

## Research Article

# Bayesian Estimation of Generalized Process Capability Indices

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Process capability indices (PCIs) aim to quantify the capability of a process of quality characteristic ( $X$ ) to meet some specifications that are related to a measurable characteristic of its produced items. One such quality characteristic is life time of items. The specifications are determined through the lower specification limit ( $L$ ), the upper specification limit ( $U$ ), and the target value ( $T$ ). Maiti et al. (2010) have proposed a generalized process capability index that is the ratio of proportion of specification conformance to proportion of desired conformance. Bayesian estimation of the index has been considered under squared error loss function. Normal, exponential (nonnormal), and Poisson (discrete) processes have been taken into account. Bayes estimates of the index have been compared with the frequentist counterparts. Data sets have been analyzed.

## 1. Introduction

The purpose of process capability index (PCI) is to provide a numerical measure on whether a production process is capable of producing items within the specification limits or not. It becomes very popular in assessing the capability of manufacturing process in practice during the past decade. More and more efforts have been devoted to studies and applications of each PCIs. For example, the  $C_p$  and  $C_{pk}$  indices have been used in Japan and in the US automotive industry. For more information on PCIs, see Hsiang and Taguchi [1], Choi and Owen [2], Pearn et al. [3], Pearn and Kotz [4], Pearn and Chen [5], Mukherjee [6], Yeh and Bhattacharya [7], Borges and Ho [8], Perakis and Xekalaki [9, 10], and Maiti et al. [11].

The usual practice is to estimate these PCIs from data and then judge the capability of the process by these estimates. Most studies on PCIs are based on the traditional frequentist point of view. The main objective of this note is to provide both point and interval estimators of the PCIs given by Maiti et al. [11] from the Bayesian point of view. We believe this effort is well justified since the Bayesian estimation has become one of popular approaches in estimation. In addition, the Bayesian approach has one great advantage over the traditional

frequentist approach: the posterior distribution is sometimes very easy to derive, and credible intervals, which are the Bayesian analogue of the classical confidence interval, can be easily obtained either by theoretical derivation or Monte Carlo methods. Lower credible limits (lcls) are constructed. Upper credible limits can also be obtained in a similar manner. However, only the case of lcls is considered as these are of greater interest (due to the fact that large values of PCIs are desirable).

The paper is organized as follows. We give a brief review on the PCIs,  $C_p$ ,  $C_{pk}$ ,  $C_{pm}$ , and  $C_{py}$  in Section 2. In Sections 3, 4, and 5, we derive the Bayes estimators for  $C_{py}$  (with process median being the process center) with respect to some chosen priors under the assumption of normal, exponential (nonnormal), and Poisson (discrete) distribution, respectively. Simulation results have been reported and discussed in Section 6. In Section 7, data sets have been analyzed to demonstrate the application of the proposed Bayesian procedure. Section 8 concludes.

## 2. Review of Some Process Capability Indices

The most popular PCIs are  $C_p$ ,  $C_{pk}$ , and  $C_{pm}$ . The  $C_p$  index is defined as

$$C_p = \frac{U - L}{6\sigma}, \quad (2.1)$$

where  $L$  and  $U$  are the lower and upper specification limits, respectively, and  $\sigma$  is the process standard deviation. Note that  $C_p$  does not depend on the process mean. The  $C_{pk}$  is then introduced to reflect the impact of  $\mu$  on the process capability indices. The  $C_{pk}$  index is defined as

$$C_{pk} = \min \left[ \frac{\mu - L}{3\sigma}, \frac{U - \mu}{3\sigma} \right]. \quad (2.2)$$

The  $C_{pm}$  index was introduced by Chan et al. [12]. This index takes into account the influence of the departure of the process mean  $\mu$  from the process target  $T$ . The  $C_{pm}$  index is defined as

$$C_{pm} = \frac{U - L}{6\sqrt{\sigma^2 + (\mu - T)^2}}. \quad (2.3)$$

Maiti et al. [11] suggested a more generalized measure which is directly or indirectly associated with all the previously defined capability indices. The measure is as follows:

$$C_{py} = \frac{p}{p_0}, \quad (2.4)$$

where  $p$  is the process yield that is,  $p = F(U) - F(L)$ ,  $F(t) = P(X \leq t)$  is the cumulative distribution function of  $X$ , and  $p_0$  is the desirable yield that is,  $p_0 = F(UDL) - F(LDL)$ ,  $LDL$  and  $UDL$  be the lower and upper desirable limit, respectively. When the process is off

centered, then  $F(L) + F(U) \neq 1$  but the proportion of desired conformance is achieved. In that case, the index is as follows:

$$C_{pyk} = \min(C_{pyu}, C_{pyl}), \quad (2.5)$$

where

$$\begin{aligned} C_{pyu} &= \frac{F(U) - F(\mu_e)}{(1/2) - \alpha_2}, \\ C_{pyl} &= \frac{F(\mu_e) - F(L)}{(1/2) - \alpha_1}, \end{aligned} \quad (2.6)$$

with  $\mu_e$  being the median of the distribution and the process center is to be located such that  $F(\mu_e) = (F(L) + F(U))/2$  that is,  $F(L) + F(U) = 1$ ,  $\alpha_1 = P(X < LDL)$ , and  $\alpha_2 = P(X > UDL)$ . It generally happens that process target  $T$  is such that  $F(T) = (F(L) + F(U))/2$ ; if  $F(T) \neq (F(L) + F(U))/2$ , the situation may be described as "generalized asymmetric tolerances" have been described by the term "asymmetric tolerances" when  $T \neq (L + U)/2$ . Under this circumstance, the index is defined as follows:

$$C_{pTk} = \min \left[ \frac{F(U) - F(T)}{(1/2) - \alpha_2}, \frac{F(T) - F(L)}{(1/2) - \alpha_1} \right]. \quad (2.7)$$

### 3. Bayes Estimate of $C_{py}$ for Normal Process

Let  $x_1, x_2, x_3, \dots, x_n$  be  $n$  observations from normal distribution with parameter  $\mu$  and  $\sigma^2$ . Then, the joint distribution of  $x_1, x_2, x_3, \dots, x_n$  is

$$L(x | \mu, \sigma) = \frac{1}{(2\pi)^{n/2} (\sigma^2)^{n/2}} e^{-(1/2\sigma^2) \sum (x_i - \mu)^2}. \quad (3.1)$$

Regarding selection of the the prior distributions, it is advisable to choose conjugate prior, since in this situation, even if prior parameters are unknown in practice, these may be estimated approximately from the likelihood functions as discussed in subsequent sections. When there is no information about the parameter(s) of the distribution, noninformative prior choice is good one.

#### 3.1. Conjugate Prior Distributions

Here we assume that the prior distribution of  $(\mu, \sigma^2)$  is of the following form

$$g(\mu, \sigma^2) = g_1(\mu | \sigma^2) g_2(\sigma^2), \quad (3.2)$$

where the  $\mu$  given  $\sigma^2$  follows normal distribution with mean  $\mu_0$  and variance  $\sigma^2$  and  $\sigma^2$  follows an inverted gamma distribution of the form:

$$f(\sigma^2) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \sigma^{-2(\alpha+1)} e^{-(\beta/\sigma^2)}; \quad \sigma^2 > 0, \alpha, \beta > 0. \quad (3.3)$$

Hence the posterior distribution of  $(\mu, \sigma^2)$  is given by

$$\begin{aligned} g(\mu, \sigma^2 | x) &= \frac{L(x/\mu, \sigma^2)g(\mu, \sigma^2)}{\int_0^\infty \int_{-\infty}^\infty L(x/\mu, \sigma^2)g(\mu, \sigma^2)d\mu d\sigma^2} \\ &= 2 \frac{\sqrt{(n+1)}}{\sigma\sqrt{(2\pi)}} \cdot e^{-((n+1)/2\sigma^2)(\mu - (\bar{x}_0/(n+1)))^2} \frac{W_1^{((n+1)/2)+\alpha}}{\Gamma(((n+1)/2)+\alpha)} \cdot \sigma^{-2((n/2)+\alpha+1)} e^{-(\beta/\sigma^2)}, \end{aligned} \quad (3.4)$$

where  $\bar{x}_0 = n\bar{x} + \mu_0$  and  $W_1 = \sum x_i^2 + 2\beta + \mu_0^2 - (\bar{x}_0^2/(n+1))$ .

If the process quality characteristic follows normal distribution with mean  $\mu$  and variance  $\sigma^2$ , then the generalized process capability index is given by

$$C_{py} = \frac{p}{p_0} = \frac{\Phi((U - \mu)/\sigma) - \Phi((L - \mu)/\sigma)}{p_0}. \quad (3.5)$$

Then, the Bayes estimate of  $C_{py}$  under squared error loss is given by

$$\begin{aligned} \hat{C}_{py} &= E(C_{py} | x) \\ &= \frac{1}{p_0} \int_0^\infty \int_{-\infty}^\infty \left\{ \Phi\left(\frac{U - \mu}{\sigma}\right) - \Phi\left(\frac{L - \mu}{\sigma}\right) \right\} g(\mu, \sigma^2 | x) d\mu d\sigma^2. \end{aligned} \quad (3.6)$$

Now,

$$E(C_{py}^2 | x) = \frac{1}{p_0^2} \int_0^\infty \int_{-\infty}^\infty \left\{ \Phi\left(\frac{U - \mu}{\sigma}\right) - \Phi\left(\frac{L - \mu}{\sigma}\right) \right\}^2 g(\mu, \sigma^2 | x) d\mu d\sigma^2 \quad (3.7)$$

and hence

$$\text{Var}(C_{py} | x) = E(C_{py}^2 | x) - E^2(C_{py} | x). \quad (3.8)$$

It is to be noted that the Bayes estimate of  $C_{py}$  and the variance depend on the parameters of the prior distribution of  $\mu$  and  $\sigma^2$ . These parameters could be estimated by means of an empirical Bayes procedure, see Lindley [13] and Awad and Gharraf [14]. Given the random samples  $(X_1, X_2, \dots, X_n)$ , the likelihood function of  $\mu$  given  $\sigma^2$  is normal density with mean  $(\bar{X})$  and the likelihood function of  $\sigma^2$  is inverted gamma with  $\beta = \sum_{i=1}^n (X_i - \bar{X})^2 / 2$  and  $\alpha = (n - 3) / 2$ . Hence it is proposed to estimate the prior parameters  $\mu_0$ , and  $\beta$  and  $\alpha$  from

the samples by  $\bar{X}$ , and  $\sum_{i=1}^n (X_i - \bar{X})^2 / 2$  and  $(n - 3) / 2$ , respectively. The variances of these estimators are  $\sigma^2 / n$ ,  $(n - 1)\sigma^4 / 2$  and 0, respectively. The expressions of  $\bar{x}_0$  and  $W_1$  will be  $(n + 1)\bar{X}$  and  $2 \sum_{i=1}^n (X_i - \bar{X})^2$ , respectively.

### 3.2. Noninformative Prior Distributions

Here we assume that the prior distribution of  $(\mu, \sigma^2)$  is noninformative of the form

$$g(\mu, \sigma^2) \propto \frac{1}{\sigma^2}. \quad (3.9)$$

Hence the posterior distribution of  $(\mu, \sigma^2)$  is of the form

$$g(\mu, \sigma^2 | x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(1/2\sigma^2)(\mu - \bar{x})^2} 2 \frac{\left(\sum (x_i - \bar{x})^2 / 2\right)^{n/2}}{\Gamma(n/2)} \sigma^{-2((n+1)/2)} e^{-(\sum (x_i - \bar{x})^2) / 2\sigma^2}. \quad (3.10)$$

Estimates are to be found out in the same way as in Section 3.1.

## 4. Bayes Estimate of $C_{Py}$ for Exponential Process

Let  $x_1, x_2, x_3, \dots, x_n$  be  $n$  observations from exponential distribution with parameter  $\lambda$ . Then, the joint distribution of  $x_1, x_2, x_3, \dots, x_n$  is

$$L(x | \lambda) = \lambda^n e^{-\lambda \sum x_i}. \quad (4.1)$$

### 4.1. Conjugate Prior Distributions

Here we assume that the prior distribution of  $\lambda$  is gamma with parameter  $(m, a)$  that is, the distribution of  $\lambda$  is given as

$$g(\lambda) = \frac{a^m}{\Gamma(m)} e^{-a\lambda} \lambda^{m-1}, \quad \lambda > 0. \quad (4.2)$$

Hence the posterior distribution of  $\lambda$  is given as

$$\begin{aligned} g(\lambda | x) &= \frac{L(x/\lambda)g(\lambda)}{\int_0^\infty L(x/\lambda)g(\lambda)d\lambda} \\ &= \frac{(\sum x_i + a)^{m+n}}{\Gamma(m+n)} \lambda^{m+n-1} e^{-\lambda(a+\sum x_i)}, \quad \lambda > 0. \end{aligned} \quad (4.3)$$

As a process whose distribution can be regarded to be the exponential distribution, the generalized process capability index is given by

$$C_{py} = \frac{p}{p_0} = \frac{\{e^{-\lambda L} - e^{-\lambda U}\}}{p_0}. \quad (4.4)$$

Then, the Bayes estimate of  $C_{py}$  under squared error loss is given by

$$\begin{aligned} \hat{C}_{py} &= E(C_{py} | x) \\ &= \frac{1}{p_0} \int_0^\infty \{e^{-\lambda L} - e^{-\lambda U}\} g(\lambda | x) d\lambda \\ &= \frac{1}{p_0} \left[ \left( \frac{a + \sum x_i}{L + a + \sum x_i} \right)^{m+n} - \left( \frac{a + \sum x_i}{U + a + \sum x_i} \right)^{m+n} \right]. \end{aligned} \quad (4.5)$$

Now,

$$\text{Var}(C_{py} | x) = E(C_{py}^2 | x) - E^2(C_{py} | x). \quad (4.6)$$

Again,

$$\begin{aligned} E(C_{py}^2 | x) &= \left( \frac{1}{p_0} \right)^2 \int_0^\infty \{e^{-2\lambda L} + e^{-2\lambda U} - 2e^{-\lambda(L+U)}\} g(\lambda | x) d\lambda \\ &= \left( \frac{1}{p_0} \right)^2 \left[ \left( \frac{a + \sum x_i}{2L + a + \sum x_i} \right)^{m+n} + \left( \frac{a + \sum x_i}{2U + a + \sum x_i} \right)^{m+n} \right. \\ &\quad \left. - 2 \left( \frac{a + \sum x_i}{L + U + a + \sum x_i} \right)^{m+n} \right]. \end{aligned} \quad (4.7)$$

Thus,

$$\begin{aligned} \text{Var}(C_{py} | x) &= \left( \frac{1}{p_0} \right)^2 \left[ \left\{ \left( \frac{a + \sum x_i}{2L + a + \sum x_i} \right)^{m+n} + \left( \frac{a + \sum x_i}{2U + a + \sum x_i} \right)^{m+n} \right. \right. \\ &\quad \left. \left. - 2 \left( \frac{a + \sum x_i}{L + U + a + \sum x_i} \right)^{m+n} \right\} \right. \\ &\quad \left. - \left\{ \left( \frac{a + \sum x_i}{L + a + \sum x_i} \right)^{m+n} - \left( \frac{a + \sum x_i}{U + a + \sum x_i} \right)^{m+n} \right\}^2 \right]. \end{aligned} \quad (4.8)$$

If we put  $L = 0$  that is, if only upper specification limit is given, then

$$C_{py} = \frac{1 - e^{-\lambda U}}{p_0} \quad (4.9)$$

with posterior distribution

$$\begin{aligned}
 g(C_{py} | x) &= \frac{(\sum x_i + a)^{m+n}}{\Gamma(m+n)U(1-p_0 \cdot C_{py})} \left( -\frac{\ln(1-p_0 \cdot C_{py})}{U} \right)^{m+n-1} e^{(\ln(1-p_0 \cdot C_{py})(\sum x_i + a)/U)} p_0 \\
 &= \frac{(\sum x_i + a)^{m+n}}{\Gamma(m+n)L \cdot C_{py}} \left( (1-p_0 \cdot C_{py})(\sum x_i + a) \right)^{1/U} \left( -\frac{\ln(1-p_0 \cdot C_{py})}{U} \right)^{m+n-1}, \\
 & \qquad \qquad \qquad 0 < C_{py} < \frac{1}{p_0},
 \end{aligned} \tag{4.10}$$

and the Bayes estimate is given by

$$\hat{C}_{py} = \frac{1}{p_0} \left[ 1 - \left( \frac{a + \sum x_i}{U + a + \sum x_i} \right)^{m+n} \right]. \tag{4.11}$$

Similarly, if  $U = \infty$  that is, if only lower specification is given, then

$$C_{py} = \frac{e^{-\lambda L}}{p_0} \tag{4.12}$$

with posterior distribution

$$\begin{aligned}
 g(C_{py} | x) &= \frac{(\sum x_i + a)^{m+n}}{\Gamma(m+n)L \cdot p_0 \cdot C_{py}} \left( -\frac{\ln(p_0 \cdot C_{py})}{L} \right)^{m+n-1} e^{(\ln(p_0 \cdot C_{py})(\sum x_i + a)/L)} p_0 \\
 &= \frac{(\sum x_i + a)^{m+n}}{\Gamma(m+n)L \cdot C_{py}} \left( p_0 \cdot C_{py} (\sum x_i + a) \right)^{1/L} \left( -\frac{\ln(p_0 \cdot C_{py})}{L} \right)^{m+n-1}, \quad 0 < C_{py} < \frac{1}{p_0},
 \end{aligned} \tag{4.13}$$

and the Bayes estimate is given by

$$\hat{C}_{py} = \frac{1}{p_0} \left( \frac{a + \sum x_i}{U + a + \sum x_i} \right)^{m+n}. \tag{4.14}$$

The Bayes estimate of  $C_{py}$  and the variance depend on the parameters of the prior distribution of  $\lambda$ . Given the random samples  $(X_1, X_2, \dots, X_n)$ , the likelihood function of  $\lambda$  is gamma density with parameters  $(n+1, \sum X_i)$ . Hence it is proposed to estimate the

prior parameters  $m$  and  $a$  from the samples by  $n + 1$  and  $\sum X_i$  with variances 0 and  $n/\lambda^2$ , respectively. Hence

$$\begin{aligned}\hat{C}_{py} &= \frac{1}{p_0} \left[ \left( \frac{2 \sum x_i}{L + 2 \sum x_i} \right)^{2n+1} - \left( \frac{2 \sum x_i}{U + 2 \sum x_i} \right)^{2n+1} \right], \\ \text{Var}(C_{py} | x) &= \left( \frac{1}{p_0} \right)^2 \left[ \left\{ \left( \frac{\sum x_i}{L + \sum x_i} \right)^{2n+1} + \left( \frac{\sum x_i}{U + \sum x_i} \right)^{2n+1} - 2 \left( \frac{2 \sum x_i}{L + U + 2 \sum x_i} \right)^{2n+1} \right\} \right. \\ &\quad \left. - \left\{ \left( \frac{2 \sum x_i}{L + 2 \sum x_i} \right)^{2n+1} - \left( \frac{2 \sum x_i}{U + 2 \sum x_i} \right)^{2n+1} \right\}^2 \right].\end{aligned}\quad (4.15)$$

#### 4.2. Noninformative Prior Distributions

In this subsection, we obtain the Bayes estimator of  $C_{py}$  under the assumption that the parameter  $\lambda$  is random variable having noninformative prior  $g(\lambda) \propto 1/\lambda$ .

Hence, the Bayes estimator with respect to squared error loss function will be

$$\begin{aligned}\hat{C}_{py} &= \frac{1}{p_0} \left[ \left( \frac{\sum x_i}{L + \sum x_i} \right)^n - \left( \frac{\sum x_i}{U + \sum x_i} \right)^n \right], \\ \text{Var}(C_{py} | x) &= \left( \frac{1}{p_0} \right)^2 \left[ \left\{ \left( \frac{\sum x_i}{2L + \sum x_i} \right)^n + \left( \frac{\sum x_i}{2U + \sum x_i} \right)^n - 2 \left( \frac{\sum x_i}{L + U + \sum x_i} \right)^n \right\} \right. \\ &\quad \left. - \left\{ \left( \frac{\sum x_i}{L + \sum x_i} \right)^n - \left( \frac{\sum x_i}{U + \sum x_i} \right)^n \right\}^2 \right].\end{aligned}\quad (4.16)$$

When only upper (lower) specification is to be given, then we will get the expressions substituting  $L = 0$  ( $U = \infty$ ).

### 5. Bayes Estimate of $C_{py}$ for Poisson Process

Let  $x_1, x_2, x_3, \dots, x_n$  be  $n$  observations from Poisson distribution with parameter  $\lambda$ . Then, the joint distribution of  $x_1, x_2, x_3, \dots, x_n$  is

$$\begin{aligned}L(x | \lambda) &= \prod \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \\ &= \frac{e^{-n\lambda} \lambda^{\sum x_i}}{\prod x_i!}.\end{aligned}\quad (5.1)$$



### 5.1. Conjugate Prior Distributions

Let the prior distribution of  $\lambda$  is assumed to be gamma with parameter  $(m, a)$ . Then the distribution of  $\lambda$  is given as

$$g(\lambda) = \frac{a^m}{\Gamma(m)} e^{-a\lambda} \lambda^{m-1}, \quad \lambda > 0. \quad (5.2)$$

Now, the posterior distribution of  $\lambda$  is given as

$$g(\lambda | x) = \frac{(n+a)^{\sum x_i + m}}{\Gamma(\sum x_i + m)} e^{-(n+a)\lambda} \lambda^{(m+\sum x_i - 1)}, \quad \lambda > 0. \quad (5.3)$$

Now, the process yield is

$$p = e^{-\lambda} \sum_{t=L}^U \frac{\lambda^t}{t!}. \quad (5.4)$$

Then,

$$\begin{aligned} \hat{C}_{py} &= E(C_{py} | x) = \frac{1}{p_0} \left[ \sum_{t=L}^U \frac{1}{t!} \int_0^\infty e^{-\lambda} \lambda^t g(\lambda | x) d\lambda \right] \\ &= \frac{1}{p_0} \left[ \sum_{t=L}^U \frac{1}{t!} \int_0^\infty \frac{(n+a)^{\sum x_i + m}}{\Gamma(\sum x_i + m)} e^{-(n+a+1)\lambda} \lambda^{\sum x_i + m + t - 1} d\lambda \right] \\ &= \frac{1}{p_0} \left[ \sum_{t=L}^U \frac{1}{t!} \left( \frac{n+a}{n+a+1} \right)^{\sum x_i + m} \left( \frac{1}{n+a+1} \right)^t \frac{\Gamma(\sum x_i + m + t)}{\Gamma(\sum x_i + m)} \right]. \end{aligned} \quad (5.5)$$

Again,

$$\begin{aligned} E(C_{py}^2 | x) &= \frac{1}{p_0^2} \left[ \sum_{t=L}^U \left( \frac{1}{t!} \right)^2 \int_0^\infty e^{-2\lambda} \lambda^{2t} g(\lambda | x) d\lambda + 2 \sum \sum \frac{1}{t!} \frac{1}{t'!} \int_0^\infty e^{-2\lambda} \lambda^t \lambda^{t'} g(\lambda | x) d\lambda \right] \\ &= \frac{1}{p_0^2} \left[ \sum_{t=L}^U \left( \frac{1}{t!} \right)^2 \int_0^\infty \frac{(n+1)^{\sum x_i + m}}{\Gamma(\sum x_i + m)} e^{-(n+a+2)\lambda} \lambda^{\sum x_i + m + 2t - 1} d\lambda \right. \\ &\quad \left. + 2 \sum \sum \frac{1}{t!} \frac{1}{t'!} \int_0^\infty \frac{(n+a)^{\sum x_i + m}}{\Gamma(\sum x_i + m)} e^{-(n+a+2)\lambda} \lambda^{\sum x_i + m + t + t' - 1} d\lambda \right] \\ &= \frac{1}{p_0^2} \left[ \sum_{t=L}^U \left( \frac{1}{t!} \right)^2 \left( \frac{n+a}{n+a+2} \right)^{\sum x_i + m} \left( \frac{1}{n+a+2} \right)^{2t} \frac{\Gamma(\sum x_i + m + 2t)}{\Gamma(\sum x_i + m)} \right. \\ &\quad \left. + 2 \sum \sum \frac{1}{t!} \frac{1}{t'!} \left( \frac{n+a}{n+a+2} \right)^{\sum x_i + m} \left( \frac{1}{n+a+2} \right)^{t+t'} \frac{\Gamma(\sum x_i + m + t + t')}{\Gamma(\sum x_i + m)} \right]. \end{aligned} \quad (5.6)$$

**Table 1:** Bayes Estimates of  $C_{py}$  and their MSEs with  $L = 0$ ,  $U = 10$ , samples generated from normal distribution.

$\mu, \sigma, C_{py}   n$	25	50	100	150
5, 3	1.01033	1.018082	1.026673	1.0277564
1.004910	0.0201438	0.0187564	0.00856342	0.0067548
5, 4	0.960722	0.964536	0.956746	0.953425
0.8763338	0.0162069	0.0087685	0.0065743	0.0056734
6, 3	0.980259	0.995643	0.996753	0.999452
0.9844874	0.0154522	0.0098564	0.0078453	0.0056734
6, 4	0.966049	0.967543	0.965746	0.967854
0.8605973	0.0112564	0.0087564	0.0065643	0.0036754

Here, the Bayes estimate of  $C_{py}$  and the variance depend on the parameters of the prior distribution of  $\lambda$ . Given the random samples  $(X_1, X_2, \dots, X_n)$ , the likelihood function of  $\lambda$  is gamma density with parameters  $(\sum X_i + 1, n)$ . Hence it is proposed to estimate the prior parameters  $m$  and  $a$  from the samples by  $\sum X_i + 1$  and  $n$  with variances  $n\lambda$  and 0, respectively. Substituting these in the above expressions, we will have the empirical Bayes estimates.

## 5.2. Noninformative Prior Distributions

In this subsection, we obtain the Bayes estimator of  $C_{py}$  under the assumption that the parameter  $\lambda$  is random variable having noninformative prior  $g(\lambda) \propto 1/\lambda$ .

Hence, the Bayes estimator with respect to squared error loss function will be

$$\begin{aligned} \hat{C}_{py} &= \frac{1}{p_0} \left[ \sum_{t=L}^U \frac{1}{t!} \left( \frac{n}{n+1} \right)^{\sum x_i} \left( \frac{1}{n+1} \right)^t \frac{\Gamma(\sum x_i + t)}{\Gamma(\sum x_i)} \right], \\ E(C_{py}^2 | x) &= \frac{1}{p_0^2} \left[ \sum_{t=L}^U \left( \frac{1}{t!} \right)^2 \left( \frac{n}{n+2} \right)^{\sum x_i} \left( \frac{1}{n+2} \right)^{2t} \frac{\Gamma(\sum x_i + 2t)}{\Gamma(\sum x_i)} \right. \\ &\quad \left. + 2 \sum_{t < t'} \frac{1}{t!} \frac{1}{t'!} \left( \frac{n}{n+2} \right)^{\sum x_i} \left( \frac{1}{n+2} \right)^{t+t'} \frac{\Gamma(\sum x_i + t + t')}{\Gamma(\sum x_i)} \right]. \end{aligned} \quad (5.7)$$

When only upper (lower) specification is to be given, then we will get the expressions substituting  $L = 0$  ( $U = \infty$ ).

## 6. Simulation and Discussion

In this section, we present some results based on the Monte Carlo Simulations to compare the performance of frequentist (maximum likelihood and minimum variance unbiased estimators) as well as the Bayesian method of estimation. All the computations were performed using *R*-software and *Mathematica*, and these are available on request from the corresponding author. The maximum likelihood estimator (MLE) and uniformly minimum variance unbiased estimator (UMVUE) and their mean square errors (MSEs) were shown in Maiti et al. [11]. We have performed the Bayes estimators and their MSEs in Tables 1–6. All the results are based on 25,000 replications.

**Table 2:** Bayes Estimates of  $C_{py}$  and their MSEs with  $L = 0, U = 8$ , samples generated from normal distribution.

$\mu, \sigma, C_{py}   n$	25	50	100	150
5, 3	0.979133	0.978665	0.979896	0.9799876
0.8817271	0.0130996	0.0087564	0.00556342	0.0027548
5, 4	0.833937	0.834536	0.836746	0.843425
0.7419143	0.0225739	0.0097685	0.0075743	0.0047865
6, 3	0.860858	0.861465	0.862612	0.864012
0.805286	0.0036421	0.0015647	0.00096745	0.00068963
6, 4	0.759651	0.754632	0.759984	0.753241
0.6940614	0.010984	0.0088576	0.0050495	0.0028967

In each cell first row indicates the Bayes estimates of  $C_{py}$  and second row indicates its MSE.

**Table 3:** Bayes Estimates of  $C_{py}$  and their MSEs with  $L = 0, U = 10$ , samples generated from exponential distribution.

$\lambda, C_{py}   n$	25	50	100	150
0.2	0.9608913	0.9612706	0.9608863	0.960665
0.9607386	0.003193873	0.001708656	0.0008960904	0.0005968731
0.5	1.00270	1.102005	1.102827	1.103064
1.103625	0.0001013043	$3.777950 \times 10^{-5}$	$1.656285 \times 10^{-5}$	$1.069774 \times 10^{-5}$
0.7	1.108919	1.10954	1.109825	1.109927
1.11098	$8.406398 \times 10^{-6}$	$2.39847 \times 10^{-6}$	$8.557008 \times 10^{-7}$	$4.58935 \times 10^{-7}$
1	1.110879	1.110982	1.111026	1.111039
1.11061	$2.113364 \times 10^{-7}$	$3.841548 \times 10^{-8}$	$8.476886 \times 10^{-9}$	$4.000626 \times 10^{-9}$

We represented the average  $C_{py}$  value and the MSE for normal process in Tables 1 and 2. We take the same set up of Maiti et al. [11] to make comparable with the Bayesian approach. We take  $p_0 = 0.90$ , for two choices of  $(L, U)$  as  $(0, 10)$  and  $(0, 8)$ , and for sample of sizes  $n = 25, 50, 100, 150$ . We generate observations from normal distributions with choices of  $(\mu, \sigma) = (5, 3), (5, 4), (6, 3)$ , and  $(6, 4)$ . First column of Tables 1 and 2 shows the values of  $(\mu, \sigma)$  and the corresponding  $C_{py}$ . Remaining columns show average  $\hat{C}_{py}$ , and its MSE, for the above-mentioned sample sizes. It is observed that in almost all the cases, MSEs of  $\hat{C}_{py}$  in the Bayesian set up using the empirical Bayes procedure of the prior parameters are larger than those obtained in the frequentist approach.  $\hat{C}_{py}$  overestimates the true  $C_{py}$  in general. Therefore, this empirical Bayes estimate is not so encouraging compared to maximum likelihood estimator or minimum variance unbiased estimator.

We represented the average  $C_{py}$  value and the MSE for exponential process in Tables 3 and 4. We simulate observations from the exponential distribution with rate  $\lambda$ . We take  $\lambda = 0.2, 0.5, 0.7$ , and  $1.0$ . From Tables 3 and 4, we find that for  $\lambda < 0.5$ , the empirical Bayes estimate of  $C_{py}$  gives better result than the ML estimate of  $C_{py}$  in MSE sense, but for  $\lambda > 0.5$ , it reverses. As soon as the mean quality characteristic gets larger (when  $\lambda < 0.5$ ), the empirical Bayes estimate becomes better in MSE sense and hence, it is recommended to use it. For smaller mean quality characteristic, the use of UMVUE of  $C_{py}$  is fair even though it is, to some extent, computation intensive.

We simulate observations from Poisson distribution with mean  $m$ . We take  $m = 1, 4, 8$ , and  $10$ . From Tables 5 and 6, we find that for  $m = 1$  and  $4$ , the UMVUE of  $C_{py}$  gives better result than the empirical Bayes estimate of  $C_{py}$  in MSE sense, but for  $m = 8$  and  $10$ , it

**Table 4:** Bayes Estimates of  $C_{py}$  and their MSEs with  $L = 0$ ,  $U = 8$ , samples generated from exponential distribution.

$\lambda, C_{py}   n$	25	50	100	150
0.2	0.8907383	0.8867816	0.8883253	0.8875391
0.8867816	0.004669863	0.002447009	0.001263855	0.0008378543
0.5	1.086258	1.088331	1.089627	1.089983
1.090760	0.0003279872	0.0001522034	$6.9765 \times 10^{-5}$	$4.591931 \times 10^{-5}$
0.7	1.104586	1.105791	1.107002	1.106608
1.107002	$4.531069 \times 10^{-5}$	$1.691586 \times 10^{-5}$	$6.925086 \times 10^{-5}$	$4.192666 \times 10^{-5}$
1	1.10095	1.110440	1.110596	1.110646
1.110738	$2.400033 \times 10^{-6}$	$6.018572 \times 10^{-7}$	$1.812796 \times 10^{-7}$	$9.911838 \times 10^{-8}$

In each, cell first row indicates the Bayes estimates of  $C_{py}$  and second row indicates its MSE.

**Table 5:** Bayes estimates of  $C_{py}$  and their MSEs with  $L = 0$ ,  $U = 8$ , samples generated from exponential distribution.

$m, C_{py}   n$	25	50	100	150
1	1.111111	1.111111	1.111111	1.111111
1.111111	$7.292996 \times 10^{-14}$	$5.136107 \times 10^{-15}$	$6.73274 \times 10^{-16}$	$2.730549 \times 10^{-16}$
4	1.106690	1.107327	1.107646	1.107764
1.107956	$1.190196 \times 10^{-5}$	$4.247376 \times 10^{-6}$	$1.758665 \times 10^{-6}$	$1.060935 \times 10^{-6}$
8	0.8990717	0.9022682	0.9045761	0.9048294
0.9065398	0.003909427	0.001932993	0.0009643427	0.0006502683
10	0.6452548	0.6471974	0.647822	0.647822
0.647822	0.007262223	0.003748991	0.001915881	0.001280117

is opposite. Here also if mean quality is getting larger and larger, like exponential process, the empirical Bayes estimate is estimated efficiently. So, it is advisable to use the empirical Bayes estimate of  $C_{py}$  when mean quality characteristic is large, but for smaller mean, use of UMVUE of  $C_{py}$  is a fair one.

It is expected that when there is prior information regarding parameters, the performance of the Bayes estimates would be better than their traditional frequentist counterpart. But here we choose empirical estimate of parameters following the approach of Lindley [13] and Awad and Gharraf [14]. Since it is an empirical approach, it may not perform uniformly better than the frequentist approach that has been reflected in simulation study. The performance is less encouraging in case of normally distributed quality characteristic whereas it performs better when the underlying distribution is exponential with larger mean and also performs better when the quality characteristic distribution is Poisson with a larger mean.

## 7. Data Analysis

This section is devoted for demonstrating inferential aspect of  $C_{py}$ , by analyzing some data sets. We choose two data sets fit approximately exponential and Poisson distribution, respectively.

(a) For demonstration purpose, we consider here the data that represent the number of miles to first and succeeding major motor failures of 191 buses (cf. Davis [15]) operated

**Table 6:** Bayes estimates of  $C_{py}$  and their MSEs with  $L = 0, U = 8$ , samples generated from exponential distribution.

$m, C_{py}   n$	25	50	100	150
1	1.111106	1.111108	1.111109	1.111109
1.111111	$1.083070 \times 10^{-10}$	$1.415413 \times 10^{-11}$	$3.215503 \times 10^{-12}$	$1.502195 \times 10^{-16}$
4	1.082719	1.085116	1.086145	1.086548
1.087374	0.0002517306	0.0001037937	$4.880645 \times 10^{-5}$	$3.151035 \times 10^{-5}$
8	0.656013	0.6564953	0.6577742	0.658386
0.658386	0.007244622	0.003718803	0.001873888	0.001267805
10	0.3747645	0.3720029	0.3706492	0.3708838
0.3697996	0.006036498	0.003085871	0.001548111	0.001035829

In each, cell first row indicates the Bayes estimates of  $C_{py}$  and second row indicates its MSE.

by a large city bus company. Failures were either abrupt, in which some part broke and the motor would not run or, by definition, when the maximum power produced, as measured by a dynamo meter, fell below a fixed percentage of the normal rated value. Failures of motor accessories which could be easily replaced were not included in these data. The bus motor failures are compared with exponential distribution, and observed chi-square index has been calculated as 3.40 with  $P$  value 0.32.

Here, we assume that the upper specification limit ( $U$ ) and lower specification limit ( $L$ ) are 75 and 15, respectively. Sample size,  $n = 85$ , sample mean  $\bar{x} = 35.17647058$ .

Then, we find out the MLE, MVUE and the Bayes estimate of the index as  $\hat{C}_{py} = 0.562372642$ ,  $\tilde{C}_{py} = 0.5623726$ , and  $\hat{C}_{py}^b = 0.5661285$ , respectively. And 95% lower confidence limit (lcl) of the Bayes estimate is 0.563874661.

Now, if we consider the case that only upper specification limit (UCL) has been given, then the MLE and MVUE of the index are  $\hat{C}_{pyu} = 0.927802988$  and  $\tilde{C}_{pyu} = 0.9280266$ , respectively. And 95% lower confidence limit (lcl) for the index  $C_{pyu}$  is given as 0.872705746 (cf. Maiti and Saha [16]). Here, we also find out the Bayes estimate of the index as  $\hat{C}_{pyu}^b = 1.042010$  and the corresponding 95% lcl, given as 1.039383292

On the other hand, if we consider the case that only lower specification limit has been given, then the MLE and MVUE of  $C_{pyl}$  are  $\hat{C}_{pyl} = 0.687201232$  and  $\tilde{C}_{pyl} = 0.6899253$ , respectively. And 95% lower confidence limit (lcl) for the index is given by 0.635114903. In this case, the Bayes estimate and corresponding 95% lcl of the index are  $\hat{C}_{pyl}^b = 0.63523$  and 0.629801423, respectively.

(b) Data on dates of repair calls on 15 hand electric drill motors are taken from Davis [15]. Mean number of days between failures for each drill was used as a milepost and frequency distribution compared with the theoretical Poisson distribution, and observed chi-square index has been calculated as 38 with  $P$  value 0.16. Here, we assume that the upper specification limit ( $U$ ) and lower specification limit ( $L$ ) are 3 and 1, respectively. Here sample size,  $n = 164$  and sample mean  $\bar{x} = 0.975609756$ . Then, the MLE, MVUE, and Bayes estimate of  $C_{py}$  are  $\hat{C}_{py} = 0.637377368$ ,  $\tilde{C}_{py} = 0.6389400$ , and  $\hat{C}_{py}^b = 0.673041$ , respectively. And 95% lower confidence limit (lcl) of the Bayes estimate is 0.659472.

Now, if we consider the case in which only upper specification limit (UCL) has been given, then the MLE of the index is  $\hat{C}_{pyu} = 1.034179963$  and the MVUE of the index is  $\tilde{C}_{pyu} = 1.034559263$ . 95% lower confidence limit (lcl) for the index  $C_{pyu}$  is given as 1.024565558.

Here the Bayes estimate and corresponding 95% lcl are  $\hat{C}_{py}^b = 1.091235$  and  $1.077885192$ , respectively.

On the other hand, if we consider the case that only lower specification limit has been given, then the MLE of the index  $\hat{C}_{pyl} = 0.655829023$  and the MVUE is  $\tilde{C}_{pyl} = 0.657012316$ . 95% lower confidence limit (lcl) for the index is given as  $0.448874947$ . In this case, we also find out the Bayes estimate and 95% lcl of the corresponding index, which are given as  $\hat{C}_{pyu}^b = 0.6929168$  and  $0.626240727$ , respectively.

## 8. Concluding Remark

In this paper, the Bayesian inference aspects of generalized PCI (cf. Maiti et al. [11]) have been presented. The Bayes estimates of the generalized process capability index have been studied for normal, exponential (nonnormal), and Poisson (discrete) processes. The empirical Bayes estimation procedure has been discussed when parameters of the prior distribution are unknown. The Bayes estimates have been compared with their most frequent counterpart, and situations have been mentioned when the Bayes estimates are better through simulation study. Because of its appealing features, examining its potential use in other types of processes often arising in connection with applications would be of practical importance. Other loss functions can be used to find out the estimates in similar fashion.

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