\ &\qquad + (1+\tau)^4\|\partial_x^4 u, \partial_t^2(v_x, u, \theta_x), \partial_x\partial_t^2(v, u, \theta), \partial_t\partial_x^3(v,\theta), \partial_x^5\theta(\tau)\|^2\}\,\mathrm{d}\tau \\ &\quad \le C\|v_0, u_0, \theta_0\|_4^2. \end{aligned} \tag{1.10}$$

In the next step we first obtain "explicit" formula of (v, θ). From the decay orders obtained in Theorem 1, the term u_t in the left-hand side of $(1.7)_2$ (the second equation of (1.7)) decays faster than the other terms. Hence, differentiating $(1.7)_2$ once in x and using $(1.7)_1$, we regard (1.7) as a parabolic system of (v, u):

$$
\begin{aligned}
&v_t - v_{xx} + b_0\theta_{xx} = -u_{xt} + g_{2x},\\
&b_0 v_t + \theta_t - \theta_{xx} = g_3,
\end{aligned}
\tag{1.11}
$$

or

$$
A\begin{pmatrix} v \\ \theta \end{pmatrix}_t - B\begin{pmatrix} v \\ \theta \end{pmatrix}_{xx} = \begin{pmatrix} -u_{xt} + g_{2x} \\ g_3 \end{pmatrix} =: \mathbf{F}, \tag{1.12}
$$

where

$$
A = \begin{pmatrix} 1 & 0 \\ b_0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -b_0 \\ 0 & 1 \end{pmatrix}. \tag{1.13}
$$

Setting

$$
\begin{pmatrix} v \\ \theta \end{pmatrix} = P\begin{pmatrix} V \\ \Theta \end{pmatrix} \tag{1.14}
$$

for a regular constant matrix P, we have

$$
\begin{pmatrix} V \\ \Theta \end{pmatrix}_t - P^{-1}A^{-1}BP\begin{pmatrix} V \\ \Theta \end{pmatrix}_{xx} = P^{-1}A^{-1}\mathbf{F}. \tag{1.15}
$$

The eigenvalues k_1, k_2 of $A^{-1}B = \begin{pmatrix} 1 & -b_0 \\ -b_0 & b_0^2 + 1 \end{pmatrix}$ are

$$
\begin{aligned}
0 < k_1 &= \frac{b_0^2 + 2 - \sqrt{(b_0^2 + 2)^2 - 4}}{2} \\
< k_2 &= \frac{b_0^2 + 2 + \sqrt{(b_0^2 + 2)^2 - 4}}{2},
\end{aligned}
\tag{1.16}
$$

and corresponding unit vectors are

$$
\begin{aligned}
&\frac{1}{\sqrt{b_0^2+(k_1-1)^2}}\begin{pmatrix}-b_0\\ k_1-1\end{pmatrix}=:\begin{pmatrix}p_{11}\\ p_{21}\end{pmatrix},\\
&\frac{1}{\sqrt{b_0^2+(k_2-1)^2}}\begin{pmatrix}-b_0\\ k_2-1\end{pmatrix}=:\begin{pmatrix}p_{12}\\ p_{22}\end{pmatrix}.
\end{aligned}
\tag{1.17}
$$

Hence, a matrix

$$
P:=\begin{pmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{pmatrix}
$$

gives the diagonalized system

$$
\begin{pmatrix}V\\ \Theta\end{pmatrix}_t-\begin{pmatrix}k_1 & 0\\ 0 & k_2\end{pmatrix}\begin{pmatrix}V\\ \Theta\end{pmatrix}_{xx}=P^{-1}A^{-1}\mathbf{F},
\tag{1.18}
$$

and hence the "explicit" formula is

$$
\begin{aligned}
\begin{pmatrix}V\\ \Theta\end{pmatrix}(x,t)&=\begin{pmatrix}G_1 & 0\\ 0 & G_2\end{pmatrix}(\cdot,t)*\begin{pmatrix}V_0\\ \Theta_0\end{pmatrix}\\
&\quad+\int_0^t\begin{pmatrix}G_1 & 0\\ 0 & G_2\end{pmatrix}(\cdot,t-\tau)*P^{-1}A\mathbf{F}(\cdot,\tau)\,\mathrm{d}\tau
\end{aligned}
\tag{1.19}
$$

where $\begin{pmatrix}V_0\\ \Theta_0\end{pmatrix}=P^{-1}\begin{pmatrix}v_0\\ \theta_0\end{pmatrix}$,

$$
G_i(x,t)=\frac{1}{\sqrt{4\pi k_i t}}\exp\left(-\frac{x^2}{4k_it}\right),\quad i=1,2
\tag{1.20}
$$

and $*$ means the convolution in x. Note that, since $A^{-1}B$ is a real symmetric matrix, P and P are orthogonal matrices and

$$
\begin{aligned}
&\sum_{i=1}^{2}p_{ij}^2=\sum_{i=1}^{2}p_{ji}^2=1,\quad j=1,2,\\
&\sum_{i=1}^{2}p_{ij}p_{ik}=\sum_{i=1}^{2}p_{ji}p_{ki}=0,\quad j\neq k.
\end{aligned}
\tag{1.21}
$$

By (1.19),

$$\begin{pmatrix} V \\ \Theta \end{pmatrix} = P^{-1} \begin{pmatrix} v \\ \theta \end{pmatrix}$$

gives

$$\begin{aligned} \begin{pmatrix} v \\ \theta \end{pmatrix}(x,t) &= P \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} P^{-1} * \begin{pmatrix} v_0 \\ \theta_0 \end{pmatrix} \\ &\quad + \int_0^t P \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} P^{-1} * A^{-1} \begin{pmatrix} -u_{xt} + g_{2x} \\ g_3 \end{pmatrix} \mathrm{d}\tau. \end{aligned} \tag{1.22}$$

From (1.17) and (1.21)

$$\begin{aligned} P \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} P^{-1} &= \begin{pmatrix} p_{11}^2 G_1 + p_{12}^2 G_2 & p_{11}p_{21}G_1 + p_{12}p_{22}G_2 \\ p_{11}p_{21}G_1 + p_{12}p_{22}G_2 & p_{21}^2 G_1 + p_{22}^2 G_2 \end{pmatrix} \\ &=: \begin{pmatrix} (1-\alpha)G_1 + \alpha G_2 & \gamma(G_1 - G_2) \\ \gamma(G_1 - G_2) & \beta G_1 + (1-\beta)G_2 \end{pmatrix} \end{aligned} \tag{1.23}$$

with $0 < \alpha, \beta, |\gamma| < 1$. Thus, we have an "explicit" formula of (v, θ):

$$\begin{aligned} \begin{pmatrix} v \\ \theta \end{pmatrix}(x,t) &= \begin{pmatrix} (1-\alpha)G_1 + \alpha G_2 & \gamma(G_1 - G_2) \\ \gamma(G_1 - G_2) & \beta G_1 + (1-\beta)G_2 \end{pmatrix}(\cdot, t) * \begin{pmatrix} v_0 \\ \theta_0 \end{pmatrix} \\ &\quad + \int_0^t \begin{pmatrix} (1-\alpha)G_1 + \alpha G_2 & \gamma(G_1 - G_2) \\ \gamma(G_1 - G_2) & \beta G_1 + (1-\beta)G_2 \end{pmatrix}(\cdot, t-\tau) \\ &\quad * \begin{pmatrix} lu_{xt} + g_{2x} \\ b_0(u_{xt} - g_{2x}) + g_3 \end{pmatrix}(\cdot, \tau)\, \mathrm{d}\tau, \end{aligned} \tag{1.24}$$

which is "explicit" in the sense that several kinds of information about u_{xt}, g_2, g_3 are already known. From $(1.7)_2$, u has the form

$$u(x,t) = v_x - b_0\theta_x - u_t + g_2. \tag{1.25}$$

From (1.24) and (1.25), (v_x, u, θ_x) instead of (v_x, u_x, θ_x) have same decay order if u_t and g_2 decay faster. From this point of view the decay orders obtained in Theorem 1 seem to be reasonable. Compare this to the result of Zheng [10]. See also [2,4].

Further, if the initial data (v_0, θ_0) is in $L^1(\mathbf{R})$, then these decay orders are improved. In fact, we have the following second main theorem:

THEOREM 2 *In addition to the assumptions in Theorem* 1, *suppose that* (v_0, θ_0) *is in* $L^1(\mathbf{R})$. *Then, the solution* (v, u, θ) *of* (1.4) *and* (1.5) *satisfies the decay estimates*

$$\begin{aligned}
&(1+t)^{1/4}\|(v,\theta)(t)\| + (1+t)^{1/2}\|(v,\theta)(t)\|_{L^\infty} \\
&\quad + (1+t)^{3/4}\|(v_x,u,\theta_x)(t)\| + (1+t)\|(v_x,u,\theta_x)(t)\|_{L^\infty} \\
&\quad + (1+t)^{5/4}\|(v_{xx},u_x,\theta_{xx})(t)\| + (1+t)^{3/2}\|(v_{xx},u_x,\theta_{xx})(t)\|_{L^\infty} \\
&\quad \le C(\|v_0,u_0,\theta_0\|_4 + \|v_0,\theta_0\|_{L^1}).
\end{aligned} \tag{1.26}$$

Remark 1 In this stage the assumption $u_0 \in L^1$ is not necessary.

Finally, consider the Cauchy problem to the original system (1.1). Taking (1.3) and the first component of (1.24) (denote by $(1.24)_1$) into consideration, we assume $w_{0x} = v_0$ with $w_0 \in H^5(\mathbf{R}) \cap L^1(\mathbf{R})$, and set

$$w(x,t) = \int_{-\infty}^{x} v(y,t)\,\mathrm{d}y. \tag{1.27}$$

By $(1.7)_1$, $w_t(x,t) = \int_{-\infty}^{x} v_t(y,t)\,\mathrm{d}y = \int_{-\infty}^{x} u_x(y,t)\,\mathrm{d}y = u(x,t)$. Hence, (w,θ) satisfies (1.1). Estimating $(1.24)_2$ and (1.27) with $(1.24)_1$, we have the following theorem:

THEOREM 3 *Suppose that* $(w_0, w_1, \theta_0) \in H^5(\mathbf{R}) \times H^4(\mathbf{R}) \times H^4(\mathbf{R})$ *is suitably small and* $w_0, w_{0x}, w_1, \theta_0$ *are in* $L^1(\mathbf{R})$, *and that* (v,u,θ) *is a solution of* (1.4) *with* $(v,u,\theta)|_{t=0} = (w_{0x}, w_1, \theta_0)$ *obtained in Theorem* 2. *Then* (w,θ) *defined by* (1.27) *and* $(1.24)_2$ *is a solution of* (1.1), *which satisfies*

$$\begin{aligned}
&(1+t)^{-1/4}\|w(t)\| + \|w(t)\|_{L^\infty} \\
&\quad + (1+t)^{1/4}\|(w_x,\theta)(t)\| + (1+t)^{1/2}\|(w_x,\theta)(t)\|_{L^\infty} \\
&\quad + (1+t)^{3/4}\|(w_{xx},w_t,\theta_x)(t)\| + (1+t)\|(w_{xx},w_t,\theta_x)\|_{L^\infty} \\
&\quad + (1+t)^{5/4}\|(w_{xxx},w_{tx},\theta_t,\theta_{xx})(t)\| \\
&\quad + (1+t)^{3/2}\|(w_{xxx},w_{tx},\theta_{xx})(t)\|_{L^\infty} \\
&\quad \le C(\|w_0\|_5 + \|w_1,\theta_0\|_4 + \|w_0,w_{0x},w_1,\theta_0\|_{L^1}).
\end{aligned} \tag{1.28}$$

2 L^2-ENERGY ESTIMATES

In this section we prove Theorem 1 employing the L^2-energy method. Our present concern is the Cauchy problem to the system of equations (1.4) with the initial data (1.5).

The global existence of the solution is given by the combination of the local existence (Proposition 2.1) and the *a priori* estimates (Proposition 2.2). This observation immediately gives the proof of Theorem 1.

By multiplying $(1.4)_1$ by $a(v,\theta)$, the resultant system becomes the symmetric hyperbolic–parabolic system. Thus, the local existence theorem below immediately follows from the general theory constructed in Kawashima [3]. The readers are referred to [8], too.

PROPOSITION 2.1 (Local Existence) *Let $s \geq 3$ be an integer. Suppose that $(v_0, u_0, \theta_0) \in H^s(\mathbf{R})$. Then, there exists a positive constant T_0, depending only on $\|(v_0, u_0, \theta_0)\|_s$, such that the initial value problem* (1.4) *and* (1.5) *has a unique solution (v, u, θ) satisfying that*

$$\begin{aligned}(u, v) &\in C^0([0, T_0]; H^s(\mathbf{R})) \cap C^1([0, T_0]; H^{s-1}(\mathbf{R})),\\ \theta &\in C^0([0, T_0]; H^s(\mathbf{R})) \cap C^1([0, T_0]; H^{s-2}(\mathbf{R}))\\ &\qquad \cap L^2([0, T_0]; H^{s+1}(\mathbf{R})).\end{aligned}$$

Our theory concerning the asymptotic states requires the solutions (v, u, θ) to be in the space $H^4(\mathbf{R})$ in the spatial variable x. Thus, we fix $s = 4$ hereafter. Then, we introduce the solution space

$$X(0, T) := \{(v, u, \theta) \mid E(t; v, u, \theta) < \infty\}$$

Also, we use the supremum of $E(t; v, u, \theta) = E_1(t; v, u, \theta) + \int_0^t E_2(\tau; v, u, \theta)\, \mathrm{d}\tau$:

$$N(T)^2 := N(T; v, u, \theta)^2 = \sup_{0 \leq t \leq T} E(t; v, u, \theta).$$

Apparently, it holds that

$$\|(v, u, \theta)(t)\|_4 \leq E(t; v, u, \theta).$$

Thus, we can combine the following *a priori* estimates with the local existence theorem.

PROPOSITION 2.2 (*A Priori* Estimates) *Let* $(v,u,\theta)\in X(0,T)$ *be a solution of* (1.7), (1.5) *satisfying* $N(T)\leq 1$. *Then, there exists a positive constant* ε_2 *such that if* $\|v_0,u_0,\theta_0\|_4\leq\varepsilon_2$, *then* (v,u,θ) *satisfies* (1.10) *for* $0\leq t\leq T$.

We now devote ourselves to the proof of Proposition 2.2, which will be done in several steps.

Step 1 We first multiply $(1.7)_1$–$(1.7)_3$ by v,u,θ, respectively, to have

$$\left(\frac{1}{2}\int v^2\,\mathrm{d}x\right)_t+\int uv_x\,\mathrm{d}x=0$$

$$\left(\frac{1}{2}\int u^2\,\mathrm{d}x\right)_t+\int(-uv_x+b_0u\theta_x+u^2)\,\mathrm{d}x=\int g_2\cdot u\,\mathrm{d}x$$

$$\left(\frac{1}{2}\int \theta^2\,\mathrm{d}x\right)_t+\int(-b_0u\theta_x+\theta_x^2)\,\mathrm{d}x=\int g_3\cdot\theta\,\mathrm{d}x.$$

Here and hereafter, the integrand **R** is often abbreviated. Adding three equations, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(v,u,\theta)(t)\|^2+\|(u,\theta_x)(t)\|^2&=\int(g_2\cdot u+g_3\cdot\theta)\,\mathrm{d}x\\&=:F_1^{(0)}(t;g).\end{aligned}\tag{2.1$_0$}$$

Integrating $(2.1)_0$ over $[0,t]$, $t\leq T$, we have first lemma.

LEMMA 2.1 *For some constant C independent of t it holds that*

$$\begin{aligned}&\|(v,u,\theta)(t)\|^2+\int_0^t\|(u,\theta_x)(\tau)\|^2\,\mathrm{d}\tau\\&\quad\leq C\left(\|v_0,u_0,\theta_0\|^2+\int_0^t F_1^{(0)}(\tau;g)\,\mathrm{d}\tau\right).\end{aligned}\tag{2.2}$$

Step 2 Multiplying $(1.7)_1$–$(1.7)_3$ by $-\partial_x^2v,-\partial_x^2u,-\partial_x^2\theta$, respectively, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(v_x,u_x,\theta_x)(t)\|^2+\|(u_x,\theta_{xx})(t)\|^2\\&\quad=\int(\partial_xg_2\cdot\partial_xu+\partial_xg_3\cdot\partial_x\theta)\,\mathrm{d}x=:F_1^{(1)}(t;g).\end{aligned}\tag{2.1$_1$}$$

We also multiply $(1.7)_2$ and $(1.7)_3$ by u_t, θ_t, respectively, and add the resultant equations to have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|(u,\theta_x)(t)\|^2 + \int (b_0\theta_x - v_x)u\,\mathrm{d}x\right] + \int (u_t^2 + \theta_t^2 - u_x^2 + 2b_0\theta_t u_x)\,\mathrm{d}x = \int (g_2\cdot u_t + g_3\cdot\theta_t)\,\mathrm{d}x. \quad (2.3)$$

Calculating $(2.1)_1 + (2.3)\times\lambda$ for a small positive constant λ, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|(v_x,u_x)(t)\|^2 + \frac{1+\lambda}{2}\|\theta_x(t)\|^2 + \frac{\lambda}{2}\|u(t)\|^2\right] \\
&+\int \lambda(b_0\theta_x - v_x)u\,\mathrm{d}x + (1-\lambda)\|u_x(t)\|^2 + \lambda\|(u_t,\theta_t)(t)\|^2 \\
&+\|\theta_{xx}(t)\|^2 + \int 2\lambda b_0 u_x\cdot\theta_t\,\mathrm{d}x \\
&=\int(\partial_x g_2\cdot\partial_x u + \lambda\partial_x g_2\cdot u_t + \partial_x g_3\cdot\partial_x\theta + \lambda g_3\cdot\theta_t)\,\mathrm{d}x =: F_2^{(1)}(t;g).
\end{aligned} \quad (2.4)_1$$

and hence

$$\|(v_x,u,u_x,\theta_x)(t)\|^2 + \int_0^t \|(u_x,u_t,\theta_t,\theta_{xx})(\tau)\|^2\,\mathrm{d}\tau \le C\left(\|v_0,u_0,\theta_0\|_1^2 + \int_0^t F_2^{(1)}(\tau;g)\,\mathrm{d}\tau\right). \quad (2.5)$$

Moreover, differentiating $(1.7)_2$ with respect to x and using $(1.7)_1$, we have

$$v_t - v_{xx} + u_{tx} + b_0\theta_{xx} = g_{2x},$$

and, by multiplying this by v,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2 + \|v_x(t)\|^2 - \int (u_t + b_0\theta_x)v_x\,\mathrm{d}x = -\int g_2\cdot v_x\,\mathrm{d}x =: F_3^{(1)}(t;g). \quad (2.6)_1$$

By (2.2), (2.5) and the Schwarz inequality

$$\|v(t)\|^2 + \int_0^t \|v_x(\tau)\|^2 \, \mathrm{d}\tau \\ \leq C\left(\|v_0, u_0, \theta_0\|_1^2 + \int_0^t (F_1^{(0)} + F_2^{(1)} + F_3^{(1)})(\tau; g) \, \mathrm{d}\tau \right). \qquad (2.7)$$

We now have had the integrability of $\|v_x(\tau)\|^2$ on $[0, t]$. Hence we turn back to $(2.4)_1$ and multiply $(2.4)_1$ by $(1 + t)$ to obtain

$$(1+t)\|(v_x, u, u_x, \theta_x)(t)\|^2 + \int_0^t (1+\tau)\|(v_x, u_t, \theta_t, \theta_{xx})(\tau)\|^2 \, \mathrm{d}\tau \\ \leq C\left(\|v_0, u_0, \theta_0\|_1^2 + \int_0^t (F_1^{(0)}(\tau; g) + (1+\tau)F_2^{(1)}(\tau; g) + F_3^{(1)}(\tau; g)) \, \mathrm{d}\tau \right) \\ =: C\left(\|v_0, u_0, \theta_0\|_1^2 + \int_0^t H_1(\tau; g) \, \mathrm{d}\tau \right). \qquad (2.8)$$

Combining (2.8) and (2.7) we have the second lemma.

LEMMA 2.2 *It holds that*

$$(1+t)\|(v_x, u, u_x, \theta_x)(t)\|^2 \\ + \int_0^t (\|v_x(\tau)\|^2 + (1+\tau)\|(u_x, u_t, \theta_t, \theta_{xx})(\tau)\|^2) \, \mathrm{d}\tau \\ \leq C\left(\|v_0, u_0, \theta_0\|_1^2 + \int_0^t H_1(\tau; g) \, \mathrm{d}\tau \right). \qquad (2.9)$$

Step 3 Estimates of higher order derivatives corresponding to $(2.1)_1$, $(2.4)_1$ and $(2.6)_1$, respectively, become

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^k(v, u, \theta)(t)\|^2 + \|\partial_x^k(u, \theta_x)(t)\|^2 \\ = \int (\partial_x^k g_2 \cdot \partial_x^k u + \partial_x^k g_3 \cdot \partial_x^k \theta) \, \mathrm{d}x := F_1^{(k)}(t; g), \qquad (2.1)_k$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|\partial_x^k(v,u)(t)\|^2 + \frac{1+\lambda}{2}\|\partial_x^k\theta(t)\|^2 + \frac{\lambda}{2}\|\partial_x^{k-1}u(t)\|^2\right]$$
$$+ \int \lambda(b_0\partial_x^k\theta - \partial_x^k v)\partial_x^{k-1}u\,\mathrm{d}x + (1-\lambda)\|\partial_x^k u(t)\|^2$$
$$+ \lambda\|\partial_x^{k-1}(u_t,\theta_t)(t)\|^2 + \|\partial_x^k\theta_x(t)\|^2 + \int 2\lambda b_0\partial_x^k u\cdot\partial_x^{k-1}\theta_t\,\mathrm{d}x$$
$$= \int (\partial_x^k g_2\cdot\partial_x^k u + \lambda\partial_x^{k-1}g_2\cdot\partial_x^{k-1}u_t + \partial_x^k g_3\cdot\partial_x^k\theta)$$
$$+ \lambda\partial_x^{k-1}g_3\cdot\partial_x^{k-1}\theta_t\,\mathrm{d}x =: F_2^{(k)}(t;g), \tag{2.4$_k$}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{k-1}v(t)\|^2 + \|\partial_x^k v(t)\|^2 - \int(\partial_x^{k-1}u_t + b_0\partial_x^k\theta)\partial_x^k v\,\mathrm{d}x$$
$$= -\int \partial_x^{k-1}g_2\cdot\partial_x^k v\,\mathrm{d}x =: F_3^{(k)}(t;g) \tag{2.6$_k$}$$

for $k=2,3,4$. Same method as that of obtaining Lemmas 2.1–2.2 yields the third lemma.

LEMMA 2.3 *It holds that*

$$(1+t)^2\|\partial_x(v_x,u,u_x,\theta_x)(t)\|^2$$
$$+ \int_0^t [(1+\tau)\|\partial_x^2 v(\tau)\|^2 + (1+\tau)^2\|\partial_x(u_x,u_t,\theta_t,\theta_{xx})(\tau)\|^2]\,\mathrm{d}\tau$$
$$\le C\left(\|v_0,u_0,\theta_0\|_2^2 + \int_0^t H_2(\tau;g)\,\mathrm{d}\tau\right), \tag{2.10}$$

$$(1+t)^3\|\partial_x^2(v_x,u,u_x,\theta_x)(t)\|^2$$
$$+ \int_0^t [(1+\tau)^2\|\partial_x^3 v(\tau)\|^2 + (1+\tau)^3\|\partial_x^2(u_x,u,\theta_t,\theta_{xx})(\tau)\|^2]\,\mathrm{d}\tau$$
$$\le C\left(\|v_0,u_0,\theta_0\|_3^2 + \int_0^t H_3(\tau;g)\,\mathrm{d}\tau\right), \tag{2.11}$$

and

$$
\begin{aligned}
&(1+t)^4\|\partial_x^3(v_x,u,u_x,\theta_x)(t)\|^2 \\
&\quad+\int_0^t[(1+\tau)^3\|\partial_x^4 v(\tau)\|^2+(1+\tau)^4\|\partial_x^3(u_x,u,\theta_t,\theta_{xx})(\tau)\|^2]\,\mathrm{d}\tau \\
&\qquad\le C\left(\|v_0,u_0,\theta_0\|_4^2+\int_0^t H_4(\tau;g)\,\mathrm{d}\tau\right), \qquad (2.12)
\end{aligned}
$$

where

$$
\begin{aligned}
H_m(\tau;g)=\sum_{k=1}^{m}\{&(1+\tau)^{k-1}F_1^{(k-1)}(\tau;g)+(1+\tau)^kF_2^{(k)}(\tau;g) \\
&+(1+\tau)^{k-1}F_3^{(k)}(\tau;g)\}. \qquad (2.13)
\end{aligned}
$$

Step 4 We next estimate the derivatives of (v,u,θ) with respect to t. Differentiate (1.7) in t once to have

$$
\begin{aligned}
&(v_t)_t-(u_t)_x=0, \\
&(u_t)_t-(v_t)_x+b_0(\theta_t)_x+u_t=g_{2t}, \qquad (2.14) \\
&(\theta_t)_t+b_0(u_t)_x-(\theta_t)_{xx}=g_{3t}.
\end{aligned}
$$

Since $\|(v_t,u_t,\theta_t)|_{t=0}\|\le C(\|v_0,u_0\|_1+\|\theta_0\|_2)$ and that $(1+\tau)\|(v_t=u_x,u_t,\theta_t)(\tau)\|^2$ is integrable on $[0,t]$ by Lemma 2.2, same way as in Lemma 2.3 yields the following lemma:

LEMMA 2.4 *It holds that*

$$
\begin{aligned}
&(1+t)^2\|\partial_t(v,u,\theta)(t)\|^2+\int_0^t(1+\tau)^2\|\partial_t(u,\theta_x)(\tau)\|^2\,\mathrm{d}\tau \\
&\quad\le C\left(\|v_0,u_0\|_1^2+\|\theta_0\|_2^2+\int_0^t\left[H_1(\tau;g)+(1+\tau)^2F_1^{(0)}(\tau;g_t)\right]\mathrm{d}\tau\right), \qquad (2.15)
\end{aligned}
$$

$$
\begin{aligned}
&(1+t)^3\|\partial_t(v_x,u,u_x,\theta_x)(t)\|^2+\int_0^t[(1+\tau)^2\|v_{tx}(\tau)\|^2 \\
&\quad+(1+\tau)^3\|\partial_t(u_x,u_t,\theta_t,\theta_{xx})(\tau)\|^2]\,\mathrm{d}\tau \\
&\qquad\le C\left(\|v_0,u_0\|_2^2+\|\theta_0\|_3^2+\int_0^t\left[H_1(\tau;g)+(1+\tau)^2H_1(\tau;g_t)\right]\mathrm{d}\tau\right) \qquad (2.16)
\end{aligned}
$$

and

$$
\begin{aligned}
&(1+t)^4\|\partial_x\partial_t(v_x,u,u_x,\theta_x)(t)\|^2 + \int_0^t \{(1+\tau)^3\|v_{txx}(\tau)\|^2 \\
&\quad + (1+\tau)^4\|\partial_x\partial_t(u_x,u_t,\theta_t,\theta_{xx}(\tau)\|^2\}\,\mathrm{d}\tau \\
&\quad \le C\left(\|v_0,u_0\|_3^2 + \|\theta_0\|_4^2 + \int_0^t \{H_1(\tau;g) + (1+\tau)^2H_2(\tau;g_t)\}\,\mathrm{d}\tau\right).
\end{aligned}
\tag{2.17}
$$

Step 5 Differentiating (2.14) in t once more, we have

LEMMA 2.5 *It holds that*

$$
\begin{aligned}
&(1+t)^4\|\partial_t^2(v,u,\theta)(t)\|^2 + \int_0^t (1+\tau)^4\|\partial_t^2(u,\theta_x)(\tau)\|^2\,\mathrm{d}\tau \\
&\quad \le C\left(\|v_0,u_0,\theta_0\|_4^2 + \int_0^t [H_1(\tau;g) + (1+\tau)^2H_1(\tau;g_t)\right. \\
&\qquad \left. + (1+\tau)^4F_1^{(0)}(\tau;g_{tt})]\,\mathrm{d}\tau\right).
\end{aligned}
\tag{2.18}
$$

Step 6 Adding all inequalities obtained in Lemmas 2.1–2.5, we have

$$
\begin{aligned}
&E_1(t;v,u,\theta) + \int_0^t E_2(\tau;v,u,\theta)\,\mathrm{d}\tau \\
&\quad \le C(\|v_0,u_0,\theta_0\|^2 + \int_0^t [H_4(\tau;g) + (1+\tau)^2H_2(\tau;g_t) \\
&\qquad + (1+\tau)^4F_1^{(0)}(\tau;g_{tt})]\,\mathrm{d}\tau.
\end{aligned}
\tag{2.19}
$$

Here we have used $F_1^{(0)}(t;g) \ll H_1(t;g) \ll H_2(t;g) \ll H_3(t;g) \ll H_4(t;g)$, where $F \ll H$ means that all terms of F are included in H.

The last term of (2.19) has higher orders of (v,u,θ) and estimated as follows:

LEMMA 2.6 *For small positive constant ν it holds that*

$$
\begin{aligned}
&C\int_0^t [H_4(\tau;g) + (1+\tau)^2H_2(\tau;g_t) + (1+\tau)^4F_1^{(0)}(\tau;g_{tt})]\,\mathrm{d}\tau \\
&\quad \le C\|v_0,u_0,\theta_0\|_4^2 + \nu\int_0^t E_2(\tau;v,u,\theta)\,\mathrm{d}\tau + CN(T)^{3/2}.
\end{aligned}
$$

The proof of lemma 2.6 is not difficult, but many and tedious calculations are necessary. So, we only show a few terms. For example, $\int_0^t H_4(\tau; g)\,\mathrm{d}\tau$ includes

$$J_1 := \int_0^t \int \frac{1}{c(v,\theta)}(b_0 - b(v,\theta))u_x \cdot \theta \,\mathrm{d}x\,\mathrm{d}\tau,$$

$$J_2 := \int_0^t (1+\tau)^4 \int (a(v,\theta) - 1) v_{xxxx} u_{xxxt}\,\mathrm{d}x\,\mathrm{d}\tau,$$

the latter of which is in $\int_0^t (1+\tau)^4 \int \partial_x^3 g_2 \cdot \partial_x^3 u_t \,\mathrm{d}x\,\mathrm{d}\tau$. J_1 is estimated as follows:

$$\begin{aligned}
&\int_0^t \int \left[-\left(\frac{\theta}{c(v,\theta)}\right)_x (b_0 - b(v,\theta))u + \frac{\theta}{c(v,\theta)} b(v,\theta)_x u \right] \mathrm{d}x\,\mathrm{d}\tau \\
&\quad \le CN(T)^{1/2} \int_0^t \int (v_x^2 + u^2 + \theta_x^2)\,\mathrm{d}x\,\mathrm{d}\tau \le CN(T)^{3/2}.
\end{aligned}$$

Since $v_t = u_x$,

$$\begin{aligned}
J_2 &= \int_0^t (1+\tau)^4 \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \int (a(v,\theta) - 1) v_{xxxx} u_{xxx}\,\mathrm{d}x \right. \\
&\quad \left. - \int (a(v,\theta)_\tau v_{xxxx} u_{xxx} - (a(v,\theta) - 1) v_{txxxx} u_{xxx})\,\mathrm{d}x \right] \mathrm{d}\tau \\
&= (1+\tau)^4 \int (a(v,\theta) - 1) v_{xxxx} u_{xxx}\,\mathrm{d}x \Big|_{\tau=0}^{\tau=t} \\
&\quad - 4 \int_0^t (1+\tau)^3 \int (a(v,\theta) - 1) v_{xxxx} u_{xxx}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad - \int_0^t (1+\tau)^4 \int (a(v,\theta)_\tau v_{xxxx} u_{xxx} + a(v,\theta)_x u_{xxxx} u_{xxx} \\
&\quad + (a(v,\theta) - 1) u_{xxxx}^2)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\le CN(T)^{3/2} + C\|v_0, u_0, \theta_0\|_4^2 + \nu \int_0^t (1+\tau)^4 \|u_{xxxx}(\tau)\|^2\,\mathrm{d}\tau \\
&\quad + CN(T)^{1/2} \int_0^t [(1+\tau)^3 \|(v_{xxxx}, u_{xxx})(\tau)\|^2 + (1+\tau)^4 \|u_{xxxx}(\tau)\|^2]\,\mathrm{d}\tau \\
&\le C\|v_0, u_0, \theta_0\|_4^2 + \nu \int_0^t (1+\tau)^4 \|u_{xxxx}(\tau)\|^2\,\mathrm{d}\tau + CN(T)^{3/2}.
\end{aligned}$$

The other terms are omitted. We now have reached to the inequality

$$N(T) \le C(\|v_0, u_0, \theta_0\|_4^2 + N(T)^{3/2}),$$

and hence

$$N(T) \le C\|v_0, u_0, \theta_0\|_4^2$$

provided that $\|v_0, u_0, \theta_0\|_4$ is suitably small. Thus, we have completed the proof of Proposition 2.2.

3 ESTIMATES IN L^1-FRAMEWORK

In this section we prove Theorem 2. Assuming $(v_0, \theta_0) \in L^1$ in addition to the assumptions in Theorem 1, we remind the "explicit" formula (1.24) of (v, θ). In order to obtain the estimates of (v, θ), it is enough to estimate $I_1 := G * v_0$, $I_2 := G * \theta_0$, $II := \int_0^t G * u_{xt}$, $III := \int_0^t G * g_{2x}\, \mathrm{d}\tau$ and $IV := \int_0^t G * g_3\, \mathrm{d}\tau$, where $G = G_1$ or G_2, and g_2, g_3, G_1, G_2 are, respectively, given by (1.9) and (1.20).

First, we seek for the L^∞-norm of v, θ. Since $\|G(t)\|_{L^\infty} \le O(t^{-1/2})$, it is easily seen that

$$|I_1| + |I_2| \le Ct^{-1/2} \tag{3.1}$$

(From now on we denote a constant depending on $\|v_0, u_0, \theta_0\|_4 + \|v_0, \theta_0\|_{L^1}$ simply by C.) Dividing the integrand $(0, t)$ into $(0, t/2) \cup (t/2, t)$ and using the Hausdorff–Young inequality, we have

$$\begin{aligned}
|II| &\le \int_0^{t/2} \|G_x(t-\tau)\|\, \|u_t(\tau)\|\, \mathrm{d}\tau + \int_{t/2}^t \|G(t-\tau)\|\, \|u_{xt}(\tau)\|\, \mathrm{d}\tau \\
&\le C\int_0^{t/2} (t-\tau)^{-3/4}(1+\tau)^{-3/2}\, \mathrm{d}\tau + C\int_{t/2}^t (t-\tau)^{-1/4}(1+\tau)^{-2}\, \mathrm{d}\tau \\
&\le Ct^{-3/4},
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
|III| &\leq \int_0^{t/2} \|G_x(t-\tau)\|_{L^\infty} \|g_2\|_{L^1} \, \mathrm{d}\tau + \int_{t/2}^t \|G(t-\tau)\|_{L^\infty} \|g_{2x}\| \, \mathrm{d}\tau \\
&\leq C \int_0^{t/2} (t-\tau)^{-3/2} \|(v,\theta)(\tau)\| \, \|(v_x,\theta_x)(\tau)\| \, \mathrm{d}\tau \\
&\quad + C \int_{t/2}^t (t-\tau)^{-1/2} (\|(v_x,\theta_x)(\tau)\|^2 + \|(v,\theta)(\tau)\| \|(v_{xx},\theta_{xx})(\tau)\| \, \mathrm{d}\tau) \\
&\leq C t^{-3/2} \int_0^{t/2} 1 \cdot (1+\tau)^{-1/2} \, \mathrm{d}\tau + C(1+t)^{-1} \int_{t/2}^t (t-\tau)^{-1/2} \, \mathrm{d}\tau \\
&\leq C t^{-1/2} \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
|IV| &\leq \int_0^t \|G(t-\tau)\|_{L^\infty} \|g_3(\tau)\|_{L^1} \, \mathrm{d}\tau \\
&\leq C \int_0^t (t-\tau)^{-1/2} \|(v,\theta)(\tau)\| \, \|(u_x,\theta_{xx})(\tau)\| \, \mathrm{d}\tau \\
&\leq C \left(\int_0^{t/2} + \int_{t/2}^t \right) (t-\tau)^{-1/2} \cdot 1 \cdot (1+\tau)^{-1/2} \, \mathrm{d}\tau \\
&\leq C t^{-1} \ln(2+t). \qquad (3.4)
\end{aligned}
$$

Hence, together with $\|(v,\theta)(t)\|_{L^\infty} \leq C$, (3.2)–(3.4) and (1.24) give

$$
\|(v,\theta)(t)\|_{L^\infty} \leq C(1+t)^{-1/2} \ln(2+t), \qquad (3.5)
$$

which will be improved soon after getting the estimates of $\|(v,\theta)(t)\|$.

Next, we seek for $\|(v,\theta)(t)\|$ in a similar fashion to the above:

$$
\|I_1\| + \|I_2\| \leq \|G(t)\| (\|v_0\|_{L^1} + \|\theta_0\|_{L^1}) \leq C t^{-1/4}, \qquad (3.6)
$$

$$
\begin{aligned}
\|II\| + \|III\| &\leq \int_0^{t/2} (\|G_x\|_{L^1} \|u_t\| + \|G_x\| \, \|g_2\|_{L^1}) \, \mathrm{d}\tau \\
&\quad + \int_{t/2}^t (\|G\|_{L^1} \|u_{xt}\| + \|G\| \, \|g_{2x}\|_{L^1}) \, \mathrm{d}\tau \\
&\leq C t^{-1/4}, \qquad (3.7)
\end{aligned}
$$

and

$$\begin{aligned}
\|IV\| &\le \int_0^t \|G\|\,\|g_3\|_{L^1}\,\mathrm{d}\tau \\
&\le C\left(\int_0^{t/2} + \int_{t/2}^t\right)(t-\tau)^{-1/4}\|(v,\theta)(\tau)\|\,\|(u_x,\theta_{xx})(\tau)\|\,\mathrm{d}\tau \\
&\le Ct^{-1/4}\ln(2+t). \qquad (3.8)
\end{aligned}$$

Hence

$$\|(v,\theta)(t)\| \le C(1+t)^{-1/4}\ln(2+t). \qquad (3.9)$$

Applying (3.9), just obtained, to (3.4) and (3.7) we have

$$\|IV\|_{L^\infty} \le C(1+t)^{-1/2}, \quad \|IV\| \le Ct^{-1/4}$$

from which we obtain the desired estimate

$$(1+t)^{1/2}\|(v,\theta)(t)\|_{L^\infty} + (1+t)^{1/4}\|(v,\theta)(t)\| \le C. \qquad (3.10)$$

By (1.24) the estimates of $I_{1x}, \dots, IV_x$ yield

$$(1+t)\|(v_x,\theta_x)(t)\|_{L^\infty} + (1+t)^{3/4}\|(v_x,\theta_x)(t)\| \le C. \qquad (3.11)$$

From (1.25), (3.11) and the Sobolev inequality

$$\begin{aligned}
\|u(t)\|_{L^\infty} &\le C(\|(v_x,\theta_x)(t)\|_{L^\infty} \\
&\quad + \|u_t(t)\|_{L^\infty} + \|(v,\theta)(t)\|_{L^\infty}\|(v_x,\theta_x)(t)\|_{L^\infty} \\
&\le C(1+t)^{-1} \qquad (3.12)
\end{aligned}$$

and

$$\begin{aligned}
\|u(t)\| &\le C(\|(v_x,\theta_x)(t)\| + \|u_t(t)\| + \|(v,\theta)(t)\|_{L^\infty}\|(v_x,\theta_x)(t)\|) \\
&\le C(1+t)^{-3/4}. \qquad (3.13)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|(v_{xx},\theta_{xx},u_x)(t)\|_{L^\infty} &\le C(1+t)^{-3/2}\\ \|(v_{xx},\theta_{xx},u_x)(t)\| &\le C(1+t)^{-5/4}.\end{aligned}\tag{3.14}$$

By $(1.7)_1$ and $(1.7)_3$ v_t and θ_t have same decay orders as (3.14). Equations (3.10)–(3.14) yield the desired estimate (1.26). Here, we note that the assumption $u_0 \in L^1$ is not necessary till now.

4 THERMOELASTIC SYSTEM OF SECOND ORDER

In the final section we consider the original second order thermoelastic system (1.1) with dissipation, and prove Theorem 3.

For the solution (v,u,θ) of (1.4) with the initial data $(v_0,u_0,\theta_0)=(w_{0x},w_1,\theta_0)$ obtained in Theorems 1 and 2, Eqs. (1.24) and (1.27) give the solution (w,θ) of (1.1) by

$$\begin{aligned}w(x,t) &= (G_{11}*w_0)(x,t)+\int_{-\infty}^{x}(G_{12}*\theta_0)(\xi,t)\,\mathrm{d}\xi\\ &\quad+\int_0^t\Big[(G_{11}+b_0G_{12})(\cdot,t-\tau)*(-u_t+g_2)(\cdot,\tau)\Big](x)\,\mathrm{d}\tau\\ &\quad+\int_{-\infty}^{x}\int_0^t\Big[G_{12}(\cdot,t-\tau)*g_3(\cdot,\tau)\Big](\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi\\ &= (1)+(2)+(3)+(4)\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}\theta(x,t) &= (G_{12}*w_{0x})(x,t)+(G_{22}*\theta_0)(x,t)\\ &\quad\times\int_0^t[G_{12}(\cdot,t-\tau)*(-u_{xt}+g_{2x})(\cdot,\tau)+G_{22}(\cdot,t-\tau)\\ &\quad*(b_0(-u_{xt}+g_{2x})+g_3)(\cdot,\tau)]\,\mathrm{d}\tau,\end{aligned}\tag{4.2}$$

where

$$\begin{aligned}G_{11} &= \alpha G_1+(1-\alpha)G_2,\quad G_{12}=\gamma(G_1-G_2)\\ G_{22} &= \beta G_1+(1-\beta)G_2.\end{aligned}\tag{4.3}$$

First, note that, for any $f \in L^1 \cap L^2$,

$$
\begin{aligned}
&\int_{-\infty}^{x} (G_{12} * f)(\xi)\,\mathrm{d}\xi \\
&= \int_{-\infty}^{x} [(G_1 - G_2) * f](\xi)\,\mathrm{d}\xi \\
&= \int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{4k_1\pi t}} \mathrm{e}^{-((\xi-y)^2/4k_1 t)} f(y)\,\mathrm{d}y \right. \\
&\quad \left. - \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k_2\pi t}} \mathrm{e}^{-((\xi-y)^2/4k_2 t)} f(y)\,\mathrm{d}y \right] \mathrm{d}\xi \\
&= \int_{-\infty}^{x} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-(\eta^2/4t)} \left(f\left(\xi + \sqrt{k_1}\eta\right) - f\left(\xi + \sqrt{k_2}\eta\right) \right) \mathrm{d}\eta\,\mathrm{d}\xi \\
&= \int_{-\infty}^{x} \int_{-\infty}^{\infty} \frac{(\sqrt{k_1} - \sqrt{k_2})\eta}{\sqrt{4\pi t}} \mathrm{e}^{-(\eta^2/4t)} \\
&\quad \times \int_0^1 f'\left(\xi + \sqrt{k_1}\eta + \lambda\left(\sqrt{k_2} - \sqrt{k_1}\right)\eta\right) \mathrm{d}\lambda\,\mathrm{d}\eta\,\mathrm{d}\xi \\
&= \left(\sqrt{k_1} - \sqrt{k_2}\right) \int_{-\infty}^{\infty} \frac{\eta}{\sqrt{4\pi t}} \mathrm{e}^{-(\eta^2/4t)} \\
&\quad \times \int_0^1 f\left(x + \left(\sqrt{k_1} + \lambda\left(\sqrt{k_2} - \sqrt{k_1}\right)\right)\eta\right) \mathrm{d}\lambda\,\mathrm{d}\eta \\
&= \left(\sqrt{k_1} - \sqrt{k_2}\right) \int_{-\infty}^{\infty} \eta \cdot G_0(\eta, t) \\
&\quad \times \int_0^1 f\left(x + \left(\sqrt{k_1} + \lambda\left(\sqrt{k_2} - \sqrt{k_1}\right)\right)\eta\right) \mathrm{d}\lambda\,\mathrm{d}\eta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left| \int_{-\infty}^{x} [(G_1 - G_2) * f](\xi)\,\mathrm{d}\xi \right| &\leq C \sup_{\mathbf{R}} |\eta \cdot G_0(\eta, t)| \cdot \|f\|_{L^1} \\
&\leq C\|f\|_{L^1}
\end{aligned} \tag{4.4}
$$

and

$$
\begin{aligned}
\left\| \int_{-\infty}^{x} [(G_1 - G_2) * f](\xi)\,\mathrm{d}\xi \right\| &\leq C\|\eta \cdot G_0(\eta, t)\|\,\|f\| \\
&\leq Ct^{1/4}\|f\|.
\end{aligned} \tag{4.5}
$$

Using (4.4) and (4.5) we estimate each term of (4.1). First two terms are easily estimated as

$$|(1)| \leq C(1+t)^{-1/2}, \quad \|(1)\| \leq C(1+t)^{-1/4} \tag{4.6}$$

and

$$|(2)| \leq C, \quad \|(2)\| \leq C(1+t)^{1/4} \tag{4.7}$$

if $\theta_0 \in L^1$. In this section, only by C denote a constant depending on $\|w_0\|_5 + \|w_1, \theta_0\|_4 + \|w_0, w_{0x}, w_1, \theta_0\|_{L^1}$. For (3) it is enough to estimate $(3)_1 := \int_0^t G * u_t \, \mathrm{d}\tau$ and $(3)_2 := \int_0^t G * g_2 \, \mathrm{d}\tau$, where $G = G_1$ or G_2. By the integration by parts in τ,

$$\begin{aligned}(3)_1 = {} & [G(t-\tau) * u(\tau)]|_{\tau=0}^{\tau=t/2} + \int_0^{t/2} G_t(t-\tau) * u(\tau) \, \mathrm{d}\tau \\ & + \int_{t/2}^{t} G(t-\tau) * u_t(\tau) \, \mathrm{d}\tau\end{aligned}$$

and hence, from Theorems 1 and 2,

$$\begin{aligned}|(3)_1| \leq {} & \|G(t/2)\| \, \|u(t/2)\| + \|G(t)\|_{L^\infty} \|w_1\|_{L^1} \\ & + \int_0^{t/2} \|G_t(t-\tau)\| \, \|u(\tau)\| \, \mathrm{d}\tau + \int_{t/2}^{t} \|G(t-\tau)\| \, \|u_t(\tau)\| \, \mathrm{d}\tau \\ \leq {} & C\biggl(t^{-1} + t^{-1/2} + \int_0^{t/2} (t-\tau)^{-5/4}(1+\tau)^{-3/4} \, \mathrm{d}\tau \\ & + \int_{t/2}^{t} (t-\tau)^{-1/4}(1+\tau)^{-3/2} \, \mathrm{d}\tau\biggr) \\ \leq {} & Ct^{-1/2}\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}&\|(3)_1\| \\ &\quad \leq \|G(t/2)\|_{L^1} \|u(t/2)\| + \|G(t)\| \, \|w_1\|_{L^1} \\ &\quad\quad + \int_0^{t/2} \|G_t(t-\tau)\|_{L^1} \|u(\tau)\| \, \mathrm{d}\tau + \int_{t/2}^{t} \|G(t-\tau)\|_{L^1} \|u_t(\tau)\| \, \mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&\leq C\left(t^{-3/4}+t^{-1/4}+\int_0^{t/2}(t-\tau)^{-1}(1+\tau)^{-3/4}\,\mathrm{d}\tau\right.\\
&\qquad\left.+\int_{t/2}^{t}1\cdot(1+\tau)^{-3/2}\,\mathrm{d}\tau\right)\\
&\leq Ct^{-1/4}.
\end{aligned}
\tag{4.9}
$$

Since

$$
\|g_2(t)\|_{L^1}\leq C\|(v,\theta)(t)\|\,\|(v_x,\theta_x)(t)\|\leq C(1+t)^{-1},
\tag{4.10}
$$

it holds that

$$
\begin{aligned}
|(3)_2|&\leq\int_0^t\|G(t-\tau)\|_{L^\infty}\|g_2(\tau)\|_{L^1}\,\mathrm{d}\tau\\
&\leq C\left(\int_0^{t/2}+\int_{t/2}^{t}\right)(t-\tau)^{-1/2}(1+\tau)^{-1}\,\mathrm{d}\tau\\
&\leq C(1+t)^{-1/2}\ln(2+t),
\end{aligned}
\tag{4.11}
$$

and that

$$
\begin{aligned}
\|(3)_2\|&\leq\int_0^t\|G(t-\tau)\|\,\|g_2(\tau)\|_{L^1}\,\mathrm{d}\tau\\
&\leq C(1+t)^{-1/4}\ln(2+t).
\end{aligned}
\tag{4.12}
$$

Estimates of the final term (4) are as follows:

$$
\begin{aligned}
|(4)|&\leq C\int_0^t\|g_3\|_{L^1}\,\mathrm{d}\tau\\
&\leq C\int_0^t(\|(v,\theta)(\tau)\|\,\|u_x(\tau)\|+\|(\theta,\theta_x)(\tau)\|\,\|\theta_{xx}\|)\,\mathrm{d}\tau\\
&\leq C\int_0^t(1+\tau)^{-1/4-5/4}\,\mathrm{d}\tau\leq C
\end{aligned}
\tag{4.13}
$$

and

$$\begin{aligned}\|(4)\| &\leq C \int_0^t (t-\tau)^{1/4} \|g_3(\tau)\| \, \mathrm{d}\tau \\ &\leq C \int_0^t (t-\tau)^{-1/4} (1+\tau)^{-1/2-5/4} \, \mathrm{d}\tau \leq C t^{1/4}. \end{aligned} \tag{4.14}$$

Combining (4.6)–(4.14) we obtain

$$(1+t)^{1/4} \|w(t)\| + \|w(t)\|_{L^\infty} \leq C.$$

The other terms $w_x = v$, $w_t = u$, θ etc. are same as the orders in Theorem 2.

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