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Comparison of Two Definitions of Lower and Upper Functions Associated to Nonlinear Second Order Differential Equations

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The notions of lower and upper functions of the second order differential equations take their beginning from the classical work by C. Scorza-Dragoni and have been investigated till now because they play an important role in the theory of nonlinear boundary value problems. Most of them define lower and upper functions as solutions of the corresponding second order differential inequalities. The aim of this paper is to compare two more general approaches. One is due to Rachůnková and Tvrdý (Nonlinear systems of differential inequalities and solvability of certain boundary value problems (*J. of Inequal. & Appl.* (to appear))) who defined the lower and upper functions of the given equation as solutions of associated systems of two differential inequalities with solutions possibly not absolutely continuous. The second belongs to Fabry and Habets (*Nonlinear Analysis*, *TMA* **10** (1986), 985–1007) and requires the monotonicity of certain integro-differential expressions.

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The method of lower and upper functions is an effective tool in the theory of nonlinear boundary value problems for the second order differential equation

$$u'' = f(t, u, u'). \tag{1}$$

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Let us note that the terminology is not uniform and some authors use the term lower and upper solutions. Until now, a lot of definitions of these notions, less or more general, have been introduced. In [3], the authors made use of the following definition of lower and upper functions of (1) where f fulfils the Carathéodory conditions on $[a,b]\times\mathbb{R}^2$, i.e. f has the following properties: (i) for each $x\in\mathbb{R}$ and $y\in\mathbb{R}$ the function $f(\cdot,x,y)$ is measurable on $[a,b]$; (ii) for almost every $t\in[a,b]$ the function $f(t,\cdot,\cdot)$ is continuous on $\mathbb{R}^2$; (iii) for each compact set $K\subset\mathbb{R}^2$ the function $m_K(t)=\sup_{(x,y)\in K}|f(t,x,y)|$ is Lebesgue integrable on $[a,b]$. By $\mathrm{Car}([a,b]\times\mathbb{R}^2)$ we denote the set of functions which satisfy the Carathéodory conditions on $[a,b]\times\mathbb{R}^2$.

DEFINITION 1 *Functions (σ,ρ) are called lower (upper) functions of (1) (on $[a,b]$) if σ is absolutely continuous on $[a,b]$ and ρ has a bounded variation on $[a,b]$, the singular part ρ^{sing} of ρ is nondecreasing (nonincreasing) on $[a,b]$ and they verify the following system of differential inequalities:*

$$\begin{aligned}&\sigma'(t)=\rho(t) && \text{a.e. on } [a,b],\\ &\rho'(t)\geq f(t,\sigma(t),\rho(t)) \quad (\rho'(t)\leq f(t,\sigma(t),\rho(t))) && \text{a.e. on } [a,b]. \end{aligned} \tag{2}$$

On the other hand, the authors of [1] introduce somewhat different definition of lower and upper functions of (1) with f continuous.

DEFINITION 2 *A lower (upper) function of (1) is a continuous function α on $[a,b]$ which possesses for all $t\in[a,b)$ the right derivative $\alpha'_+(t)$ and for all $t\in(a,b]$ the left derivative $\alpha'_-(t)$ such that $\alpha'_+(t)$ is right-continuous on $[a,b)$,*

$$\alpha'_-(t)\leq\alpha'_+(t)\quad(\alpha'_-(t)\geq\alpha'_+(t))\qquad \text{for all } t\in(a,b)$$

and

$$\alpha'_+(t)-\int_a^t f(s,\alpha(s),\alpha'_+(s))\,\mathrm{d}s\qquad\left(-\alpha'_+(t)+\int_a^t f(s,\alpha(s),\alpha'_+(s))\,\mathrm{d}s\right)$$

is nondecreasing on $[a,b)$.

At one glance the relationship of Definitions 1 and 2 is not clear. Their comparison will be given by the following two theorems.

THEOREM 1 *Let* $f \in \mathrm{Car}([a,b] \times \mathbb{R}^2)$. *Then, if* α *is a lower (upper) function of* (1) *in the sense of Definition* 2, *the functions* (σ, ρ), *where*

$$\sigma(t) \equiv \alpha(t) \text{ on } [a,b] \quad \text{and}$$
$$\rho(t) = \begin{cases} \alpha'_+(t) & \text{if } t \in [a,b), \\ \lim_{s\to b-} \alpha'_+(s) & \text{if } t = b, \end{cases} \tag{3}$$

are lower (upper) functions of (1) *with respect to Definition* 1.

THEOREM 2 *Let the functions* (σ, ρ) *be lower (upper) functions of* (1) *in the sense of Definition* 1. *Then the function* α *defined by*

$$\alpha(t) = \sigma(t) \quad \text{for } t \in [a,b]$$

is a lower (upper) function of (1) *in the sense of Definition* 2.

Let us start with the proof of Theorem 1. To this aim the following three lemmas are helpful.

LEMMA 1 *Let a function g be defined and continuous on the interval* $[a,b)$ *and assume*

(i) $g'_+(t)$ *exists for all* $t \in [a,b)$,
(ii) $g'_+(t)$ *is right-continuous on* $[a,b)$ *and*
(iii) *there exists a function h continuous on* $[a,b)$ *and such that* $g'_+(t) - h(t)$ *is nondecreasing on* $[a,b)$.

Then the function g is absolutely continuous on every interval $[a,c]$, $c \in (a,b)$.

Proof Choose $c \in (a,b)$. Condition (iii) implies

$$g'_+(t) - h(t) \le g'_+(c) - h(c) \quad \text{for } t \in [a,c]$$

so that

$$g'_+(t) \le g'_+(c) + h(t) - h(c) \le g'_+(c) + \max_{t\in[a,c]}[h(t) - h(c)].$$

Similarly, for $t \in [a,c]$ we have

$$g'_+(t) \ge g'_+(a) + h(t) - h(a) \ge g'_+(a) + \min_{t\in[a,c]}[h(t) - h(a)],$$

i.e. $g'_+(t)$ is bounded. Define

$$\tilde{g}(t) = \int_a^t g'_+(s)\,\mathrm{d}s \quad \text{for } t \in [a,b).$$

Due to (ii) (continuity from the right of $g'_+(t)$), for any $t \in [a,b)$ the derivative $\tilde{g}'_+(t)$ is defined and $\tilde{g}'_+(t) = g'_+(t)$.

Now, the proof of the lemma will be completed by the proof of the following relation:

$$g(t) - \tilde{g}(t) \equiv g(a) - \tilde{g}(a) \text{ on } [a,b).$$

Denote $\Delta(t) = g(t) - \tilde{g}(t)$. Then $\Delta(t)$ is continuous on $[a,b)$ and $\Delta'_+(t) = 0$ for any $t \in [a,b)$. Assume that there is a point $s \in (a,b)$ such that $\Delta(s) > \Delta(a)$ and define

$$p(t) = \frac{1}{3}[\Delta(s) - \Delta(a) + \frac{\Delta(s) - \Delta(a)}{s-a}(t-a)] - (\Delta(t) - \Delta(a)). \quad (4)$$

Certainly, $p(a) > 0$ and $p(s) < 0$. Let t^* be the greatest point in (a,s) fulfilling $p(t^*) = 0$. Equality (4) yields

$$p'_+(t^*) = \frac{1}{3}\frac{\Delta(s) - \Delta(a)}{s-a} > 0, \quad (5)$$

Since $p(t) < 0$ for $t > t^*$, we have

$$\frac{p(t) - p(t^*)}{t - t^*} < 0$$

and hence $p'_+(t^*) \leq 0$, a contradiction with (5). The case $\Delta(s) < \Delta(a)$ is symmetric.

LEMMA 2 *Let α be a lower function of (1) according to Definition 2. Then the limit*

$$\alpha'_+(b-) = \lim_{t \to b-} \alpha'_+(t)$$

exists and is finite.

Proof Let us define the function $r:[a,b)\mapsto\mathbb{R}$ by the relation

$$\alpha'_+(t)=\int_a^t f(s,\alpha(s),\alpha'_+(s))\,\mathrm{d}s+r(t)\quad\text{for } t\in[a,b). \tag{6}$$

Due to Definition 2, r is nondecreasing and right-continuous on $[a,b)$. Consequently, the limits

$$\lim_{s\to t-}\alpha'_+(s)=\alpha'_+(t-),\quad t\in(a,b) \tag{7}$$

exist and

$$\alpha'_+(t-)\le\alpha'_+(t)\quad\text{on } (a,b). \tag{8}$$

Now, let

$$\liminf_{t\to b-}\alpha'_+(t)<\limsup_{t\to b-}\alpha'_+(t).$$

Let us fix the numbers $c_1, c_2\in\mathbb{R}$ in such a way that

$$\liminf_{t\to b-}\alpha'_+(t)<c_1<c_2<\limsup_{t\to b-}\alpha'_+(t).$$

Let $t_1, t_2\in(a,b)$ be such that

$$\alpha'_+(t_1)\ge c_2\quad\text{and}\quad\alpha'_+(t_2)\le c_1.$$

Consider a family $\mathcal{I}$ of closed intervals $I=[\tau_1,\tau_2]$ fulfilling

$$I\subset[t_1,t_2],\quad\alpha'_+(\tau_1)\ge c_2,\quad\alpha'_+(\tau_2)\le c_1. \tag{9}$$

Let I_n be a decreasing sequence of intervals from $\mathcal{I}$. Its intersection $I=\bigcap_{n=1}^{\infty}I_n$ is a nonempty interval $I=[\xi_1,\xi_2]$ and the existence of the limits (7) together with the relations (8) and the continuity from the right of α'_+ ensure that conditions (9) are fulfilled for I, either, i.e. $I\in\mathcal{I}$. It means that a minimal interval $I_{\min}=[z_1,z_2]$ exists in the family $\mathcal{I}$. Minimality yields $\alpha'_+(t)\in(c_1,c_2)$ for $t\in I_{\min}$. We conclude (put $K = \sup_{t\in[a,b]} |\alpha(t)|$) that

$$\int_{z_1}^{z_2} \sup_{|x|\le K, y\in[c_1,c_2]} |f(s,x,y)|\,\mathrm{d}s \ge -\int_{z_1}^{z_2} f(s,\alpha(s),\alpha'_+(s))\,\mathrm{d}s$$
$$= -(\alpha'_+(z_2) - \alpha'_+(z_1)) + (r(z_2) - r(z_1)) \ge c_2 - c_1.$$

This is in contradiction with the Carathéodory property of f since infinitely many of such disjoint intervals can be constructed. Hence

$$\liminf_{t\to b-} \alpha'_+(t) = \limsup_{t\to b-} \alpha'_+(t) = \lim_{t\to b-} \alpha'_+(t).$$

If

$$\lim_{t\to b-} \alpha'_+(t) = \infty$$

were valid, then by Lemma 1 we would have

$$\alpha'_-(b) = \lim_{t\to b-} \frac{\alpha(b)-\alpha(t)}{b-t} = \lim_{t\to b-}\left(\lim_{\tau\to b-} \frac{\alpha(\tau)-\alpha(t)}{\tau-t}\right)$$
$$= \lim_{t\to b-}\left(\lim_{\tau\to b-} \int_t^\tau \frac{\alpha'_+(s)}{\tau-t}\,\mathrm{d}s\right) = \infty,$$

a contradiction.

Similarly, if

$$\lim_{t\to b-} \alpha'_+(t) = -\infty$$

held, we would obtain $\alpha'_-(b) = -\infty$, again a contradiction.

LEMMA 3 *Let α be a lower function of* (1) *according to Definition* 2 *and let ρ be given by* (3). *Then ρ has a bounded variation on* $[a,b]$.

Proof Lemma 2 yields that the nondecreasing function r given by (6) has a finite limit $r(b-) = \lim_{t\to b-} r(t)$, i.e. it has a bounded variation on $[a,b]$. Denoting

$$h(t) = \int_a^t f(s,\alpha(s),\rho(s))\,\mathrm{d}s,$$

we can write

$$\sum_i |\rho(t_{i+1}) - \rho(t_i) - h(t_{i+1}) + h(t_i)| = \rho(b) - \rho(a) - h(b) + h(a)$$
$$= r(b-) - r(a).$$

for an arbitrary partition $\{t_i\}$ of the interval $[a, b]$. Thus

$$\text{var}_a^b \rho \le \text{var}_a^b h + r(b-) - r(a).$$

Proof of Theorem 1 Let α be a lower function of (1) with respect to Definition 2 and let the function r be again given by (6). In the proof of Lemma 3 it has been already shown that the limit $r(b-)$ exists and is finite. Furthermore, integrating (6) and making use of Lemma 1 we get

$$\alpha(t) = \alpha(a) + \int_a^t \int_a^\tau f(s, \alpha(s), \alpha'_+(s))\, \mathrm{d}s\, \mathrm{d}\tau$$
$$+ \int_a^t r(s)\, \mathrm{d}s \quad \text{on } [a, b],$$

i.e. α is absolutely continuous on $[a, b]$. Now, let σ and ρ be defined by (3). Thus, σ is absolutely continuous on $[a, b]$ and, according to Lemma 3, ρ has a bounded variation on $[a, b]$. Moreover, in virtue of (3) and Definition 2, the function

$$\rho(t) - \int_a^t f(s, \sigma(s), \rho(s))\, \mathrm{d}s$$

is nondecreasing on $[a, b)$. It means that the couple (σ, ρ) verifies the inequalities (2) and, moreover, the singular part of ρ is also nondecreasing on $[a, b)$ (cf. e.g. [2, Theorem 125] or [4, II.25]), i.e. (σ, ρ) are lower functions for (1) according to Definition 1, either.

Analogously we would argue in the case of upper functions.

Proof of Theorem 2 Let (σ, ρ) be lower functions of (1) with respect to Definition 1. Since the function ρ has a finite variation and since

$$\sigma(t) = \sigma(a) + \int_a^t \rho(s)\, \mathrm{d}s \text{ on } [a, b],$$

the limits $\rho(t+)$, $\rho(t-)$ exist. It follows that the function σ has the one-sided derivatives $\sigma'_+(t) = \rho(t+)$ and $\sigma'_-(t) = \rho(t-)$ in each $t \in [a, b)$ or

$t \in (a, b]$, respectively. As $\rho(t+)$ is right continuous on $[a, b)$, σ'_+ is right continuous on $[a, b)$ either.

Denote ρ^{ac} and ρ^{sing} the absolute and singular parts of ρ, respectively. Since $(\rho^{\mathrm{sing}})'(t) = 0$ almost everywhere (see [4, II.25]) we have

$$(\rho^{\mathrm{ac}})'(t) \geq f(t, \sigma(t), \rho(t)) \quad \text{a.e. on } [a, b]$$

and since ρ^{sing} is nondecreasing on $[a, b]$, for $a \leq t_1 < t_2 \leq b$ we have

$$\begin{aligned} \rho(t_2) - \rho(t_1) &= \rho^{\mathrm{ac}}(t_2) - \rho^{\mathrm{ac}}(t_1) + \rho^{\mathrm{sing}}(t_2) - \rho^{\mathrm{sing}}(t_1) \\ &\geq \int_{t_1}^{t_2} f(s, \sigma(s), \rho(s))\, \mathrm{d}s. \end{aligned}$$

Substituting α instead of σ and α'_+ instead of ρ and considering the fact that $\rho(t+) \neq \rho(t)$ and hence also $\alpha'_+(t) \neq \rho(t)$ can happen only in at most countably many points $t \in [a, b]$, we get the monotonicity on $[a, b)$ of the function

$$\alpha'_+(t) - \int_a^t f(s, \alpha(s), \alpha'_+(s))\, \mathrm{d}s = \rho(t) - \int_a^t f(s, \sigma(s), \rho(s))\, \mathrm{d}s$$

required by Definition 2. Finally, the relations $\alpha'_-(t) \leq \alpha'_+(t)$, $t \in (a, b)$, follow from the fact that ρ^{sing} is by Definition 1 nondecreasing on $[a, b]$.

Analogously we would argue in the case of upper functions.

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