

Inequalities for Zeros of Entire Functions*

M. I. GIL'

*Department of Mathematics, Ben Gurion University of Negev,
P.O. Box 653, Beer-Sheva 84105, Israel*

(Received 8 December 1999; In final form 3 March 2000)

Let f be an entire function and $z_k(f)$ ($k = 1, 2, \dots$) be the zeros of f . Inequalities for the sums

$$\sum_{k=1}^j |z_k(f)|^{-1} \quad (j = 1, 2, \dots)$$

are derived. Under some restrictions they improve the Hadamard theorem.

Keywords: Entire functions; Estimates for zeros

AMS (MOS) Subject Classifications: 30D20, 30C15

1. STATEMENT OF THE MAIN RESULT

Consider an entire function of the form

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{a_k \lambda^k}{(k!)^\gamma} \quad (\lambda \in \mathbb{C}, a_0 = 1, \gamma \geq 0) \quad (1)$$

with complex, in general, coefficients. Assume that

$$\theta_f \equiv \left[\sum_{k=1}^{\infty} |a_k|^2 \right]^{1/2} < \infty. \quad (2)$$

*This research was in supported by the Israel Ministry of Science and Technology.

If $\gamma = 0$, *i.e.*,

$$f(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k \quad (a_0 = 1)$$

then condition (2) is always fulfilled. If $\gamma > 0$, then under (2), f has the finite order.

Everywhere below $\{z_k(f)\}_{k=1}^m$ ($m \leq \infty$) is the set of all the zeros of f taken with their multiplicities and numerated in the following way:

$$|z_k(f)| \leq |z_{k+1}(f)| \quad (k = 1, \dots, m-1).$$

Since $a_0 = 1$, we have $z_k(f) \neq 0$ ($k = 1, 2, \dots, m$). The aim of the present paper is to prove the following

THEOREM 1 *Let f be an entire function of the form (1). Then under condition (2), the inequalities*

$$\sum_{k=1}^j |z_k(f)|^{-1} \leq \theta_f + \sum_{k=1}^j (k+1)^{-\gamma} \quad (j = 1, 2, \dots, m) \quad (3)$$

are valid.

The proof of this theorem is presented below.

In particular, if in (1) $\gamma = 0$, then Theorem 1 yields

$$\sum_{k=1}^j |z_k(f)|^{-1} \leq \theta_f + j \quad (j = 1, 2, \dots, m)$$

As it is shown below, under some restriction, Theorem 1 improves the Hadamard theorem, *cf.* (Levin, 1996, p. 18).

2. PROOF OF THEOREM 1

To prove Theorem 1, for a natural $n \geq 2$ consider the polynomial

$$p_n(\lambda) = \sum_{k=0}^n \frac{a_k \lambda^{n-k}}{(k!)^\gamma} \quad (4)$$

with the zeros $z_k(p_n)$ taken with their multiplicities and ordered in the following way:

$$|z_k(p_n)| \geq |z_{k+1}(p_n)| \quad (k = 1, \dots, n).$$

Set

$$\theta(p_n) \equiv \left[\sum_{k=1}^n |a_k|^2 \right]^{1/2}.$$

LEMMA 2 *The zeros of p_n satisfy the inequalities*

$$\sum_{k=1}^j |z_k(p_n)| \leq \theta(p_n) + \sum_{k=1}^j (k+1)^{-\gamma} \quad (j = 1, \dots, n-1).$$

and

$$\sum_{k=1}^n |z_k(p_n)| \leq \theta(p_n) + \sum_{k=1}^{n-1} (k+1)^{-\gamma}.$$

Proof Introduce the $n \times n$ -matrix

$$B_n = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n \\ 1/2^\gamma & 0 & 0 & \dots & 0 & 0 \\ 0 & 1/3^\gamma & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1/n^\gamma & 0 \end{pmatrix}$$

The direct calculations show that $p_n(\lambda) = \det (B_n - \lambda I_n)$ ($\lambda \in \mathbb{C}$), where I_n is the unit matrix. So

$$z_k(p_n) = \lambda_k(B_n) \tag{5}$$

where $\lambda_k(B)$ ($k = 1, \dots, n$) mean the eigenvalues of an $n \times n$ -matrix B with their multiplicities. Due to the Weil theorem (Marcus and Minc, 1964, Section II.4.2),

$$\sum_{k=1}^j |\lambda_k(B_n)| \leq \sum_{k=1}^j s_k(B_n) \quad (j = 1, \dots, n) \tag{6}$$

where $s_k(B)$ denote the singular numbers of an $n \times n$ -matrix B :

$$s_k^2(B) = \lambda_k(BB^*) \quad (k = 1, \dots, n),$$

ordered in the decreasing way. Here and below the asterisk means the adjointness. Obviously, $B_n = D + C$, where

$$C = \begin{pmatrix} 0 & -a_1 & -a_2 & \dots & -a_n \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1/2^\gamma & 0 & 0 & \dots & 0 & 0 \\ 0 & 1/3^\gamma & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1/n^\gamma & 0 \end{pmatrix}$$

Clearly,

$$CC^* = \begin{pmatrix} \theta^2(p_n) & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and

$$DD^* = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1/2^{2\gamma} & 0 & \dots & 0 & 0 \\ 0 & 0 & 1/3^{2\gamma} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1/n^{2\gamma} \end{pmatrix}$$

Since the diagonal entries of diagonal matrices are the eigenvalues, we can write out

$$s_1(C) = \theta(p_n), \quad s_k(C) = 0 \quad (k = 2, \dots, n).$$

In addition,

$$s_k(D) = 1/(k+1)^\gamma \quad (k = 1, \dots, n-1), \quad s_n(D) = 0.$$

Take into account that

$$\sum_{k=1}^j s_k(B_n) = \sum_{k=1}^j s_k(D + C) \leq \sum_{k=1}^j s_k(D) + \sum_{k=1}^j s_k(C)$$

cf. (Gohberg and Krein, 1969, Lemma II.4.2). So

$$\sum_{k=1}^j s_k(B_n) \leq \theta(p_n) + \sum_{k=1}^j (k + 1)^{-\gamma} \quad (j = 1, \dots, n - 1)$$

and

$$\sum_{k=1}^n s_k(B_n) \leq \theta(p_n) + \sum_{k=1}^{n-1} (k + 1)^{-\gamma}.$$

Now (5) and (6) yield the required result. ■

Proof of Theorem 1 Consider the polynomial

$$q_n(z) = 1 + a_1z + \dots + \frac{a_n z^n}{(n!)^\gamma} \quad (2 \leq n < \infty)$$

with the zeros $z_k(q_n)$ ordered in the following way:

$$|z_k(q_n)| \leq |z_{k+1}(q_n)| \quad (k \leq n - 1).$$

With a fixed $k \leq n$ and $z = z_k(q_n)$, we get

$$q_n(z) = z^n \sum_{k=0}^n \frac{a_k}{z^{n-k} (k!)^\gamma} = 0.$$

Hence,

$$z_k^{-1}(q_n) = z_k(p_n) \quad (k = 1, \dots, n)$$

where p_n is defined by (4). Now Lemma 2 yields the inequalities

$$\sum_{k=1}^j |z_k(q_n)|^{-1} \leq \theta(p_n) + \sum_{k=1}^j (k + 1)^{-\gamma} \leq \theta_f + \sum_{k=1}^j (k + 1)^{-\gamma} \quad (j = 1, \dots, n - 1). \quad (7)$$

But the zeros of entire functions continuously depend on its coefficients. So for any fixed natural $j \leq m$,

$$\sum_{k=1}^j |z_k(q_n)|^{-1} \rightarrow \sum_{k=1}^j |z_k(f)|^{-1}$$

as $n \rightarrow \infty$. Now (7) implies the required result. ■

3. APPLICATIONS OF THEOREM 1

Put

$$\tau_1 = \theta_f + 2^{-\gamma} \quad \text{and} \quad \tau_k = (k+1)^{-\gamma} \quad (k = 2, 3, \dots).$$

The well-known Lemma II.3.4 (Gohberg and Krein, 1969) and Theorem 1 imply

COROLLARY 3 *Let $\phi(t)$ ($0 \leq t < \infty$) be a convex scalar-valued function, such that $\phi(0) = 0$. Then under (2) the inequalities*

$$\sum_{k=1}^j \phi(|z_k(f)|^{-1}) \leq \sum_{k=1}^j \phi(\tau_k) \quad (j = 1, 2, \dots, m)$$

are valid. In particular, for any real $r \geq 1$,

$$\sum_{k=1}^j |z_k(f)|^{-r} \leq \sum_{k=1}^j \tau_k^r = (\theta_f + 2^{-\gamma})^r + \sum_{k=2}^j (k+1)^{-r\gamma} \quad (j = 2, \dots, m). \tag{8}$$

Assume that the conditions

$$r\gamma > 1 \quad \text{and} \quad r \geq 1 \tag{9}$$

hold. Then the series

$$\sum_{k=1}^{\infty} \tau_k^r = (\theta_f + 2^{-\gamma})^r + \sum_{k=2}^{\infty} (k+1)^{-r\gamma} = (\theta_f + 2^{-\gamma})^r + \zeta(r\gamma) - 1 - 2^{-r\gamma}$$

converges. Here $\zeta(\cdot)$ is the Riemann zeta-function. Now relation (8) yields

COROLLARY 4 Under the conditions (2), (9) and $m = \infty$, the inequality

$$\sum_{k=1}^{\infty} |z_k(f)|^{-r} \leq (\theta_f + 2^{-\gamma})^r + \zeta(\gamma r) - 1 - 2^{-\gamma r} \tag{10}$$

is valid. In particular, if $\gamma > 1$ and $m = \infty$, then

$$\sum_{k=1}^{\infty} |z_k(f)|^{-1} \leq \theta_f + \zeta(\gamma) - 1. \tag{11}$$

This corollary under condition (9) improves the above mentioned Hadamard theorem, since it not only asserts the convergence of the series in the left hand parts of (10) and (11), but gives us the estimate for the sums of the zeros.

Consider now a positive scalar-valued function $\Phi(t_1, t_2, \dots, t_j)$ ($j \leq m$) defined on the domain

$$0 \leq t_j \leq t_{j-1} \leq t_2 \leq t_1 < \infty$$

and satisfying

$$\frac{\partial \Phi}{\partial t_1} > \frac{\partial \Phi}{\partial t_2} > \dots > \frac{\partial \Phi}{\partial t_j} > 0 \quad \text{for } t_1 > t_2 > \dots > t_j. \tag{12}$$

Then Theorem 1 and the well-known Lemma II.3.5 (Gohberg and Krein, 1969) yield.

COROLLARY 5 Under conditions (2) and (12), the inequality

$$\Phi(|z_1(f)|^{-1}, |z_2(f)|^{-1}, \dots, |z_j(f)|^{-1}) \leq \Phi(\tau_1, \tau_2, \dots, \tau_j)$$

is valid.

In particular, let $\{d_k\}_{k=1}^{\infty}$ be a decreasing sequence of non-negative numbers. Take

$$\Phi(t_1, t_2, \dots, t_j) = \sum_{k=1}^j d_k t_k.$$

Then Corollary 5 yields

$$\sum_{k=1}^j d_k |z_k(f)|^{-1} \leq \sum_{k=1}^j \tau_k d_k = d_1 \theta_f + \sum_{k=2}^j d_k (k+1)^{-\gamma} \tag{13}$$

$(j = 2, \dots, m).$

For instance, let $0 < \gamma < 1$. Take $d_k = (k+1)^{-1-\varepsilon+\gamma}$ with an arbitrary positive ε . Then

$$\sum_{k=1}^j (k+1)^{-1-\varepsilon+\gamma} |z_k(f)|^{-1} \leq 2^{-1-\varepsilon+\gamma} \theta_f + \sum_{k=1}^j (k+1)^{-1-\varepsilon} \quad (j = 2, \dots, m).$$

Hence, with $m = \infty$ we get.

$$\sum_{k=1}^{\infty} (k+1)^{-1-\varepsilon+\gamma} |z_k(f)|^{-1} \leq 2^{-1-\varepsilon+\gamma} \theta_f + \zeta(1+\varepsilon) - 1.$$

Take $\gamma = 0$. Then relation (13) yields

$$\sum_{k=1}^j d_k |z_k(f)|^{-1} \leq d_1 \theta_f + \sum_{k=2}^j d_k \quad (j = 2, 3, \dots).$$

Assume that

$$d_1 = 1, \quad \alpha \equiv \sum_{k=2}^{\infty} d_k < \infty (d_{k+1} \leq d_k). \quad (14)$$

COROLLARY 6 *Let in (1) $\gamma = 0$ and the set of zeros of f be infinite. Then under assumption (14), the inequality*

$$\sum_{k=1}^{\infty} d_k |z_k(f)|^{-1} \leq \theta_f + \alpha \quad (15)$$

is valid.

In particular, taking in (15) $d_k = k^{-2}$, with $\gamma = 0$ we get

$$\sum_{k=1}^{\infty} k^{-2} |z_k(f)|^{-1} \leq \theta_f + \zeta(2) - 1.$$

If $\gamma = 0$ and $d_k = 2^{1-k}$, then inequality (15) implies

$$2 \sum_{k=1}^{\infty} \frac{1}{2^k |z_k(f)|} \leq \theta_f + 1.$$

References

- [1] Gohberg, I. C. and Krein, M. G. (1969). Introduction to the Theory of Linear Nonselfadjoint Operators, *Trans. of Math. Monographs*, Vol. 18, *Amer. Math. Soc.*, R.I.
- [2] Levin, B. Ya. (1996). Lectures on Entire Functions, *Trans. of Math. Monographs*, Vol. 150. *Amer. Math. Soc.*, R.I.
- [3] Marcus, M. and Minc, H. (1964). *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston.