J. of Inequal. & Appl., 2001, Vol. 6, pp. 227–245
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Nonlinear Variational Inequalities of Semilinear Parabolic Type

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(Received 10 August 1999; Revised 29 October 1999)

The existence of solutions for the nonlinear functional differential equation governed by the variational inequality is studied. The regularity and a variation of solutions of the equation are also given.

Keywords: Variational inequalities of semilinear parabolic type;
Existence of solutions; Regularity for solutions; Analytic semigroup;
Subdifferential operator

1991 Mathematics Subject Classification: Primary 35F25; Secondary 35K55

1. INTRODUCTION

In this paper, we deal with the existence and a variational of constant formula for solutions of the nonlinear functional differential equation governed by the variational inequality in Hilbert spaces.

Let H and V be two complex Hilbert spaces. Assume that V is dense subspace in H and the injection of V into H is continuous. The norm on V (resp. H) will be denoted by $||\cdot||$ (resp. $|\cdot|$) respectively. Let A be a continuous linear operator from V into V^* which is assumed to satisfy

$$(Au, u) \geq \omega_1 ||u||^2 - \omega_2 |u|^2,$$

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where $\omega_1 > 0$ and ω_2 is a real number and let $\phi: V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Then we study the following the variational inequality problem with nonlinear term:

$$\begin{aligned} &\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t} + Ax(t), x(t) - z\right) + \phi(x(t)) - \phi(z) \\ &\quad \le (f(t, x(t)) + k(t), x(t) - z), \quad \text{a.e. } 0 < t \le T,\ z \in V, \\ &x(0) = x_0. \end{aligned} \tag{VIP}$$

Noting that the subdifferential operator $\partial\phi$ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \le \phi(y) + (x^*, x - y), \quad y \in V\}$$

where $(\cdot,\cdot)$ denotes the duality pairing between V^* and V, the problem (VIP) is represented by the following nonlinear functional differential problem on H:

$$\begin{aligned} &\frac{\mathrm{d}x(t)}{\mathrm{d}t} + Ax(t) + \partial\phi(x(t)) \ni f(t, x(t)) + k(t), \quad 0 < t \le T, \\ &x(0) = x_0. \end{aligned} \tag{NDE}$$

The existence and regularity for the parabolic variational inequality in the linear case ($f \equiv 0$), which was first investigated by Brézis [5,6], has been developed as seen in Section 4.3.2 of Barbu [3] (also see Section 4.3.1 in [2]).

When the nonlinear mapping f is a Lipschitz continuous from $\mathbb{R} \times V$ into H, we will obtain the existence for solutions of (NDE) by converting the problem into the contraction mapping principle and the norm estimate of a solution of the above nonlinear equation on $L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \cap C([0, T]; H)$. Consequently, in view of the monotonicity of $\partial\phi$, we show that the mapping

$$H \times L^2(0, T; V^*) \ni (x_0, k) \mapsto x \in L^2(0, T; V) \cap C([0, T]; H)$$

is continuous. An example illustrated the applicability of our work is given in the last section.

2. PRELIMINARIES

Let V and H be complex Hilbert space forming Gelfand triple $V \subset H \subset V^*$ with pivot space H. For the sake of simplicity, we may consider

$$\|u\|_* \leq |u| \leq \|u\|, \quad u \in V$$

where $\|\cdot\|_*$ is the norm of the element of V^*. We also assume that there exists a constant C_1 such that

$$\|u\| \leq C_1 \|u\|_{D(A)}^{1/2} |u|^{1/2} \tag{2.1}$$

for every $u \in D(A)$, where

$$\|u\|_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$. Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\operatorname{Re} a(u,u) \geq \omega_1 \|u\|^2 - \omega_2 |u|^2, \tag{2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number.

Let A be the operator associated with the sesquilinear form $a(\cdot,\cdot)$:

$$(Au, v) = a(u, v), \quad u, v \in V.$$

Then A is a bounded linear operator from V to V^* and $-A$ generates an analytic semigroup in both of H and V^* as is seen in [9; Theorem 3.6.1]. The realization for the operator A in H which is the restriction of A to

$$D(A) = \{u \in V; Au \in H\}$$

be also denoted by A.

The following L^2-regularity for the abstract linear parabolic equation

$$\begin{aligned} &\frac{dx(t)}{dt} + Ax(t) = k(t), \quad 0 < t \leq T, \\ &x(0) = x_0 \end{aligned} \tag{LE}$$

has a unique solution x in $[0, T]$ for each $T > 0$ if $x_0 \in (D(A), H)_{1/2,2}$ and $k \in L^2(0, T; H)$ where $(D(A), H)_{1/2,2}$ is the real interpolation space between $D(A)$ and H. Moreover, we have

$$\|x\|_{L^2(0,T;D(A))\cap W^{1,2}(0,T,H)} \leq C_2(\|x_0\|_{(D(A),H)_{1/2,2}} + \|k\|_{L^2(0,T;H)}) \qquad (2.3)$$

where C_2 depends on T and M (see Theorem 2.3 of [4,8]).

If an operator A is bounded linear from V to V^* associated with the sesquilinear form $a(\cdot,\cdot)$ then it is easily seen that

$$H = \left\{ x \in V^*: \int_0^T \|Ae^{-tA}x\|_*^2 \, dt < \infty \right\},$$

for the time $T > 0$. Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{1/2,2} = H$$

and obtain the following results.

PROPOSITION 2.1 *Let $x_0 \in H$ and $k \in L^2(0, T; V^*)$, $T > 0$. Then there exists a unique solution x of* (LE) *belonging to*

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$\|x\|_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_2(|x_0| + \|k\|_{L^2(0,T;V^*)}), \qquad (2.4)$$

where C_2 is a constant depending on T.

Let $\phi: V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Then the subdifferential operator $\partial\phi$ of ϕ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \leq \phi(y) + (x^*, x - y), \quad y \in V\}.$$

First, let us concern with the following perturbation of subdifferential operator:

$$\begin{aligned} &\frac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) \ni k(t), \quad 0 < t \leq T, \\ &x(0) = x_0. \end{aligned} \qquad \text{(VE)}$$

Using the regularity for the variational inequality of parabolic type as seen in [3; Section 4.3] we have the following result on the Eq. (VE). We denote the closure in H of the set $D(\phi)=\{u\in V\colon \phi(u)<\infty\}$ by $\overline{D(\phi)}$.

PROPOSITION 2.2 (1) *Let* $k\in L^2(0,T;V^*)$ *and* $x_0\in\overline{D(\phi)}$. *Then the Eq.* (VE) *has a unique solution*

$$x\in L^2(0,T;V)\cap C([0,T];H),$$

which satisfies

$$x'(t)=(k(t)-Ax(t)-\partial\phi(x(t)))^0$$

and

$$\|x\|_{L^2\cap C}\leq C_3(1+|x_0|+\|k\|_{L^2(0,T;V^*)}) \tag{2.5}$$

where C_3 *is some positive constant and* $L^2\cap C=L^2(0,T;V)\cap C([0,T];H)$.

(2) *Let A be symmetric and let us assume that there exist $h\in H$ such that for every $\epsilon>0$ and any $y\in D(\phi)$*

$$J_\epsilon(y+\epsilon h)\in D(\phi)\ \text{ and }\ \phi(J_\epsilon(y+\epsilon h))\leq\phi(y)$$

where $J_\epsilon=(I+\epsilon A)^{-1}$. *Then for* $k\in L^2(0,T;H)$ *and* $x_0\in\overline{D(\phi)}\cap V$ *the Eq.* (VE) *has a unique solution*

$$x\in L^2(0,T;D(A))\cap W^{1,2}(0,T;H)\cap C([0,T];H),$$

which satisfies

$$\|x\|_{L^2\cap W^{1,2}\cap C}\leq C_3(1+\|x_0\|+\|k\|_{L^2(0,T;H)}). \tag{2.6}$$

Here, we remark that if $D(A)$ is compactly embedded in V and $x\in L^2(0,T;D(A))$ (or the semigroup operator $S(t)$ generated by A is compact), the following embedding

$$L^2(0,T;D(A))\cap W^{1,2}(0,T;H)\subset L^2(0,T;V)$$

is compact in view of Theorem 2 of Aubin [1]. Hence, the mapping $k\mapsto x$ is compact from $L^2(0,T;H)$ to $L^2(0,T;V)$, which is also applicable to optimal control problem.

3. EXISTENCE OF SOLUTIONS

Let f be a nonlinear single valued mapping from $[0,\infty)\times V$ into H. We assume that

$$|f(t,x_1)-f(t,x_2)|\leq L\|x_1-x_2\|, \tag{F}$$

for every $x_1,x_2\in V$.

The following Lemma is from Brézis [6; Lemma A.5].

LEMMA 3.1 *Let $m\in L^1(0,T;\mathcal{R})$ satisfying $m(t)\geq 0$ for all $t\in(0,T)$ and $a\geq 0$ be a constant. Let b be a continuous function on $[0,T]\subset\mathcal{R}$ satisfying the following inequality:*

$$\frac{1}{2}b^2(t)\leq\frac{1}{2}a^2+\int_0^t m(s)b(s)\,\mathrm{d}s,\quad t\in[0,T].$$

Then,

$$|b(t)|\leq a+\int_0^t m(s)\,\mathrm{d}s,\quad t\in[0,T].$$

Proof Let

$$\beta_\epsilon(t)=\frac{1}{2}(a+\epsilon)^2+\int_0^t m(s)b(s)\,\mathrm{d}s,\quad \epsilon>0.$$

Then

$$\frac{\mathrm{d}\beta_\epsilon(t)}{\mathrm{d}t}=m(t)b(t),\quad \tau\in(0,T),$$

and

$$\frac{1}{2}b^2(t)\leq\beta_0(t)\leq\beta_\epsilon(t),\quad t\in[0,T]. \tag{3.1}$$

Hence, we have

$$\frac{\mathrm{d}\beta_\epsilon(t)}{\mathrm{d}t}\leq m(t)\sqrt{2}\sqrt{\beta_\epsilon(t)}.$$

Since $t \longrightarrow \beta_\epsilon(t)$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\beta_\epsilon(t)} = \frac{1}{2\sqrt{\beta_\epsilon(t)}}\frac{\mathrm{d}\beta_\epsilon(t)}{\mathrm{d}t}$$

for all $t \in (0, T)$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\beta_\epsilon(t)} \leq \frac{1}{\sqrt{2}}m(t),$$

that is,

$$\sqrt{\beta_\epsilon(t)} \leq \sqrt{\beta_\epsilon(0)} + \frac{1}{\sqrt{2}}\int_0^t m(s)\,\mathrm{d}s, \quad t \in (0, T).$$

Therefore, combining this with (3.1), we conclude that

$$\begin{aligned} |b(t)| &\leq \sqrt{2}\sqrt{\beta_\epsilon(t)} \leq \sqrt{2}\sqrt{\beta_\epsilon(0)} + \int_0^t m(s)\,\mathrm{d}s \\ &= a + \epsilon + \int_0^t m(s)\,\mathrm{d}s, \quad t \in [0, T] \end{aligned}$$

for arbitrary $\epsilon > 0$.

We establish the following results on the solvability of (NDE).

THEOREM 3.1 *Let the assumption* (F) *be satisfied. Assume that* $k \in L^2(0, T; V^*)$ *and* $x_0 \in \overline{D(\phi)}$. *Then, the Eq.* (NDE) *has a unique solution*

$$x \in L^2(0, T; V) \cap C([0, T]; H)$$

and there exists a constant C_4 *depending on* T *such that*

$$\|x\|_{L^2 \cap C} \leq C_4(1 + |x_0| + \|k\|_{L^2(0,T;V^*)}). \tag{3.2}$$

Furthermore, if $k \in L^2(0, T; H)$ *then the solution* x *belongs to* $W^{1,2}(0, T; H)$ *and satisfies*

$$\|x\|_{W^{1,2}(0,T;H)} \leq C_4(1 + |x_0| + \|k\|_{L^2(0,T;H)}). \tag{3.3}$$

Proof Invoking Proposition 2.2, we obtain that the problem

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + Ay(t) + \partial\phi(y(t)) \ni f(t, x(t)) + k(t), \quad 0 < t \le T,$$
$$y(0) = x_0$$

has a unique solution $y \in L^2(0, T; V) \cap C([0, T]; H)$.

Assume that (2.2) holds for $\omega_2 \neq 0$. Let us fix $T_0 > 0$ such that

$$\frac{L^2}{4\omega_1\omega_2}(\mathrm{e}^{2\omega_2 T_0} - 1) < 1. \tag{3.4}$$

For $i = 1, 2$, we consider the following equation:

$$\begin{aligned} &\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} + Ay_i(t) + \partial\phi(y_i(t)) \ni f(t, x_i(t)) + k(t), \quad 0 < t \le T, \\ &y_i(0) = x_0. \end{aligned} \tag{3.5}$$

We are going to show that $x \mapsto y$ is strictly contractive from $L^2(0, T_0; V)$ to itself if the condition (3.4) is satisfied. Let y_1, y_2 be the solutions of (3.5) with x replaced by $x_1, x_2 \in L^2(0, T_0; V)$ respectively. From (3.5) it follows that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(y_1(t) - y_2(t)) + A(y_1(t) - y_2(t)) + \partial\phi(y_1(t)) - \partial\phi(y_2(t)) \\ &\quad \ni f(t, x_1(t)) - f(t, x_2(t)), \quad t > 0. \end{aligned}$$

Multiplying on both sides of $y_1(t) - y_2(t)$ and using the monotonicity of $\partial\phi$, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|y_1(t) - y_2(t)|^2 + a(y_1(t) - y_2(t), y_1(t) - y_2(t)) \\ &\quad \le (f(t, x_1(t)) - f(t, x_2(t)), y_1(t) - y_2(t)), \end{aligned}$$

and hence,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|y_1(t) - y_2(t)|^2 + \omega_1\|y_1(t) - y_2(t)\|^2 \\ &\quad \le \omega_2|y_1(t) - y_2(t)|^2 + L\|x_1(t) - x_2(t)\|\,|y_1(t) - y_2(t)|. \end{aligned} \tag{3.6}$$

Putting

$$G(t) = L\|x_1(t) - x_2(t)\| \, |y_1(t) - y_2(t)|$$

and integrating (3.6) over $(0, t)$, this yields that

$$\begin{aligned} &\frac{1}{2}|y_1(t) - y_2(t)|^2 + \omega_1 \int_0^t \|y_1(s) - y_2(s)\|^2 \, \mathrm{d}s \\ &\quad \le \omega_2 \int_0^t |y_1(s) - y_2(s)|^2 \, \mathrm{d}s + \int_0^t G(s) \, \mathrm{d}s. \end{aligned} \tag{3.7}$$

From (3.7) it follows that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathrm{e}^{-2\omega_2 t} \int_0^t |y_1(s) - y_2(s)|^2 \, \mathrm{d}s\right\} \\ &\quad = 2\mathrm{e}^{-2\omega_2 t}\left\{\frac{1}{2}|y_1(t) - y_2(t)|^2 - \omega_2 \int_0^t |y_1(s) - y_2(s)|^2 \, \mathrm{d}s\right\} \\ &\quad \le 2\mathrm{e}^{-2\omega_2 t} \int_0^t G(s) \, \mathrm{d}s. \end{aligned} \tag{3.8}$$

Integrating (3.8) over $(0, t)$ we have

$$\begin{aligned} &\mathrm{e}^{-2\omega_2 t} \int_0^t |y_1(s) - y_2(s)|^2 \, \mathrm{d}s \le 2 \int_0^t \mathrm{e}^{-2\omega_2 \tau} \int_0^\tau G(s) \, \mathrm{d}s \, \mathrm{d}\tau \\ &\quad = 2 \int_0^t \int_s^t \mathrm{e}^{-2\omega_2 \tau} \, \mathrm{d}\tau G(s) \, \mathrm{d}s = 2 \int_0^t \frac{\mathrm{e}^{-2\omega_2 s} - \mathrm{e}^{-2\omega_2 t}}{2\omega_2} G(s) \, \mathrm{d}s \\ &\quad = \frac{1}{\omega_2} \int_0^t (\mathrm{e}^{-2\omega_2 s} - \mathrm{e}^{-2\omega_2 t}) G(s) \, \mathrm{d}s, \end{aligned}$$

thus, we get

$$\omega_2 \int_0^t |y_1(s) - y_2(s)|^2 \, \mathrm{d}s \le \int_0^t (\mathrm{e}^{2\omega_2(t-s)} - 1) G(s) \, \mathrm{d}s. \tag{3.9}$$

From (3.7) and (3.9) it follows that

$$\begin{aligned} &\frac{1}{2}|y_1(t) - y_2(t)|^2 + \omega_1 \int_0^t \|y_1(s) - y_2(s)\|^2 \, \mathrm{d}s \\ &\quad \le \int_0^t \mathrm{e}^{2\omega_2(t-s)} G(s) \, \mathrm{d}s \\ &\quad = \int_0^t \mathrm{e}^{2\omega_2(t-s)} L\|x_1(s) - x_2(s)\| \, |y_1(s) - y_2(s)| \, \mathrm{d}s, \end{aligned} \tag{3.10}$$

which implies

$$\frac{1}{2}(e^{-\omega_2 t}|y_1(t) - y_2(t)|)^2 + \omega_1 e^{-2\omega_2 t} \int_0^t \|y_1(s) - y_2(s)\|^2 \, ds$$
$$\leq L \int_0^t e^{-\omega_2 s} \|x_1(s) - x_2(s)\| e^{-\omega_2 s} |y_1(s) - y_2(s)| \, ds.$$

By using Lemma 3.1, we obtain that

$$e^{-\omega_2 t}|y_1(t) - y_2(t)| \leq \int_0^t L e^{-\omega_2 s} \|x_1(s) - x_2(s)\| \, ds. \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$\begin{aligned}
&\frac{1}{2}|y_1(t) - y_2(t)|^2 + \omega_1 \int_0^t \|y_1(s) - y_2(s)\|^2 \, ds \\
&\leq L^2 \int_0^t e^{2\omega_2(t-s)} \|x_1(s) - x_2(s)\| \int_0^s e^{\omega_2(s-\tau)} \|x_1(\tau) - x_2(\tau)\| \, d\tau \, ds \\
&= L^2 e^{2\omega_2 t} \int_0^t e^{-\omega_2 s} \|x_1(s) - x_2(s)\| \int_0^s e^{-\omega_2 \tau} \|x_1(\tau) - x_2(\tau)\| \, d\tau \, ds \\
&= L^2 e^{2\omega_2 t} \int_0^t \frac{1}{2} \frac{d}{ds} \left\{ \int_0^s e^{-\omega_2 \tau} \|x_1(\tau) - x_2(\tau)\| \, d\tau \right\}^2 ds \\
&= \frac{1}{2} L^2 e^{2\omega_2 t} \left\{ \int_0^t e^{-\omega_2 \tau} \|x_1(\tau) - x_2(\tau)\| \, d\tau \right\}^2 \\
&\leq \frac{1}{2} L^2 e^{2\omega_2 t} \int_0^t e^{-2\omega_2 \tau} \, d\tau \int_0^t \|x_1(\tau) - x_2(\tau)\|^2 \, d\tau \\
&= \frac{1}{2} L^2 e^{2\omega_2 t} \frac{1 - e^{-2\omega_2 t}}{2\omega_2} \int_0^t \|x_1(\tau) - x_2(\tau)\|^2 \, d\tau \\
&= \frac{L^2}{4\omega_2} (e^{2\omega_2 t} - 1) \int_0^t \|x_1(s) - x_2(s)\|^2 \, ds.
\end{aligned} \tag{3.12}$$

Starting from initial value $x_0(t) = x_0$, consider a sequence $\{x_n(\cdot)\}$ satisfying

$$\frac{d}{dt} x_{n+1}(t) + A x_{n+1}(t) + \partial\phi(x_{n+1}(t)) \ni f(t, x_n(t)) + k(t), \quad 0 < t \leq T,$$
$$x_{n+1}(0) = x_0.$$

Then from (3.12) it follows that

$$\frac{1}{2}|x_{n+1}(t) - x_n(t)|^2 + \omega_1 \int_0^t \|x_{n+1}(s) - x_n(s)\|^2 \, ds$$
$$\leq \frac{L^2}{4\omega_2}(e^{2\omega_2 t} - 1) \int_0^t \|x_n(s) - x_{n-1}(s)\|^2 \, ds. \qquad (3.13)$$

So by virtue of the condition (3.4) the contraction principle gives that there exists $x(\cdot) \in L^2(0, T_0; V)$ such that

$$x_n(\cdot) \to x(\cdot) \quad \text{in } L^2(0, T_0; V),$$

and hence, from (3.13) there exists $x(\cdot) \in C([0, T_0]; H)$ such that

$$x_n(\cdot) \to x(\cdot) \quad \text{in } C(0, T_0; H).$$

Next we establish the estimates of solution. Let y be the solution of

$$\frac{dy(t)}{dt} + Ay(t) + \partial\phi(y(t)) \ni k(t), \quad 0 < t \leq T_0,$$
$$y(0) = x_0.$$

Then, since

$$\frac{d}{dt}(x(t) - y(t)) + A(x(t) - y(t)) + \partial\phi(x(t)) - \partial\phi(y(t)) \ni f(t, x(t)),$$

by multiplying by $x(t) - y(t)$ and using the monotonicity of $\partial\phi$ and (2.2), we obtain

$$\frac{1}{2}\frac{d}{dt}|x(t) - y(t)|^2 + \omega_1\|x(t) - y(t)\|^2$$
$$\leq \omega_2|x(t) - y(t)|^2 + L\|x(t)\|\,|x(t) - y(t)|. \qquad (3.14)$$

By integrating on (3.14) over $(0, t)$ we have

$$\frac{1}{2}|x(t) - y(t)|^2 + \omega_1 \int_0^t \|x(s) - y(s)\|^2 \, ds$$
$$\leq \omega_2 \int_0^t |x(s) - y(s)|^2 \, ds + L \int_0^t \|x(s)\|\,|x(s) - y(s)| \, ds. \qquad (3.15)$$

By the procedure similar to (3.12) we have

$$\begin{aligned}\frac{1}{2}|x(t)-y(t)|+\omega_1\int_0^t\|x(s)-y(s)\|^2\,\mathrm{d}s\\ \le L^2\int_0^t \mathrm{e}^{2\omega_2(t-s)}\|x(s)\|\int_0^s \mathrm{e}^{\omega_2(s-\tau)}\|x(\tau)\|\,\mathrm{d}\tau\,\mathrm{d}s\\ =\frac{L^2}{4\omega_2}(\mathrm{e}^{2\omega_2 t}-1)\int_0^t\|x(s)\|^2\,\mathrm{d}s.\end{aligned}$$

Put

$$N=\frac{L^2}{4\omega_1\omega_2}(\mathrm{e}^{2\omega_2 T_0}-1).$$

Then it holds

$$\|x-y\|_{L^2(0,T_0;V)}\le N^{1/2}\|x\|_{L^2(0,T_0;V)}$$

and hence, from (2.5) in Proposition 2.2, we have that

$$\begin{aligned}\|x\|_{L^2(0,T_0;V)}&\le\frac{1}{1-N^{1/2}}\|y\|_{L^2(0,T_0;V)}\\ &\le\frac{C_2}{1-N^{1/2}}(1+\|x_0\|+\|k\|_{L^2(0,T_0;V^*)})\\ &\le C_4(1+\|x_0\|+\|k\|_{L^2(0,T_0;V^*)})\end{aligned}\tag{3.16}$$

for some positive constant C_4.

If $\omega_2=0$, replace (3.4) by $L^2T_0/2<1$, the results mentioned above still hold.

Acting on both side of (NDE) by $x'(t)$ and by using

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(x(t))=\left(g(t),\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right),\quad \text{a.e. } 0<t,$$

for all $g(t)\in\partial\phi(x(t))$, it holds

$$\begin{aligned}\int_0^t|x_n'(t)|^2+\frac{1}{2}(Ax_n(t),x_n(t))+\phi(x_n(t))\\ \le\frac{1}{2}(Ax_0,x_0)+\phi(x_0)+\int_0^t|f(s,x_n(s))+k(s)|\,|x_n'(s)|\,\mathrm{d}s,\end{aligned}\tag{3.17}$$

thus, we obtain the norm estimate of x in $W^{1,2}(0,T;H)$ satisfying (3.3). Since the condition (3.4) is independent of initial values and we can derive from (3.17) that $\phi(x(nT_0))<\infty$, the solution of (NDE) can be extended the internal $[0,nT_0]$ for natural number n, i.e., for the initial $x(nT_0)$ in the interval $[nT_0,(n+1)T_0]$, as analogous estimate (3.16) holds for the solution in $[0,(n+1)T_0]$. Furthermore, the estimate (3.2) is easily obtained from (3.15) and (3.16).

THEOREM 3.2 *Let the assumption* (F) *be satisfied and* $(x_0,k)\in H\times L^2(0,T;H)$, *then the solution* x *of the Eq.* (NDE) *belongs to* $x\in L^2(0,T;V)\cap C([0,T];H)$ *and the mapping*

$$H\times L^2(0,T;H)\ni(x_0,k)\mapsto x\in L^2(0,T;V)\cap C([0,T];H)$$

is continuous.

Proof If $(x_0,k)\in H\times L^2(0,T;H)$ then x belongs to $L^2(0,T;V)\cap C([0,T];H)$ from Theorem 3.1. Let $(x_{0i},k_i)\in H\times L^2(0,T;H)$ and x_i be the solution of (NDE) with (x_{0i},k_i) in place of (x_0,k) for $i=1,2$. Multiplying on (NDE) by $x_1(t)-x_2(t)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x_1(t)-x_2(t)|^2+\omega_1\|x_1(t)-x_2(t)\|^2\\&\quad\le\omega_2|x_1(t)-x_2(t)|^2+|f(t,x_1(t))-f(t,x_2(t))|\,|x_1(t)-x_2(t)|\\&\qquad+|k_1(t)-k_2(t)|\,\|x_1(t)-x_2(t)\|.\end{aligned}\tag{3.18}$$

Put

$$H(t)=(L\|x_1(t)-x_2(t)\|+|k_1(t)-k_2(t)|)|x_1(t)-x_2(t)|.$$

Then

$$\begin{aligned}&\frac{1}{2}|x_1(t)-x_2(t)|^2+\omega_1\int_0^t\|x_1(s)-x_2(s)\|^2\,\mathrm{d}s\\&\quad\le\frac{1}{2}|x_{01}-x_{02}|+\omega_2\int_0^t|x_1(s)-x_2(s)|^2\,\mathrm{d}s+\int_0^t H(s)\,\mathrm{d}s\end{aligned}\tag{3.19}$$

and we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathrm{e}^{-2\omega_2 t}\int_0^t |x_1(s)-x_2(s)|^2\,\mathrm{d}s\right\}$$
$$\le 2\mathrm{e}^{-2\omega_2 t}\left\{\frac{1}{2}|x_{01}-x_{02}|^2+\int_0^t H(s)\right\}\mathrm{d}s,$$

thus, by the similar way to (3.9) we have

$$\omega_2\int_0^t |x_1(s)-x_2(s)|^2\,\mathrm{d}s$$
$$\le\frac{1}{2}(\mathrm{e}^{2\omega_2 t}-1)|x_{01}-x_{02}|^2+\int_0^t(\mathrm{e}^{2\omega_2(t-s)}-1)H(s)\,\mathrm{d}s.$$

Combining this and (3.19) it holds that

$$\frac{1}{2}|x_1(t)-x_2(t)|^2+\omega_1\int_0^t \|x_1(s)-x_2(s)\|^2\,\mathrm{d}s$$
$$\le\frac{1}{2}\mathrm{e}^{2\omega_2 t}|x_{01}-x_{02}|^2+\int_0^t \mathrm{e}^{2\omega_2(t-s)}H(s)\,\mathrm{d}s. \tag{3.20}$$

By Lemma 3.1 the following inequality

$$\frac{1}{2}(\mathrm{e}^{-\omega_2 t}|x_1(t)-x_2(t)|)^2+\omega_1\mathrm{e}^{-2\omega_2 t}\int_0^t \|x_1(s)-x_2(s)\|^2\,\mathrm{d}s$$
$$\le\frac{1}{2}|x_{01}-x_{02}|^2+\int_0^t \mathrm{e}^{-\omega_2 s}(L\|x_1(s)-x_2(s)\|$$
$$+|k_1(s)-k_2(s)|)\mathrm{e}^{-\omega_2 s}|x_1(s)-x_2(s)|\,\mathrm{d}s$$

implies that

$$\mathrm{e}^{-\omega_2 t}|x_1(t)-x_2(t)|$$
$$\le|x_{01}-x_{02}|+\int_0^t \mathrm{e}^{-\omega_2 s}(L\|x_1(s)-x_2(s)\|+|k_1(s)-k_2(s)|)\,\mathrm{d}s. \tag{3.21}$$

From (3.20) and (3.21) it follows that

$$\frac{1}{2}|x_1(t)-x_2(t)|^2+\omega_1\int_0^t\|x_1(s)-x_2(s)\|^2\,ds$$
$$\le\frac{1}{2}e^{2\omega_2 t}|x_{01}-x_{02}|^2$$
$$+\int_0^t e^{2\omega_2(t-s)}(L\|x_1(s)-x_2(s)\|+|k_1(s)-k_2(s)|)|x_{01}-x_{02}|\,ds$$
$$+\int_0^t e^{2\omega_2(t-s)}(L\|x_1(s)-x_2(s)\|+|k_1(s)-k_2(s)|)$$
$$\times\int_0^s e^{\omega_2(s-\tau)}(L\|x_1(\tau)-x_2(\tau)\|+|k_1(\tau)-k_2(\tau)|)\,d\tau\,ds. \tag{3.22}$$

The third term of the right of (3.22) is estimated as

$$\frac{(e^{2\omega_2 t}-1)}{4\omega_2}\int_0^t L^2(\|x_1(s)-x_2(s)\|^2+|k_1(s)-k_2(s)|^2)\,ds. \tag{3.23}$$

Let $T_1 < T$ be such that

$$\omega_1-\frac{L^2}{4\omega_2}(e^{2\omega_2 T_1}-1)>0.$$

Then we can choose a constant $c>0$ such that

$$\omega_1-\frac{L^2}{4\omega_2}(e^{2\omega_2 T_1}-1)-Le^{2\omega_2 T_1}\frac{c}{2}>0$$

and

$$|x_{01}-x_{02}|\,\|x_1(s)-x_2(s)\|\le\frac{1}{2c}|x_{01}-x_{02}|^2+\frac{c}{2}\|x_1(s)-x_2(s)\|^2.$$

Thus, the second term of the right of (3.22) is estimated as

$$T_1 e^{2\omega_2 T_1}\frac{L+c}{2c}|x_{01}-x_{02}|^2$$
$$+\frac{e^{2\omega_2 T_1}}{2}\int_0^{T_1}(cL\|x_1(s)-x_2(s)\|^2+|k_1(s)-k_2(s)|^2)\,ds. \tag{3.24}$$

Hence, from (3.22) to (3.24) it follows that there exists a constant $C>0$ such that

$$\frac{1}{2}|x_1(T_1)-x_2(T_1)|^2+\omega_1\int_0^{T_1}\|x_1(s)-x_2(s)\|^2\,\mathrm{d}s \leq C\left(|x_{01}-x_{02}|^2+\int_0^{T_1}|k_1(s)-k_2(s)|^2\,\mathrm{d}s\right). \quad (3.25)$$

Suppose $(x_{0n},k_n)\to(x_0,k)$ in $H\times L^2(0,T_1;V^*)$, and let x_n and x be the solutions (NDE) with (x_{0n},k_n) and (x_0,k), respectively. Then, by virtue of (3.24), we see that $x_n\to x$ in $L^2(0,T_1,V)\cap C([0,T_1];H)$. This implies that $x_n(T_1)\to x(T_1)$ in H. Therefore the same argument shows that $x_n\to x$ in

$$L^2(T_1,\min\{2T_1,T\};V)\cap C([T_1,\min\{2T_1,T\}];H).$$

Repeating this process, we conclude that $x_n\to x$ in $L^2(0,T;V)\cap C([0,T];H)$.

Remark 3.1 Under the condition that either the nonlinear term $f(\cdot,x)$ is uniformly bounded or $\omega_1-L>0$, we can show that the mapping

$$H\times L^2(0,T;V^*)\ni(x_0,k)\mapsto x\in L^2(0,T;V)\cap C([0,T];H)$$

is continuous for any $k\in L^2(0,T;V^*)$.

4. EXAMPLE

Let Ω be a region in an n-dimensional Euclidean space $\mathbb{R}^n$ with boundary $\partial\Omega$ and closure $\bar{\Omega}$. For an integer $m\geq 0$, $C^m(\Omega)$ is the set of all m-times continuously differentiable functions in Ω, and $C_0^m(\Omega)$ is its subspace consisting of functions with compact supports in Ω. If $m\geq 0$ is an integer and $1\leq p\leq\infty$, $W^{m,p}(\Omega)$ is the set of all functions f whose derivative $D^\alpha f$ up to degree m in the distribution sense belong to $L^p(\Omega)$. As usual, the norm of $W^{m,p}(\Omega)$ is given by

$$\|f\|_{m,p}=\left(\sum_{|\alpha|\leq m}\|D^\alpha f\|_p^p\right)^{1/p}=\left\{\sum_{|\alpha|\leq m}\int_\Omega|D^\alpha f(x)|^p\,\mathrm{d}x\right\}^{1/p},$$

where $1 \le p < \infty$ and $D^0 f = f$. In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ with the norm $\|\cdot\|_{0,p}$. $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. For $p' = p/(p-1)$, $1 < p < \infty$, $W^{-m,p}(\Omega)$ stands the dual space $W_0^{m,p'}(\Omega)$ of $W_0^{m,p'}(\Omega)$ whose norm is denoted by $\|\cdot\|_{-m,p}$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We take $V = W_0^{m,2}(\Omega)$, $H = L^2(\Omega)$ and $V^* = W^{-m,2}(\Omega)$ and consider a nonlinear differential operator of the form

$$Ax = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(u, x, \ldots, D^m x),$$

where $A_\alpha(u, \xi)$ are real functions defined on $\Omega \times \mathbb{R}^N$ and satisfy the following conditions:

(1) A_α are measurable in u and continuous in ξ. There exists $k \in L^2(\Omega)$ and a positive constant C such that

$$A_\alpha(u, 0) = 0,$$
$$|A_\alpha(u, \xi)| \le C(|\xi| + k(u)), \quad \text{a.e. } u \in \Omega,$$

where $\xi = (\xi_\alpha; |\alpha| \le m)$.

(2) For every $(\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N$ and for almost every $u \in \Omega$ the following condition holds:

$$\sum_{|\alpha| \le m} (A_\alpha(u, \xi) - A_\alpha(u, \eta))(\xi_\alpha - \eta_\alpha) \ge \omega_1 \|\xi - \eta\|_{m,2} - \omega_2 \|\xi - \eta\|_{0,2}$$

where $\omega_2 \in \mathbb{R}$ and ω_1 is a positive constant.

Let the sesquilinear form $a : V \times V \to \mathbb{R}$ be defined by

$$a(x, y) = \sum_{|\alpha| \le m} \int_\Omega A_\alpha(u, x, \ldots, D^m x)(D)^\alpha y \, du.$$

Then by Lax-Milgram theorem we know that the associated operator $A : V \to V^*$ defined by

$$(Ax, y) = a(x, y), \quad x, y \in V$$

is monotone and semicontinuous and satisfies conditions (A1) and (A2) in Section 2.

Let $g(t,u,x,p)$, $p \in \mathbb{R}^m$, be assumed that there is a continuous $\rho(t,r)$: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ and a real constant $1 \le \gamma$ such that

(f1) $g_1(t,u,0,0) = 0$,

(f2) $|g_1(t,u,x,p) - g_1(t,u,x,q)| \le \rho(t,|x|)(1 + |p|^{\gamma-1} + |q|^{\gamma-1})|p-q|$,

(f3) $|g_1(t,u,x,p) - g_1(t,u,y,p)| \le \rho(t,|x| + |y|)(1 + |p|^{\gamma})|x-y|$.

Let

$$g(t,x)(u) = g_1(t,u,x,dx,D^2x,\ldots,D^m x).$$

Then noting that

$$\begin{aligned}\|g(t,x) - g(t,y)\|_{0,2}^2 \le 2\int_\Omega |g_1(t,u,x,p) - g_1(t,u,x,q)|^2\,\mathrm{d}u \\ + 2\int_\Omega |g_1(t,u,x,q) - g_1(t,u,y,q)|^2\,\mathrm{d}u,\end{aligned}$$

where $p = (Dx,\ldots,D^m x)$ and $q = (Dy,\ldots,D^m y)$, it follows from (f1), (f2) and (f3) that

$$\|g(t,x) - g(t,y)\|_{0,2}^2 \le L(\|x\|_{m,2}, \|y\|_{m,2})\|x-y\|_{m,2},$$

where $L(\|x\|_{m,2}, \|y\|_{m,2})$ is a constant depending on $\|x\|_{m,2}$ and $\|y\|_{m,2}$. We set

$$f(t,x) = \int_0^t k(t-s)g(s,x(s))\,\mathrm{d}s,$$

where k belongs to $L^2(0,T)$. Let $\phi: V \to (-\infty,+\infty]$ be a lower semi-continuous, proper convex function. For every $x_0 \in \overline{D(\phi)}$ and $k \in L^2(0,T;V^*)$ the following nonlinear problem:

$$\begin{aligned}&\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t} + Ax(t), x(t) - z\right) + \phi(x(t)) - \phi(z) \\ &\quad\le (f(t,x(t)) + k(t), x(t) - z), \quad \text{a.e. } 0 < t \le T,\ z \in H, \\ &x(0) = x_0\end{aligned}$$

has a unique solution

$$x \in L^2(0, T; W_0^{m,2}(\Omega)) \cap C([0, T]; L^2(\Omega)).$$

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