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Research Article

Heat Conduction in Lenses

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We consider several heat conduction problems for glass lenses with different boundary conditions. The problems dealt with in Sections 2 to 4 are motivated by the problem of an airborne digital camera that is initially too cold and must be heated up to reach the required image quality. The problem is how to distribute the heat to the different lenses in the system in order to reach acceptable operating conditions as quickly as possible. The problem of Section 5 concerns a space borne laser altimeter for planetary exploration. Will a coating used to absorb unwanted parts of the solar spectrum lead to unacceptable heating? In this paper, we present analytic solutions for idealized cases that help in understanding the essence of the problems qualitatively and quantitatively, without having to resort to finite element computations. The use of dimensionless quantities greatly simplifies the picture by reducing the number of relevant parameters. The methods used are classical: elementary real analysis and special functions. However, the boundary conditions dictated by our applications are not usually considered in classical works on the heat equation, so that the analytic solutions given here seem to be new. We will also show how energy conservation leads to interesting sum formulae in connection with Bessel functions. The other side of the story, to determine the deterioration of image quality by given (inhomogeneous) temperature distributions in the optical system, is not dealt with here.

1. Introduction

We consider several heat conduction problems motivated by industrial applications. Even though the use of finite elements is much en vogue today, we try to solve the problems with analytical methods for three reasons. First, finite element software is quite expensive

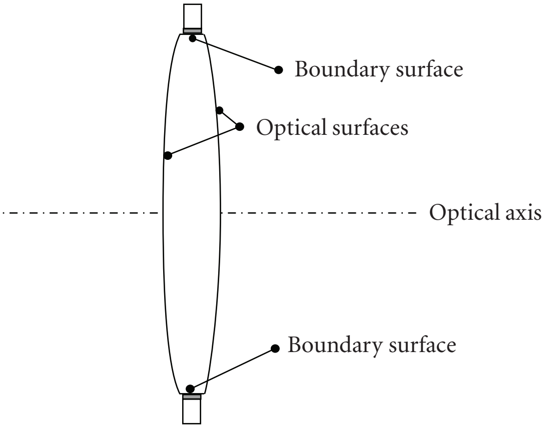

Figure 1.1. Sketch of a lens. The glue between the mounting and the glass is shown in gray.

and therefore not available to everybody, second, the reduction to dimensionless variables, a step that is often left off when finite element software is used, yields insight that one should not miss, third, trying to find analytic solutions one can have fun with the mathematics involved. Because of space limitations, we will not say much about the industrial background, just enough for an understanding of the mathematical problems. For the convenience of the reader, the mathematical derivations are relegated to the appendices.

Of course, most real-world problems must be simplified in order to allow for an analytic solution. Our case is not an exception. For most of the time we will simplify the optical lens as having constant thickness (in effect being optically useless). However, it is often possible to find a good approximation to the "correct" model, so that an analytic solution can be found. For instance in Section 5.5, we seek a stationary solution for a problem involving a "real" lens. A simple (and precise) approximation of the thickness profile allows for an explicit solution, while the same problem for a spherical lens can only be practically solved by iterated numerical quadrature.

Other and similar classical methods for deriving explicit solutions of heat equations have been employed in [1] (operational calculus), [2] (Laplace transform), [3] (Green's functions), [4] (fundamental solutions "heat kernel," Fourier series, operational calculus). Modern treatments of heat conduction can be found, for example, in [5] (classical methods, Laplace transform), [6] (computational/numerical methods), and many others.

1.1. General assumptions and preliminaries. A simple lens together with its mounting is shown in Figure 1.1. We distinguish between the "optical surfaces" and the "boundary surface" of the lens.

Initially, at time $t = 0$, the lens is uniformly cold. It is then heated up through its boundary surface, either by keeping the mounting at a higher temperature, which is then constant in time (Section 2), or by delivering a constant heat flux through the mounting up to some time $t_0 > 0$ (Section 3). In Section 4, part of the boundary surface will be kept at a given temperature, while another part will deliver a constant heat flux up to time t_0, the remaining part of the boundary being thermally isolated.

We make the following simplifying assumptions.

(i) The lens exchanges no heat through its optical surfaces ($\partial u/\partial n = 0$ at the optical surfaces).

(ii) The temperature of the lens is constant in the direction of its optical axis.

(iii) The lens and the mounting are radially symmetric, and, depending on the boundary condition.

(iv) The mounting keeps its temperature independently of the quantity of heat it must deliver.

or

(iv′) The heat source holds the heat flux constant independently of the temperature of the lens.

Assumption (ii) is satisfied when the lens is a *cylinder* and the mounting covers the whole mantle surface; if not, we will enforce it by averaging the temperature in the axis direction. This assumption reduces the problem to a spatially two-dimensional one (the lens is modeled as a circular disk) and the radial symmetry (iii) will eliminate another dimension.

In general, using *cylinder coordinates* r, φ, z, the temperature of the lens is described by the function

$$u = u(r,\varphi,z;t), \quad 0 \le r \le R,\ 0 \le \varphi \le 2\pi,\ 0 \le z \le h,\ t \ge 0, \tag{1.1}$$

where R denotes the radius of the lens and h its thickness, and u is 2π-periodic in φ. In fact, since our problems are radially symmetric and independent of z, the solution will be independent of φ and z,

$$u = u(r,t), \quad 0 \le r \le R,\ t \ge 0. \tag{1.2}$$

This follows from the fact that u satisfies the heat equation

$$\frac{\partial u}{\partial t} = \kappa \cdot \Delta u + Q(r,t) = \kappa \cdot \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}\right] + Q(r,t) \tag{1.3}$$

and is uniquely determined when initial and boundary conditions are imposed. Note that the *source term* Q is independent of φ and z and in fact will be nonzero only in Section 5. Averaging (1.3) over φ and z and using (i) and the periodicity in φ implies that the average temperature,

$$\overline{u}(r,t) := \frac{1}{2\pi h}\int_0^{2\pi}\int_0^h u(r,\varphi,z;t)\,dz\,d\varphi, \tag{1.4}$$

is another solution of (1.3), satisfying the same initial and boundary conditions when those are also independent of φ and z. Hence $u \equiv \overline{u}$, and we will assume (1.2) whenever appropriate.

In Section 5, we will consider a lens with a coating on its front surface that heats up by absorbing part of the solar spectrum. In this case the heating will come from the coating and not from the mounting. Since for simplicity we still want (ii) to hold, we will again average the temperature along the z-direction, so that the heat absorption will give rise to the source term in (1.3).

For completeness we mention that the constant κ in (1.3) is the *thermal conductivity*, which is related to other thermal parameters as follows (for uniqueness the SI units are given in brackets):

$$\begin{gathered}\kappa = \frac{\lambda}{c\rho}: \text{ thermal conductivity } [\mathrm{m}^2\,\mathrm{s}^{-1}],\\ \lambda: \text{ heat conductivity } [\mathrm{W}\,\mathrm{m}^{-1}\mathrm{K}^{-1}],\\ c: \text{specific heat } [\mathrm{J}\,\mathrm{kg}^{-1}\mathrm{K}^{-1}],\\ \rho: \text{ density } [\mathrm{kg}\,\mathrm{m}^{-3}].\end{gathered} \tag{1.5}$$

In general, the total *heat flux* $w(t)$ (measured in Watt) into a domain $G \subset \mathbf{R}^3$, inside which the heat equation (1.3) holds, can be computed as follows:

$$w(t) = \frac{\lambda}{\kappa}\int_G \frac{\partial u}{\partial t} dv \equiv \lambda \int_{\partial G} \vec{\nabla} u \cdot d\vec{\sigma} + \frac{\lambda}{\kappa}\int_G Q\, dv, \tag{1.6}$$

where ∂G denotes the boundary of G, $d\vec{\sigma}$ the infinitesimal surface element orthogonal to ∂G pointing out of G, and dv the infinitesimal volume element. The identity derives from an application of the Gauss integral theorem to the gradient vector field $\vec{\nabla} u$ and using $\vec{\nabla} \cdot \vec{\nabla} u = \Delta u = (\partial u/\partial t - Q)/\kappa$; its physical interpretation is energy conservation. In our special case, G is a cylinder of radius R and height h, on its boundary $\vec{\nabla} u$ vanishes except on the mantle surface, where it points in the direction of the surface normal, and u is radially symmetric, hence (1.6) specializes to

$$w(t) = \frac{\lambda}{\kappa} 2\pi h \int_0^R \frac{\partial u}{\partial t}(r,t) \cdot r\, dr \equiv 2\pi R h \lambda \cdot \frac{\partial u}{\partial r}(R,t) + \frac{\lambda}{\kappa} 2\pi h \int_0^R Q(r,t) \cdot r\, dr. \tag{1.7}$$

The total energy supplied to the lens is

$$E(t) = \int_0^t w(t')dt' \tag{1.8}$$

and it equals the change in internal energy of the lens

$$E(t) = \frac{\lambda}{\kappa}\int_G [u(r,t) - u(r,0)]dv, \tag{1.9}$$

as follows from (1.6) and (1.8) by changing the order of integration. Comparison of (1.7), (1.8) with (1.9) will lead to interesting infinite sums in Sections 2.5, 4.2, and 5.3.

2. Constant temperature of the mounting

2.1. Statement of the problem. We denote the initial temperature of the lens by u_0 and the *temperature of the mounting* by u_m. As explained in Section 1, we may assume (1.2), and the temperature u is determined by the following partial differential equation problem:

$$\frac{\partial u}{\partial t} = \kappa \cdot \Delta u = \kappa \cdot \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right], \tag{2.1a}$$

$$u(r,0) = u_0, \tag{2.1b}$$

$$D \cdot \frac{\partial u}{\partial r}(R,t) + u(R,t) - u_m = 0. \tag{2.1c}$$

The boundary condition (2.1c) is justified below. The quantities κ and λ defined in (1.5) both refer to glass (the lens). The length constant D in the boundary condition is determined by the contact between the stainless steel of the mounting and the glass. In fact, there is some *layer of glue* between these two materials. If the thickness of this layer is d and the heat conductivity of the glue is λ_{glue}, then the heat flux into the lens through the cylinder mantle of area $A = 2\pi Rh$ is

$$\lambda \cdot A \cdot \frac{\partial u}{\partial r}(R,t) = \lambda_{\text{glue}} \cdot A \cdot \frac{u_m - u(R,t)}{d}, \tag{2.2}$$

since the surface normal points in the radial direction, hence we get the boundary condition (2.1c) with

$$D = d \cdot \frac{\lambda}{\lambda_{\text{glue}}}. \tag{2.3}$$

If the thickness d goes to zero we get the extreme case $D = 0$. This case is slightly unphysical, because it implies that the flux of energy into the glass starts with a delta distribution at time $t = 0$. It is nevertheless an interesting limiting case, because it slightly simplifies the formulas.

2.2. Scaling. Before solving (2.1), let us introduce dimensionless quantities in order to simplify the notation and, more importantly, to pin down the relevant parameter combinations. We define

$$\widetilde{r} = \frac{r}{R}, \qquad T = \frac{R^2}{\kappa}, \qquad \widetilde{t} = \frac{t}{T}, \qquad \widetilde{D} = \frac{D}{R} = \frac{d}{R}\frac{\lambda}{\lambda_{\text{glue}}}, \qquad \widetilde{u}(\widetilde{r},\widetilde{t}) = \frac{u(r,t) - u_m}{u_m - u_0}. \tag{2.4}$$

Problem (2.1) then becomes

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}} \frac{\partial \tilde{u}}{\partial \tilde{r}}, \tag{2.5a}$$

$$\tilde{u}(\tilde{r},0) \equiv -1, \tag{2.5b}$$

$$\tilde{D} \cdot \frac{\partial \tilde{u}}{\partial \tilde{r}}(1,\tilde{t}) + \tilde{u}(1,\tilde{t}) = 0. \tag{2.5c}$$

Now *all* the quantities appearing in (2.5) are dimensionless. From (2.4), we see that instead of the five original parameters, κ, D, R, u_0, and u_m in (2.1), only the two combinations $T = R^2/\kappa$ and $\tilde{D} = D/R$ matter. The subtraction of u_m in the last entry of (2.4) makes the boundary condition (2.5c) homogeneous, which is necessary for the separation method used in the next section to work.

2.3. Analytic solution. The solution of the dimensionless problem (2.5) is given by the *Fourier-Bessel series*

$$\tilde{u}(\tilde{r},\tilde{t}) = \sum_{j=1}^{\infty} \frac{-2e^{-k_j^2 \tilde{t}}}{(\tilde{D}^2 k_j^2 + 1) \cdot k_j J_1(k_j)} J_0(k_j \tilde{r}) \tag{2.6}$$

with the constants $0 < k_1 < k_2 < \cdots$ determined by the transcendental equation

$$f(x) := J_0(x) + \tilde{D} x J_0'(x) \equiv J_0(x) - \tilde{D} x J_1(x) = 0. \tag{2.7}$$

For a proof, see Appendix A. The special case $\tilde{D} = 0$ is already known from [2, equation (12)].

The solution to the original problem (2.1) in physical units is obtained by using the replacements (2.4). For the convergence properties of Fourier-Bessel series we refer the reader to [7, Section 18]. The fast convergence of the series (2.6) is due to the following fact.

Proposition 2.1. *Equation (2.7) has only simple roots and in fact has infinitely many roots, satisfying*

$$\lim_{j \to \infty} k_{j+1} - k_j = \pi. \tag{2.8}$$

For a proof see Appendix A. For later use we note that from (2.7) and the well-known identities $J_1(x) + xJ_1'(x) = xJ_0(x)$ and $J_0'(x) = -J_1(x)$, see, for example, [8, equation (26.3)] and [9, Section 9.1.28], respectively, we get

$$f'(x) = J_0'(x) - \tilde{D}\big(J_1(x) + xJ_1'(x)\big) = J_0'(x) - \tilde{D} x J_0(x) = -J_1(x) - \tilde{D} x J_0(x). \tag{2.9}$$

2.4. Numerical considerations and examples. For the limiting case $d = \tilde{D} = 0$ (no glue), the constants k_j are just the positive zeros of J_0. They are tabulated, for example, in [9, Table 9.5]. For the *case* $d > 0$, hence $\tilde{D} > 0$, the k_j must be computed from (2.7). This

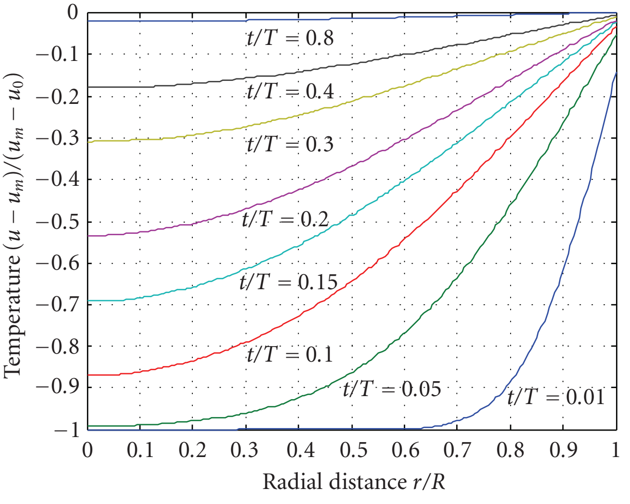

Figure 2.1. Solution of problem (2.1) for $\tilde{D} = 0.027$. That the temperature u at the boundary $r = R$ is below u_m and grows with time is due to the glue layer, see (2.1c).

can be done by Newton iteration. From (2.7) and (2.9) we have

$$-\frac{f(x)}{f'(x)} = \frac{J_0(x) - \tilde{D}xJ_1(x)}{J_1(x) + \tilde{D}xJ_0(x)}. \tag{2.10}$$

The *Newton iteration* is thus

$$k_j^{(i+1)} = k_j^{(i)} + \frac{J_0(k_j^{(i)}) - \tilde{D}k_j^{(i)}J_1(k_j^{(i)})}{J_1(k_j^{(i)}) + \tilde{D}k_j^{(i)}J_0(k_j^{(i)})} \quad (i = 0, 1, 2, \dots). \tag{2.11}$$

Usually $\tilde{D} \ll 1$, then the solutions k_j of (2.7) are close to the zeros of J_0, and one can use those as starting values for the iteration. In the general case we just sample the function f and search for sign changes. A uniform sampling grid of 3000 points in the interval $[0, 65]$ is sufficient to provide reliable starting values for $k_1, \dots, k_{20}$. A few iterations of (2.11) suffice to compute the k_j accurately. Numerical experiments show that for $\tilde{D} \le 0.02$ and $j = 1, \dots, 20$, four iterations yield residuals less than 10^{-15}.

With the k_j computed as described above, it is easy to compute the solution $\tilde{u}(\tilde{r}, \tilde{t})$ according to (2.6). Note that the smaller $\tilde{t}$, the more terms are needed in the series. Because the k_j are asymptotically equidistant, (2.8), the terms decrease exponentially. Moreover, numerical experiments show that the series is essentially (but not exactly) alternating. The last term used is thus an approximate upper bound on the truncation error, except near $\tilde{r} = 1$. Figure 2.1 shows some results for a typical value of $\tilde{D}$.

Example 2.2. If we want the temperature difference from the center of the lens to its boundary to decrease to about 18% of its original value, then we must wait the time

$t = 0.4 \cdot T$, which in a realistic situation can be about 1 hour. After half the time the temperature difference is only reduced to 53% of the original value.

2.5. Energy conservation and a sum formula. From (1.7) with $Q = 0$ the total heat flux is

$$w(t) = \frac{\lambda}{\kappa} 2\pi h \int_0^R \frac{\partial u}{\partial t}(r,t) \cdot r\, dr \equiv 2\pi R h \lambda \cdot \frac{\partial u}{\partial r}(R,t). \tag{2.12}$$

Using the scaling formulae (2.4) and

$$W := 2\pi h \lambda (u_m - u_0), \qquad \widetilde{w}(\widetilde{t}) := \frac{w(t)}{W}, \tag{2.13}$$

we get the dimensionless version of (2.12)

$$\widetilde{w}(\widetilde{t}) = \int_0^1 \frac{\partial \widetilde{u}}{\partial \widetilde{t}}(\widetilde{r},\widetilde{t}) \cdot \widetilde{r}\, d\widetilde{r} \equiv \frac{\partial \widetilde{u}}{\partial \widetilde{r}}(1,\widetilde{t}). \tag{2.14}$$

Using either form of $\widetilde{w}(\widetilde{t})$, we get from (2.6)

$$\widetilde{w}(\widetilde{t}) = \sum_{j=1}^{\infty} \frac{2e^{-k_j^2 \widetilde{t}}}{(\widetilde{D}^2 k_j^2 + 1)}. \tag{2.15}$$

As one would expect, the total heat energy $E(t)$ supplied to the lens up to time t remains bounded

$$\begin{aligned} E(t) &= WT \int_0^{t/T} \widetilde{w}(\widetilde{t})\, d\widetilde{t} \\ &= WT \sum_{j=1}^{\infty} \frac{2}{(\widetilde{D}^2 k_j^2 + 1) k_j^2} \left(1 - e^{-k_j^2 t/T}\right) \xrightarrow[t\to\infty]{} WT \sum_{j=1}^{\infty} \frac{2}{(\widetilde{D}^2 k_j^2 + 1) k_j^2}. \end{aligned} \tag{2.16}$$

Using (2.1b) and $\lim_{t\to\infty} u(r,t) = u_m$ (see (2.6) and (2.4)) we get from (1.9)

$$E(t) \xrightarrow[t\to\infty]{} = \frac{\lambda}{\kappa}(u_m - u_0) \cdot \pi R^2 h = \frac{WT}{2}. \tag{2.17}$$

Comparison with (2.16) yields the *sum formula*

$$\sum_{j=1}^{\infty} \frac{1}{(\widetilde{D}^2 k_j^2 + 1) k_j^2} = \frac{1}{4}. \tag{2.18}$$

The only reference to a similar formula we have found is [8, Example (27), page 134]

$$\sum_{j=1}^{\infty} \frac{1}{\alpha_j^2} = \frac{1}{4(\nu + 1)}, \tag{2.19}$$

where $0 < \alpha_1 < \alpha_2 < \cdots$ are the positive zeroes of J_ν, $\nu > -1$. The special case $\nu = 0$ of (2.19) agrees with the special case $\widetilde{D} = 0$ of (2.18).

3. Prescribing heat flux instead of temperature

3.1. Problem statement. Since any high-quality optical system consists of lenses of different size and thickness, the heating times computed in Section 2 will vary quite a lot among the lenses. It is therefore desirable to have an individual heat source for each lens, which can then be adjusted to the size of the lens. This just changes the boundary condition in our heat conduction problem.

We make the simplifying assumptions (i), (ii), (iii), and (iv′) from Section 1. As explained there, we expect a radially symmetric solution of the form $u = u(r,t)$ $(0 \le r \le R,\ t \ge 0)$. We want to turn on the heat flux from time 0 to some time $t_0 > 0$. Denoting by q [unit: $\mathrm{W/m^2}$] the constant *heat flux density* at the interface between the glass and the glue, applied from time 0 to t_0, we now get the following problem:

$$\frac{\partial u}{\partial t} = \kappa \cdot \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right], \tag{3.1a}$$

$$u(r,0) = u_0, \tag{3.1b}$$

$$\lambda \cdot \frac{\partial u}{\partial r}(R,t) = q \cdot H(t_0 - t), \tag{3.1c}$$

where $H(t)$ is the Heaviside function (0 for $t < 0$, 1 for $t \ge 0$).

3.2. Scaling. As in Section 2.2, we introduce dimensionless quantities. This time the natural temperature scale of the problem is qR/λ. We define

$$\widetilde{r} = \frac{r}{R}, \quad T = \frac{R^2}{\kappa}, \qquad \widetilde{t} = \frac{t}{T}, \qquad \widetilde{t}_0 = \frac{t_0}{T}, \qquad U = \frac{qR}{\lambda}, \qquad \widetilde{u}(\widetilde{r},\widetilde{t}) = \frac{u(r,t) - u_0}{U}. \tag{3.2}$$

The problem (3.1) is thus equivalent to

$$\frac{\partial \widetilde{u}}{\partial \widetilde{t}} = \frac{\partial^2 \widetilde{u}}{\partial \widetilde{r}^2} + \frac{1}{\widetilde{r}}\frac{\partial \widetilde{u}}{\partial \widetilde{r}}, \tag{3.3a}$$

$$\widetilde{u}(\widetilde{r},0) = 0, \tag{3.3b}$$

$$\frac{\partial \widetilde{u}}{\partial \widetilde{r}}(1,\widetilde{t}) = H(\widetilde{t}_0 - \widetilde{t}). \tag{3.3c}$$

All the quantities in (3.3) are dimensionless. Note that instead of the six original parameters κ, λ, R, u_0, q, and t_0, we now have three relevant physical parameters: $T = R^2/\kappa$, $U = qR/\lambda$, and t_0.

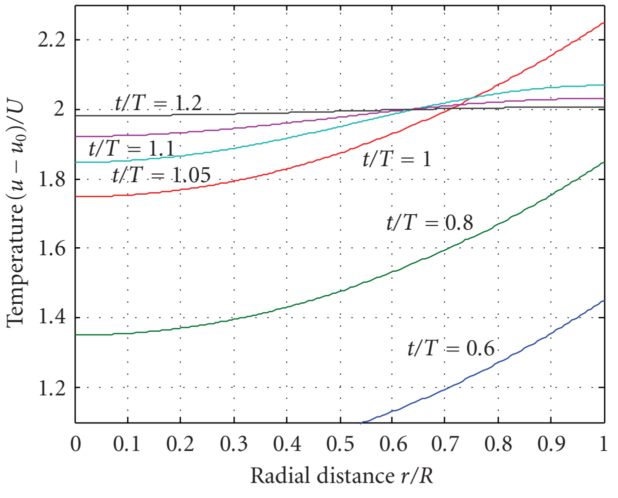

Figure 3.1. Solution of problem (3.1) for $t_0 = T$. After turning the heat flux off at time $t = T$, heat diffusion takes over.

3.3. Analytic solution. As shown in Appendix B, the solution of problem (3.3) is given by

$$\tilde{u}(\tilde{r},\tilde{t}) = \begin{cases} \dfrac{\tilde{r}^2}{2} + 2\tilde{t} - \dfrac{1}{4} - \displaystyle\sum_{j=1}^{\infty} \frac{2e^{-k_j^2\tilde{t}}}{k_j^2 J_0(k_j)} J_0(k_j\tilde{r}), & 0 \le \tilde{t} < \tilde{t}_0 \\ 2\tilde{t}_0 + \displaystyle\sum_{j=1}^{\infty} \frac{2}{k_j^2 J_0(k_j)} \left(1 - e^{-k_j^2\tilde{t}_0}\right) e^{-k_j^2(\tilde{t}-\tilde{t}_0)} J_0(k_j\tilde{r}), & \tilde{t} > \tilde{t}_0, \end{cases} \tag{3.4}$$

where the k_j are the positive zeros of the Bessel function J_1 in ascending order. For the solution to the original problem (3.1) in physical units, use the replacements (3.2).

The *energy* flowing into the lens is easily computed: from (1.7) with $Q = 0$ the heat flux is

$$w(t) = 2\pi Rh\lambda \frac{\partial u}{\partial r}(R,t) = 2\pi Rhq \cdot H(t_0 - t), \tag{3.5}$$

hence

$$E(t) = \int_0^t w(t')dt' = 2\pi Rhq \cdot \min\{t, t_0\} \quad (t \ge 0). \tag{3.6}$$

3.4. Numerical results. Computing the solution $\tilde{u}$ according to (3.4) yields the sample results shown in Figure 3.1. After the heat flux is turned off at time $t = T$, the temperature distribution gets more and more uniform.

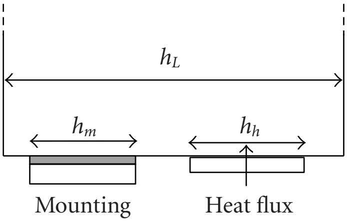

Figure 4.1. Boundary of the lens. The optical axis is horizontal and the glue is shown in gray.

4. Heating only part of the cylinder mantle

4.1. Statement of the problem and scaling. Here is a more general problem combining the two previous ones: the lens is still a cylinder of radius R and height h_L. Part of the cylinder mantle, a ring of width h_m, is kept at a constant temperature u_m by the mounting, while another part of the cylinder mantle, a ring of width h_h, is heated by delivering a constant heat flux density q for a time t_0 and then the heating is turned off, that is, this part of the mantle surface is then isolated. The remaining mantle surface (of width $h_L - (h_m + h_h)$) and all other surfaces of the lens are thermally isolated. Figure 4.1 shows a sketch of the situation.

To simplify the problem and hence allow for an analytic solution we *disregard* the variation of temperature along the axis direction, that is, we replace the temperature by its *average* along the z-coordinate, $u = u(r,t)$, $0 \le r \le R$, $t \ge 0$. This average temperature then satisfies

$$\frac{\partial u}{\partial t} = \kappa \cdot \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right] \tag{4.1a}$$

$$u(r,0) = u_0. \tag{4.1b}$$

For the boundary condition at $r = R$, we consider the energy balance through the surface $A = 2\pi R h_L$ of the cylinder mantle:

$$q \cdot H(t_0 - t) \cdot \left(A\frac{h_h}{h_L}\right) + \lambda_{\text{glue}}\left(A\frac{h_m}{h_L}\right) \cdot \frac{u_m - u(R,t)}{d} = \lambda \cdot A \cdot \frac{\partial u}{\partial r}(R,t), \tag{4.2}$$

where d denotes the thickness of the glue layer and H is the Heaviside function. Rewriting yields

$$d\frac{\lambda}{\lambda_{\text{glue}}}\frac{h_L}{h_m} \cdot \frac{\partial u}{\partial r}(R,t) + u(R,t) - \left(u_m + H(t_0 - t) \cdot q\frac{h_h}{h_m}\frac{d}{\lambda_{\text{glue}}}\right) = 0. \tag{4.1c}$$

For $t \le t_0$, our problem (4.1a), (4.1b), and (4.1c) reduces to problem (2.1) with

$$D := d\frac{\lambda}{\lambda_{\text{glue}}}\frac{h_L}{h_m} \tag{4.3}$$

and with u_m replaced by the "effective temperature"

$$u_e := u_m + q\frac{h_h}{h_m}\frac{d}{\lambda_{\text{glue}}}. \tag{4.4}$$

With the scaling

$$\begin{gathered}\tilde{r} = \frac{r}{R}, \qquad T = \frac{R^2}{\kappa}, \qquad \tilde{t} = \frac{t}{T}, \qquad \tilde{t}_0 = \frac{t_0}{T}, \qquad \tilde{D} = \frac{D}{R} = \frac{d}{R}\frac{\lambda}{\lambda_{\text{glue}}}\frac{h_L}{h_m}, \\ \tilde{u}(\tilde{r},\tilde{t}) = \frac{u(r,t) - u_m}{u_e - u_0}, \qquad \gamma = \frac{u_e - u_m}{u_e - u_0}\end{gathered} \tag{4.5}$$

problem (4.1a), (4.1b), and (4.1c) transforms into its dimensionless form

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{u}}{\partial \tilde{r}}, \tag{4.6a}$$

$$\tilde{u}(\tilde{r},0) \equiv \gamma - 1, \tag{4.6b}$$

$$\tilde{D} \cdot \frac{\partial \tilde{u}}{\partial \tilde{r}}(1,\tilde{t}) + \tilde{u}(1,\tilde{t}) = \gamma \cdot H(\tilde{t}_0 - \tilde{t}). \tag{4.6c}$$

For later use we note that the last equation in (4.5) is equivalent to each of

$$1 - \gamma = \frac{u_m - u_0}{u_e - u_0}, \qquad u_e - u_m = \frac{\gamma}{1-\gamma}(u_m - u_0). \tag{4.7}$$

4.2. Analytic solution. By Appendix C, the solution of the dimensionless problem (4.6a)–(4.6c) is given by

$$\tilde{u}(\tilde{r},\tilde{t}) = \begin{cases} \gamma - \displaystyle\sum_{j=1}^{\infty} \frac{2e^{-k_j^2\tilde{t}}}{(\tilde{D}^2k_j^2+1)\cdot k_jJ_1(k_j)}J_0(k_j\tilde{r}), & 0 \le \tilde{t} \le \tilde{t}_0 \\ \displaystyle\sum_{j=1}^{\infty} \frac{2}{(\tilde{D}^2k_j^2+1)\cdot k_jJ_1(k_j)} \cdot \left[\gamma e^{-k_j^2(\tilde{t}-\tilde{t}_0)} - e^{-k_j^2\tilde{t}}\right] \cdot J_0(k_j\tilde{r}), & \tilde{t} \ge \tilde{t}_0 \end{cases} \tag{4.8}$$

with $0 < k_1 < k_2 < \cdots$ the roots of (2.7). For the solution of the original problem (4.1a), (4.1b), and (4.1c) in physical units use the scaling (4.5) and the definition (4.4).

Note that for $q = 0$, we have $\gamma = 0$ and (4.8) reproduces (2.6) as a special case (with $\tilde{D}$ as in (2.4) if $h_m = h_L$). The analytic solution further shows that instead of the *twelve* original physical parameters of the problem, only the *four* parameters T, $\tilde{t}_0$, $\tilde{D}$, and γ really matter.

Further note that the jump discontinuity in the boundary condition (4.6c) necessarily leads to a corresponding discontinuity of the boundary temperature $\tilde{u}(1,\tilde{t})$ at $\tilde{t} = \tilde{t}_0$, but there is of course no discontinuity for $\tilde{r} < 1$.

We invite the reader to do the *energy* computation. From (1.7), (1.8), (4.5), (4.8) one gets

$$E(t)=\begin{cases} WT\sum_{j=1}^{\infty}\frac{2}{(\widetilde{D}^2k_j^2+1)k_j^2}\left[1-e^{-k_j^2t/T}\right], & 0\le t\le t_0,\\ WT\sum_{j=1}^{\infty}\frac{2}{(\widetilde{D}^2k_j^2+1)k_j^2}\left[\gamma e^{-k_j^2(t-t_0)/T}+1-\gamma-e^{-k_j^2t/T}\right], & t\ge t_0\end{cases} \tag{4.9}$$

with $W:=2\pi h\lambda(u_e-u_0)$. From (1.9) and $\lim_{t\to\infty}u(r,t)=u_m$ (see (4.8) and (4.5)) we get

$$E(t)=\frac{\lambda}{\kappa}2\pi h\int_0^R[u(r,t)-u_0]r\,dr\xrightarrow[t\to\infty]{}\frac{\lambda}{\kappa}2\pi h\frac{R^2}{2}(u_m-u_0)=\frac{WT}{2}(1-\gamma), \tag{4.10}$$

and comparison with (4.9) again yields the sum formula (2.18).

4.3. Numerical results. Again, using the Newton iteration (2.11) for the solution of (2.7), it is easy to compute the solution $\tilde{u}$ according to (4.8). Some results are displayed in Figure 4.2, with the series truncated after the first 20 terms. The *Gibbs phenomenon* shown is due to the truncation in the identity

$$\sum_{j=1}^{\infty}\frac{2}{(\widetilde{D}^2k_j^2+1)\cdot k_jJ_1(k_j)}J_0(k_j\tilde{r})\equiv 1, \tag{4.11}$$

which follows from (C.5) and (C.6). It only appears when $t-t_0$ is positive and small. For larger t it disappears due to the exponential decay of the terms in (4.8). In [7, Section 18.12] it is shown that (4.11) (in fact a more general identity) holds for $0\le\tilde{r}\le 1$ if $\widetilde{D}>0$ (the variable H used there equals $1/\widetilde{D}$).

Applying the heat flux q according to (4.4) makes the temperature u approach u_m much faster. In the example of Figure 4.2, at $t=T/4$ we have (using (4.7))

$$\left|u\left(0,\frac{T}{4}\right)-u_m\right|=0.0515\cdot|u_e-u_0|=0.0515\cdot\frac{|u_m-u_0|}{1-\gamma}=0.103\cdot|u_m-u_0|. \tag{4.12}$$

Without the heat flux, we get $|u(0,T/4)-u_m|=0.388\cdot|u_m-u_0|$.

5. Heating through absorption in the coating

5.1. Statement of the problem and scaling. The present problem concerns the receiver optics of a laser altimeter to be used by the European Space Agency for the exploration of the planet Mercury. We still model each lens as a cylinder of radius R and height h. The mounting, which is attached to the cylinder mantle by a glue layer of thickness d, is kept at a constant temperature u_0. But now one of the optical surfaces is coated in order to absorb unwanted parts of the solar spectrum, which implies that this surface is heated up. We model this effect by a constant and homogeneous heat flux density q [unit: W/m^2],

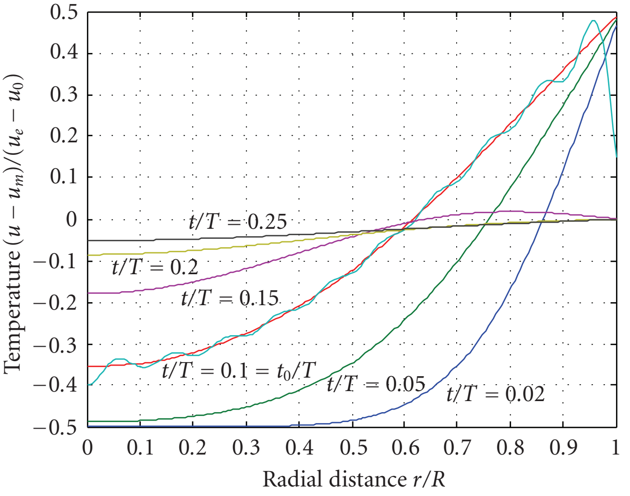

Figure 4.2. Solution to problem (4.1) for $t_0/T = 0.1$, $\gamma = 0.5$, and $\widetilde{D} = 0.01$. The wiggly line is for $t/T = 0.1001$ and shows the Gibbs phenomenon with 20 terms in the sum (4.8).

which is turned on for a time t_0 and then is turned off. The other optical surface of the lens is thermally isolated.

The problem is rotationally symmetric around the cylinder axis. To simplify it further, we again replace the temperature by its *average* along the z-coordinate, $u = u(r,t)$, $0 \le r \le R$, $t \ge 0$. The heat equation now contains a source term proportional to q,

$$\frac{\partial u}{\partial t} = \kappa \cdot \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{q}{\lambda h}H(t_0 - t)\right], \tag{5.1a}$$

where H is the Heaviside function. The initial and boundary conditions are

$$u(r,0) = u_0, \tag{5.1b}$$

$$D \cdot \frac{\partial u}{\partial r}(R,t) + u(R,t) = u_0 \tag{5.1c}$$

with

$$D = d \cdot \frac{\lambda}{\lambda_{\text{glue}}} \tag{5.2}$$

as in Section 2.1. With the scaling

$$\begin{gathered} \tilde{r} = \frac{r}{R}, \qquad T = \frac{R^2}{\kappa}, \qquad \tilde{t} = \frac{t}{T}, \qquad \tilde{t}_0 = \frac{t_0}{T}, \\ \tilde{u}(\tilde{r},\tilde{t}) = \frac{u(r,t) - u_0}{U}, \qquad U = \frac{qR}{\lambda}, \qquad \tilde{q} = \frac{R}{h}, \qquad \widetilde{D} = \frac{D}{R}, \end{gathered} \tag{5.3}$$

problem (5.1a)–(5.1c) transforms into its dimensionless form

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}} \frac{\partial \tilde{u}}{\partial \tilde{r}} + \tilde{q} \cdot H(\tilde{t}_0 - \tilde{t}), \tag{5.4a}$$

$$\tilde{u}(\tilde{r}, 0) = 0, \tag{5.4b}$$

$$\tilde{D} \cdot \frac{\partial \tilde{u}}{\partial \tilde{r}}(1, \tilde{t}) + \tilde{u}(1, \tilde{t}) = 0. \tag{5.4c}$$

Before solving these equations, we first consider the stationary state.

5.2. Stationary case. If the heating is never turned off ($\tilde{t}_0 = \infty$), then for large times the temperature will approach a stationary distribution $\tilde{u}(\tilde{r}, \infty) =: \tilde{u}(\tilde{r})$, which will give an *upper bound* on the temperature and which will be used in the derivation of the general case, see (D.1). From (5.4a), we have for the stationary case

$$\tilde{u}''(\tilde{r}) + \frac{\tilde{u}'(\tilde{r})}{\tilde{r}} + \tilde{q} = 0 \tag{5.5}$$

with the general solution

$$\tilde{u}(\tilde{r}) = -\frac{\tilde{q}}{4}\tilde{r}^2 + c_1 \cdot \log \tilde{r} + c_2. \tag{5.6}$$

The fact that $\tilde{u}(0)$ is finite implies $c_1 = 0$ and by (5.4c), $\tilde{D} \cdot \tilde{u}'(1) + \tilde{u}(1) = 0$, hence $c_2 = (\tilde{q}/4)(1 + 2\tilde{D})$, thus

$$\tilde{u}(\tilde{r}) = \frac{\tilde{q}}{4}(1 + 2\tilde{D} - \tilde{r}^2). \tag{5.7}$$

In the original physical parameters the stationary solution is, using (5.3),

$$u(r) = u_0 + \frac{q}{4\lambda h}(R^2 + 2DR - r^2) = u_0 + \frac{P}{4\pi\lambda h}\left(1 + 2\frac{D}{R} - \frac{r^2}{R^2}\right), \tag{5.8}$$

where

$$P = \pi R^2 q \tag{5.9}$$

is the total heating power.

5.3. Analytic solution for the general case. The solution of dimensionless problem (5.4a), (5.4b), and (5.4c) is shown in Appendix D to be

$$\tilde{u}(\tilde{r}, \tilde{t}) = \begin{cases} \dfrac{\tilde{q}}{4}(1 + 2\tilde{D} - \tilde{r}^2) - \displaystyle\sum_{j=1}^{\infty} \frac{2\tilde{q}e^{-k_j^2 \tilde{t}}}{(\tilde{D}^2 k_j^2 + 1) k_j^3 J_1(k_j)} J_0(k_j \tilde{r}), & 0 \le \tilde{t} \le \tilde{t}_0 \\ \displaystyle\sum_{j=1}^{\infty} \frac{2\tilde{q}}{(\tilde{D}^2 k_j^2 + 1) k_j^3 J_1(k_j)} \cdot \left[e^{-k_j^2(\tilde{t} - \tilde{t}_0)} - e^{-k_j^2 \tilde{t}}\right] \cdot J_0(k_j \tilde{r}), & \tilde{t} \ge \tilde{t}_0 \end{cases} \tag{5.10}$$

with k_j the positive roots of (2.7) in ascending order. For the solution of the original problem (5.1a)–(5.1c) in physical units use the scaling (5.3) and the definition (5.2). The analytic solution shows that from the *nine* original physical parameters of the problem, κ, q, λ, h, R, t_0, u_0, d, λ_{glue}, only the *five* combinations $T = R^2/\kappa$, $U = qR/\lambda$, $\widetilde{q} = R/h$, $\widetilde{t}_0 = t_0/T$, $\widetilde{D} = D/R = d/R \cdot \lambda/\lambda_{\text{glue}}$ really matter.

The *energy* computation will now give us another sum formula. From (1.7) with $Q = \kappa(q/\lambda h)H(t_0 - t)$, we get the heat flux

$$w(t) = 2\pi Rh\frac{\partial u}{\partial r}(R,t) + \pi R^2 q \cdot H(t_0 - t), \tag{5.11}$$

hence (5.10), (5.9), (5.3), and integration yield the energy

$$E(t) = \begin{cases} 4PT \cdot \displaystyle\sum_{j=1}^{\infty} \frac{1 - e^{-k_j^2 t/T}}{(\widetilde{D}^2 k_j^2 + 1)k_j^4}, & 0 \le t \le t_0, \\ 4PT \cdot \displaystyle\sum_{j=1}^{\infty} \frac{e^{-k_j^2 (t-t_0)/T} - e^{-k_j^2 t/T}}{(\widetilde{D}^2 k_j^2 + 1)k_j^4}, & t > t_0. \end{cases} \tag{5.12}$$

For finite t_0, we have $\lim_{t\to\infty} u(r,t) = u_0$ and $\lim_{t\to\infty} E(t) = 0$ in accord with (1.9). However, if $t_0 = \infty$, then

$$\lim_{t\to\infty} u(r,t) = u_0 + \frac{\widetilde{q}}{4}U(1 + 2\widetilde{D} - \widetilde{r}^2) = u_0 + \frac{P}{4\pi h\lambda}\left(1 + 2\widetilde{D} - \frac{r^2}{R^2}\right), \tag{5.13}$$

$$\lim_{t\to\infty} E(t) = 4PT \cdot \sum_{j=1}^{\infty} \frac{1}{(\widetilde{D}^2 k_j^2 + 1)k_j^4}. \tag{5.14}$$

Comparison with (1.9),

$$E(t) = \frac{\lambda}{\kappa}\int_G [u(r,t) - u_0]dv \xrightarrow[t\to\infty]{} \frac{P}{2\kappa}\int_0^R \left(1 + 2\widetilde{D} - \frac{r^2}{R^2}\right) r\,dr = \frac{PT}{2}\left(\widetilde{D} + \frac{1}{4}\right) \tag{5.15}$$

yields the *sum formula*

$$\sum_{j=1}^{\infty} \frac{1}{(\widetilde{D}^2 k_j^2 + 1)k_j^4} = \frac{\widetilde{D} + 1/4}{8}. \tag{5.16}$$

We could not find this formula in the standard references on Bessel functions, the closest match is [8, Example (27), page 134]

$$\sum_{j=1}^{\infty} \frac{1}{\alpha_j^4} = \frac{1}{16(\nu+1)^2(\nu+2)}, \tag{5.17}$$

where $0 < \alpha_1 < \alpha_2 < \cdots$ are the positive zeroes of J_ν, $\nu > -1$. The special case $\nu = 0$ of (5.17) agrees with the special case $\widetilde{D} = 0$ of (5.16).

Open question. We have assumed a constant thickness of the lens. In Section 5.5, we will treat the stationary solution for a varying thickness profile. For which "physically reasonable" thickness profiles can the general (nonstationary) case be solved analytically?

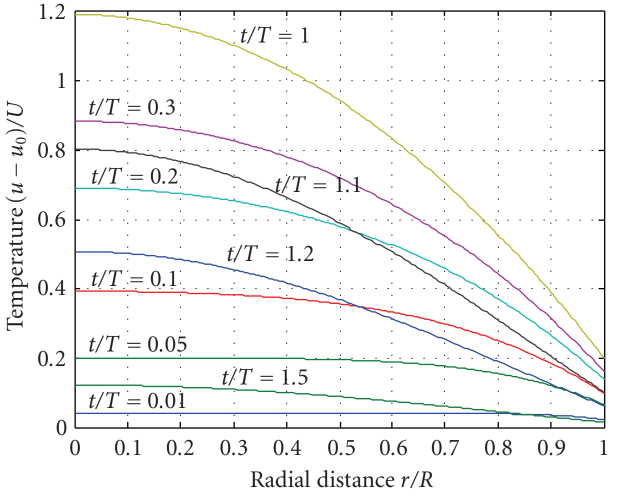

Figure 5.1. Solution to problem (5.1) for $\tilde{t}_0 \equiv t_0/T = 1$, $\tilde{q} \equiv R/h = 4$, and $\tilde{D} \equiv D/R = 0.1$. At every point the temperature u grows until $t = t_0$, then decreases again towards the mounting temperature u_0.

5.4. Numerical results. To compute the solution $\tilde{u}$ according to (5.10), we again use the Newton iteration (2.11) to solve (2.7). Some exemplary results for $\tilde{t}_0 = 1$, $\tilde{q} = 4$, $\tilde{D} = 0.1$ are displayed in Figure 5.1, with the series truncated after the first 20 terms. For $0 \le \tilde{t} \le \tilde{t}_0 = 1$ the lens heats up, reaching a center temperature of $\tilde{u}(0,1) = 1.1886$ at $\tilde{t} = 1$, which is already close to the upper bound $\tilde{q}/4 + 2\tilde{D} = 1.2$ given by the stationary solution (5.7). After this point the heating is turned off and the lens quickly cools down, the center temperature $\tilde{u}(0,\tilde{t})$ reaches 0.1217 already at $\tilde{t} = 1.5$.

5.5. Stationary solution for a lens of variable thickness. So far we have assumed that the lens has constant thickness, in order to allow for an analytic solution of the heat equation. Here we dismiss this assumption and try to generalize the results for the stationary case to a real lens. The thickness profile of a spherical biconvex lens is

$$h(r) = h_0 - r_1 - r_2 + \sqrt{r_1^2 - r^2} + \sqrt{r_2^2 - r^2}, \tag{5.18}$$

where h_0 is the central thickness and r_1 and r_2 are the radii of the lens surfaces. Our problem is still given by (5.1a)–(5.1c), but h now depends on r. The dimensionless form of (5.18) is

$$\tilde{h}(\tilde{r}) := \frac{h(R\tilde{r})}{R} = \tilde{h}_0 - \tilde{r}_1 - \tilde{r}_2 + \sqrt{\tilde{r}_1^2 - \tilde{r}^2} + \sqrt{\tilde{r}_2^2 - \tilde{r}^2}, \quad \tilde{h}_0 = \frac{h_0}{R}, \qquad \tilde{r}_i = \frac{r_i}{R} \quad (i = 1,2). \tag{5.19}$$

Dimensionless equations (5.4a), (5.4b), and (5.4c) still hold if we generalize $\tilde{q}$ to the function

$$\tilde{q} \equiv \tilde{q}(\tilde{r}) := \frac{1}{\tilde{h}(\tilde{r})} = \frac{R}{h(R\tilde{r})}. \tag{5.20}$$

The ordinary differential equation for the stationary case is thus

$$\tilde{u}''(\tilde{r}) + \frac{\tilde{u}'(\tilde{r})}{\tilde{r}} + \tilde{q}(\tilde{r}) = 0 \tag{5.21}$$

with the boundary condition

$$\tilde{D} \cdot \tilde{u}'(1) + \tilde{u}(1) = 0. \tag{5.22}$$

In Appendix D, it will be shown that the solution of (5.21), (5.22), the stationary solution for an arbitrary lens profile, is

$$\tilde{u}(\tilde{r}) = \tilde{D} \cdot g(1) - \int_1^{\tilde{r}} \frac{g(x)}{x} dx, \quad g(x) := \int_0^x x' \tilde{q}(x') dx'. \tag{5.23}$$

Trying to *symbolically* perform the integration in the definition of g with the Matlab "Symbolic Math Toolbox" did work, but a pretty print of the result fills many screens, being virtually useless. Even for the special case $\tilde{r}_1 = \tilde{r}_2$ it fills a screen and the other integration in (5.23) cannot be done symbolically. Of course we can easily perform the integrations in (5.23) *numerically*. However, it is easy to *approximate* (5.19) by a simpler function so that the integrations can be reduced to known special functions.

The quadratic polynomial

$$\tilde{h}_{\text{approx}}(\tilde{r}) = \tilde{h}_0 (1 - c\tilde{r}^2) \tag{5.24}$$

is a good approximation to (5.19) for a suitable choice of the parameter c.

Example 5.1. For $\tilde{r}_1 = \tilde{r}_2 = 10$, $\tilde{h}_0 = 0.2$, the choice $c = 0.5010403$ yields a maximal approximation error (for $0 \le \tilde{r} \le 1$) of $4.3 \cdot 10^{-5}$, which is negligible for practical purposes. This also works when $\tilde{r}_1 \ne \tilde{r}_2$. For an asymmetric lens with $\tilde{r}_1 = 8$, $\tilde{r}_2 = 13$, $\tilde{h}_0 = 0.2$ (of similar optical power (5.33)), the choice $c = 0.506063$ yields a maximal approximation error of $5.2 \cdot 10^{-5}$.

For our approximation (5.24), we get from (5.20) and (5.23)

$$g(x) = \frac{1}{\tilde{h}_0} \int_0^x \frac{x'}{1 - cx'^2} dx' = -\frac{1}{2\tilde{h}_0 c} \log(1 - cx^2), \tag{5.25}$$

$$\begin{aligned} \tilde{u}(\tilde{r}) &= \frac{1}{2\tilde{h}_0 c} \int_1^{\tilde{r}} \frac{\log(1 - cx^2)}{x} dx - \frac{\tilde{D}}{2\tilde{h}_0 c} \log(1 - c) \\ &= \frac{1}{4\tilde{h}_0 c} [\mathrm{Li}_2(c) - \mathrm{Li}_2(c\tilde{r}^2)] - \frac{\tilde{D}}{2\tilde{h}_0 c} \log(1 - c), \end{aligned} \tag{5.26}$$

where the *dilogarithm* Li_2 is defined by [10, Section 6.15]

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-x)}{x}\,dx = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \quad (|z| < 1). \tag{5.27}$$

Since any real lens has positive thickness, we may assume $\widetilde{h}_{\mathrm{approx}}(1) > 0$, hence $c < 1$. Because the dilogarithm is not commonly implemented in scientific software, it may be best to just do the integration in (5.26) numerically, or one might use the integral representation [10, Section 6.15.3]

$$\mathrm{Li}_2(z) = z \cdot \int_0^{\infty} \frac{t}{e^t - z}\,dt = z \cdot \int_0^{\pi/2} \frac{\tan s}{e^{\tan s} - z}(1 + \tan^2 s)\,ds, \quad z \notin [1, \infty). \tag{5.28}$$

For small c (weak lens), we get from (5.26) and (5.27)

$$u(\widetilde{r}) = \frac{1}{4\widetilde{h}_0}\left\{1 + 2\widetilde{D} - \widetilde{r}^2 + \frac{c}{4}(1 + 4\widetilde{D} - \widetilde{r}^4) + \frac{c^2}{9}(1 + 6\widetilde{D} - \widetilde{r}^6) + O(c^3)\right\} \tag{5.29}$$

in agreement with (5.7) for $c = 0$. Note that the *optical power* of a thin lens is given by

$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right), \tag{5.30}$$

where f is the focus length and n is the index of refraction of the glass. The second factor in (5.30) is essentially the curvature or the second derivative of $\widetilde{h}$ at the origin (since $\widetilde{h}'(0) = 0$)

$$\frac{1}{\widetilde{r}_1} + \frac{1}{\widetilde{r}_2} = -\widetilde{h}''(0). \tag{5.31}$$

For the approximation (5.24), we have

$$\widetilde{h}''_{\mathrm{approx}}(0) = -2\widetilde{h}_0\, c, \tag{5.32}$$

hence the power of the “approximating lens” is

$$\frac{1}{f} = \frac{2(n-1)}{R}\widetilde{h}_0 c, \tag{5.33}$$

so that (5.29) can be interpreted as an *expansion in powers of the optical power*.

Finally we show the *effect of the lens profile* on the equilibrium temperature. To do this, we use the same spherical lens as in the example above: $\widetilde{r}_1 = \widetilde{r}_2 = 10$, $\widetilde{h}_0 = 0.2$, approximated by (5.24) with $c = 0.5010403$; as before, we set $\widetilde{D} = 0.1$. We compare the equilibrium temperature of the real lens with that for a “flat lens” (constant thickness) of equal volume, as well as with that for a flat lens of equal thickness at the center. Figure 5.2 shows the three temperature curves. They have been computed using the trapezoid rule for the integral in (5.26).

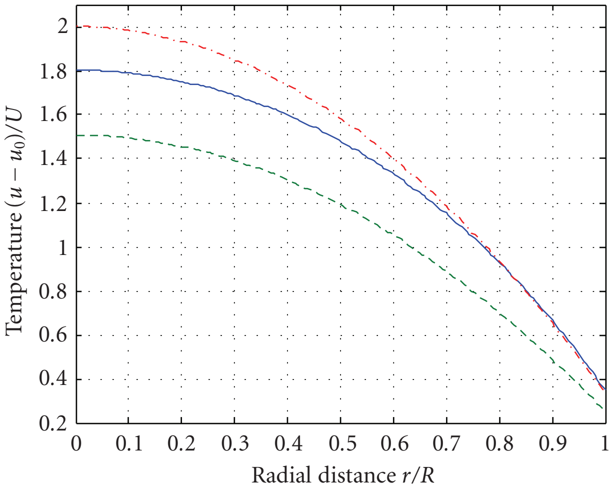

Figure 5.2. Stationary solution to problem (5.1) with $\tilde{D} = 0.1$ for a biconvex spherical lens (solid) and for "flat lenses" with equal central thickness (dashed) and equal volume (dash-dotted), respectively.

As one would expect, compared with the flat lens of equal volume, the spherical lens has slightly higher temperature at the boundary and lower temperature at the center. The flat lens with equal central thickness has much higher volume (33% in our case) and therefore lower temperature everywhere.

6. Conclusions

We have given analytic solutions in terms of Fourier-Bessel series to four idealized heat conduction problems motivated by industrial applications. It is shown how, by introducing dimensionless quantities, the number of relevant parameters can be reduced, so that a single simulation applies to a multiparameter family of physical situations. We have further shown how our formulae, involving the transcendental equation (2.7), can be used numerically. In each case, we have computed the energy $E(t)$ supplied to the lens up to time t. It is shown how energy conservation leads to apparently new sum formulae of mathematical interest.

In one case (Section 5.5) the idealizing assumption was greatly relaxed and still an explicit solution could be given for the stationary case in terms of the dilogarithm Li_2.

These results are an important first step in understanding the practical problem of how to heat up a lens system so that an acceptably homogeneous temperature is reached as quickly as possible.

Appendices

A. Mathematical derivations for Section 2

We first prove the proposition from Section 2.3.

Proof. Assume f had a double root, $f(x) = f'(x) = 0$. Then from (2.7) and (2.9), we get the system

$$\begin{bmatrix} 1 & \widetilde{D}x \\ -\widetilde{D}x & 1 \end{bmatrix} \begin{bmatrix} J_0(x) \\ J_0'(x) \end{bmatrix} = \mathbf{0} \tag{A.1}$$

with determinant $1 + (\widetilde{D}x)^2 > 1$. Hence x would be a double root of J_0, which is impossible.

From the asymptotic formulae for J_0 and J_1 [9, Section 9.2.1], we get

$$f(x) = \begin{cases} -\widetilde{D}\sqrt{\dfrac{2x}{\pi}}\left\{\cos\left(x - \dfrac{3\pi}{4}\right) + O(x^{-1})\right\}, & \widetilde{D} > 0 \\ \sqrt{\dfrac{2}{\pi x}}\left\{\cos\left(x - \dfrac{\pi}{4}\right) + O(x^{-1})\right\}, & \widetilde{D} = 0 \end{cases} \quad (x \longrightarrow +\infty). \tag{A.2}$$

This implies (2.8) and completes the proof. □

We now derive the Fourier-Bessel series (2.6). With the usual separation ansatz

$$\widetilde{u}(\widetilde{r},\widetilde{t}) = y(\widetilde{r}) \cdot g(\widetilde{t}), \tag{A.3}$$

the time dependence can be separated. From (2.5a) and (A.3), it follows

$$\frac{g'}{g} = \frac{y'' + y'/\widetilde{r}}{y} \equiv -a \quad (= \text{constant}), \tag{A.4}$$

because the left-hand side only depends on $\widetilde{t}$ and the right-hand side only depends on $\widetilde{r}$. Hence,

$$g(\widetilde{t}) = e^{-a\widetilde{t}}, \tag{A.5}$$

$$y'' + \frac{y'}{\widetilde{r}} + a \cdot y = 0. \tag{A.6}$$

Due to (A.5), we have $a > 0$. Equation (A.6) is equivalent to the *Bessel equation* [9, Section 9.1.1]

$$x^2 y'' + xy' + (k^2x^2 - n^2)y = 0 \tag{A.7}$$

with $x = \widetilde{r}$, index $n = 0$, and $a = k^2$. The solutions of (A.6) are therefore the Bessel functions $J_0(k\widetilde{r})$ and $Y_0(k\widetilde{r})$, where $k = \sqrt{a}$. The function of the second kind, Y_0, is out of the question, because it is singular at zero, hence

$$y(\widetilde{r}) = J_0(k\widetilde{r}). \tag{A.8}$$

The boundary condition (2.5c) is now

$$\widetilde{D}kJ_0'(k) + J_0(k) = 0. \tag{A.9}$$

Let $0 < k_1 < k_2 < \cdots$ denote the *positive* solutions of (A.9) and hence of (2.7). (Note that with k also $-k$ is a solution, but $J_0(k\tilde{r})$ and $J_0(-k\tilde{r})$ are linearly dependent.) The general solution of (2.5a) and (2.5c) can then be written as the *Fourier-Bessel series* (or *Dini series*)

$$\tilde{u}(\tilde{r},\tilde{t}) = \sum_{j=1}^{\infty} A_j e^{-k_j^2\tilde{t}} J_0(k_j\tilde{r}). \tag{A.10}$$

For the convergence properties of such series we refer the reader to [7, Section 18], where a series defined by (A.10), (A.9) with $\tilde{D} \neq 0$ is called Dini's series of Bessel functions.

The coefficients A_j must be determined from (2.5b),

$$\sum_{j=1}^{\infty} A_j J_0(k_j\tilde{r}) \equiv -1. \tag{A.11}$$

Note that in the case $\tilde{D} = 0$, whatever the coefficients A_j, the sum on the left is zero for $\tilde{r} = 1$, because of (A.9). Hence (A.11) only holds for $\tilde{r} < 1$, and more and more terms are needed for $\tilde{r} \to 1$. This is due to the original problem formulation, when $\tilde{D} = 0$ the initial and boundary conditions (2.5b) and (2.5c) are contradictory for $\tilde{r} = 1$, $\tilde{t} = 0$. As long as $\tilde{r} < 1$, our solution (A.10) is correct even for $\tilde{t} = 0$ (cf. [7, Section 18.12]), but the closer $\tilde{r}$ gets to 1, the more it is difficult to numerically sum the infinite series! For increasing $\tilde{t}$ the problem rapidly disappears due to the exponential decay of the terms in (A.10).

To determine the A_j from (A.11) we can use the following orthogonality relation:

$$\int_0^1 rJ_0(k_i r)J_0(k_j r)\,dr = \frac{\tilde{D}^2k_j^2+1}{2\tilde{D}^2k_j^2} J_0(k_j)^2 \cdot \delta_{i,j} \equiv \frac{\tilde{D}^2k_j^2+1}{2} J_1(k_j)^2 \cdot \delta_{i,j}, \tag{A.12}$$

see for instance [9, Section 11.4.5], [11, Section 3.12.4], or [8, Section 35]. It follows that

$$A_j = \frac{-2\tilde{D}^2k_j^2}{(\tilde{D}^2k_j^2+1)\cdot J_0(k_j)^2} \cdot \int_0^1 rJ_0(k_j r)\,dr \equiv \frac{-2}{(\tilde{D}^2k_j^2+1)\cdot J_1(k_j)^2} \cdot \int_0^1 rJ_0(k_j r)\,dr. \tag{A.13}$$

For the integral, we can use

$$\int_0^1 rJ_0(kr)\,dr = \frac{J_1(k)}{k} \quad (k \neq 0), \tag{A.14}$$

see, for example, [9, Section 11.3.20], [11, Section 3.8.1], or [8, equation (32.3)], and get

$$A_j = \frac{-2\tilde{D}^2k_j}{\tilde{D}^2k_j^2+1} \cdot \frac{J_1(k_j)}{J_0(k_j)^2} \equiv \frac{-2}{\tilde{D}^2k_j^2+1} \cdot \frac{1}{k_jJ_1(k_j)} \equiv \frac{-2\tilde{D}}{\tilde{D}^2k_j^2+1} \cdot \frac{1}{J_0(k_j)}, \tag{A.15}$$

which together with (A.10) implies (2.6).

B. Mathematical derivations for Section 3

We give a derivation of (3.4).

We first consider the case $0 \le \tilde{t} \le \tilde{t}_0$. Before we can use the separation method we must make the boundary condition (3.3c) homogeneous. This can be done by subtracting a special solution of (3.3a) and (3.3c), that is, by setting

$$v(\tilde{r},\tilde{t}) := \tilde{u}(\tilde{r},\tilde{t}) - \frac{\tilde{r}^2}{2} - 2\tilde{t}. \tag{B.1}$$

This transforms the problem (3.3) (for $0 \le \tilde{t} \le \tilde{t}_0$) into

$$\frac{\partial v}{\partial \tilde{t}} = \frac{\partial^2 v}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial v}{\partial \tilde{r}}, \tag{B.2a}$$

$$v(\tilde{r},0) = -\frac{\tilde{r}^2}{2}, \tag{B.2b}$$

$$\frac{\partial v}{\partial \tilde{r}}(1,\tilde{t}) = 0. \tag{B.2c}$$

Using a separation ansatz as in Appendix A, we get

$$v(\tilde{r},\tilde{t}) = \sum_{j=0}^{\infty} A_j e^{-k_j^2\tilde{t}} J_0(k_j\tilde{r}), \tag{B.3}$$

where now by the boundary condition (B.2c) the k_j are the nonnegative zeros of J_1,

$$0 =: k_0 < k_1 < k_2 < \cdots \quad (J_1(k_j) = 0). \tag{B.4}$$

The coefficients A_j must be determined from the initial condition (B.2b)

$$\sum_{j=0}^{\infty} A_j J_0(k_j\tilde{r}) = -\frac{\tilde{r}^2}{2}. \tag{B.5}$$

Note that, similarly as discussed in connection with (A.11), (B.5) can only be satisfied for $\tilde{r} < 1$, and more and more terms are needed for $\tilde{r} \to 1$. Again, this is caused by the initial and boundary conditions (3.3b) and (3.3c), which are contradictory for $\tilde{r} = 1$, $\tilde{t} = 0$. In (B.3), the problem disappears for increasing $\tilde{t}$ due to the exponential decay of the terms.

To solve (B.5) for A_j, we use the orthogonality relation ([8, Section 35] or (A.12) for $\tilde{D} \to \infty$),

$$\int_0^1 rJ_0(k_ir)J_0(k_jr)\,dr = \frac{1}{2}J_0(k_j)^2 \cdot \delta_{i,j}. \tag{B.6}$$

Note that this also holds if one or both of i and j are zero. This follows from $J_0(0) = 1$, (A.14), and $J_1(k_j) = 0$. It follows that

$$A_j = \frac{-1}{J_0(k_j)^2}\int_0^1 r^3 J_0(k_jr)\,dr \quad (j = 0,1,2,\ldots). \tag{B.7}$$

For the integral, we can use the formula

$$\int_0^1 r^3 J_0(kr)\,dr = \frac{2}{k^2}\left[J_0(k) + \left(\frac{k}{2} - \frac{2}{k}\right)J_1(k)\right] \quad (k \neq 0) \tag{B.8}$$

for $j \neq 0$, see, for example, [8, equation (32.4)]. Hence, since in our case $J_1(k_j) = 0$, we get

$$A_j = \frac{-2}{k_j^2 J_0(k_j)} \quad (j = 1,2,\ldots), \qquad A_0 = -\frac{1}{4}. \tag{B.9}$$

Finally, taking together (B.1), (B.3), and (B.9), yields (3.4) for $0 \le \tilde{t} \le \tilde{t}_0$.

For $\tilde{t} \ge \tilde{t}_0$ we have the limiting case $\tilde{D} \to \infty$ of the problem of Section 2, but with a different initial condition at $\tilde{t} = \tilde{t}_0$. In analogy to Section 2.3, we get

$$\tilde{u}(\tilde{r},\tilde{t}) = \sum_{j=0}^{\infty} A_j e^{-k_j^2(\tilde{t}-\tilde{t}_0)} J_0(k_j \tilde{r}) \quad (\tilde{t} \ge \tilde{t}_0), \tag{B.10}$$

where $0 =: k_0 < k_1 < k_2 < \cdots$ are the zeros of J_1, and the coefficients A_j are determined by

$$\sum_{j=0}^{\infty} A_j J_0(k_j \tilde{r}) = \tilde{u}(\tilde{r},\tilde{t}_0). \tag{B.11}$$

From (3.4) for $\tilde{t} = \tilde{t}_0$, we have

$$\tilde{u}(\tilde{r},\tilde{t}_0) = \frac{\tilde{r}^2}{2} + 2\tilde{t}_0 - \frac{1}{4} - \sum_{j=1}^{\infty} \frac{2e^{-k_j^2 \tilde{t}_0}}{k_j^2 J_0(k_j)} J_0(k_j \tilde{r}), \tag{B.12}$$

where again the k_j are the positive zeros of J_1. Note that the series (B.11) converges slower and slower when $\tilde{r}$ approaches 1, which leads to the Gibbs phenomenon shown in Figure 4.2. The cause of it is the boundary condition (3.3c), which demands a jump in the derivative $\partial\tilde{u}/\partial\tilde{r}$ at $\tilde{r} = 1$, $\tilde{t} = \tilde{t}_0$.

Equations (B.11), (B.12), and the orthogonality relation (B.6) imply

$$\begin{aligned} \frac{A_j}{2} J_0(k_j)^2 = {} & \int_0^1 \frac{r^3}{2} J_0(k_j r)\,dr + \left(2\tilde{t}_0 - \frac{1}{4}\right) \cdot \int_0^1 r J_0(k_j r)\,dr \\ & - \sum_{i=1}^{\infty} \frac{2e^{-k_i^2 \tilde{t}_0}}{k_i^2 J_0(k_i)} \int_0^1 r J_0(k_i r) J_0(k_j r)\,dr. \end{aligned} \tag{B.13}$$

For $j = 0$, we get, using $J_0(k_0 r) = J_0(0) = 1$ and (B.6),

$$\frac{A_0}{2} = \int_0^1 \frac{r^3}{2}\,dr + \left(2\tilde{t}_0 - \frac{1}{4}\right) \cdot \int_0^1 r\,dr, \tag{B.14}$$

hence

$$A_0 = 2\tilde{t}_0. \tag{B.15}$$

For $j \geq 1$ we can make use of (B.8), $J_1(k_j) = 0$, and (B.13), yielding

$$A_j = \frac{2}{k_j^2 J_0(k_j)} \left(1 - e^{-k_j^2 \tilde{t}_0}\right) \quad (j \geq 1). \tag{B.16}$$

We finally get (3.4) for $\tilde{t} \geq \tilde{t}_0$ from (B.10), (B.15), and (B.16).

C. Mathematical derivations for Section 4

Here we derive (4.8). From (4.6a)–(4.6c) it follows that $\tilde{u} - \gamma$ satisfies (2.5a)–(2.5c) for $\tilde{t} \leq \tilde{t}_0$, hence the solution is given by (2.6)

$$\tilde{u}(\tilde{r}, \tilde{t}) = \gamma - \sum_{j=1}^{\infty} \frac{2e^{-k_j^2 \tilde{t}}}{(\tilde{D}^2 k_j^2 + 1) \cdot k_j J_1(k_j)} J_0(k_j \tilde{r}) \quad (0 \leq \tilde{t} \leq \tilde{t}_0), \tag{C.1}$$

and the constants $0 < k_1 < k_2 < \cdots$ are the roots of (2.7).

For $\tilde{t} > \tilde{t}_0$ the boundary condition (4.6c) reads

$$\tilde{D} \cdot \frac{\partial \tilde{u}}{\partial \tilde{r}}(1, \tilde{t}) + \tilde{u}(1, \tilde{t}) = 0. \tag{C.2}$$

The initial condition (at $\tilde{t} = \tilde{t}_0$) is given by (C.1)

$$\tilde{u}(\tilde{r}, \tilde{t}_0) = \gamma - \sum_{j=1}^{\infty} \frac{2e^{-k_j^2 \tilde{t}_0}}{(\tilde{D}^2 k_j^2 + 1) \cdot k_j J_1(k_j)} J_0(k_j \tilde{r}). \tag{C.3}$$

Since for $\tilde{t} \geq \tilde{t}_0$, $\tilde{u}$ satisfies (4.6a) and (C.2), we have as in (A.10)

$$\tilde{u}(\tilde{r}, \tilde{t}) = \sum_{j=1}^{\infty} A_j e^{-k_j^2 (\tilde{t} - \tilde{t}_0)} J_0(k_j \tilde{r}) \quad (\tilde{t} \geq \tilde{t}_0). \tag{C.4}$$

According to (C.3) the coefficients A_j are determined by

$$\sum_{j=1}^{\infty} A_j J_0(k_j \tilde{r}) = \gamma - \sum_{j=1}^{\infty} \frac{2e^{-k_j^2 \tilde{t}_0}}{(\tilde{D}^2 k_j^2 + 1) \cdot k_j J_1(k_j)} J_0(k_j \tilde{r}). \tag{C.5}$$

In view of (A.11), (A.15), and the orthogonality property (A.12), we get

$$A_j = \frac{2}{(\tilde{D}^2 k_j^2 + 1) \cdot k_j J_1(k_j)} \cdot \left[\gamma - e^{-k_j^2 \tilde{t}_0}\right] \tag{C.6}$$

in agreement with (4.8) for $\tilde{t} \geq \tilde{t}_0$.

D. Mathematical derivations for Section 5

We now give a derivation of (5.10). We first concentrate on the *case* $0 \leq \tilde{t} \leq \tilde{t}_0$, when the heating is turned on. Since the differential equation must be homogeneous in order for the separation technique to work, we must first transform (5.4a)–(5.4c) by using the stationary solution. Defining

$$v(\tilde{r},\tilde{t}) := \frac{\tilde{q}}{4}(1+2\tilde{D}-\tilde{r}^2) - \tilde{u}(\tilde{r},\tilde{t}), \tag{D.1}$$

the function v satisfies the homogeneous problem (for $0 \le \tilde{t} \le \tilde{t}_0$),

$$\frac{\partial v}{\partial \tilde{t}} = \frac{\partial^2 v}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial v}{\partial \tilde{r}}, \tag{D.2a}$$

$$v(\tilde{r},0) = \frac{\tilde{q}}{4}(1+2\tilde{D}-\tilde{r}^2), \tag{D.2b}$$

$$\tilde{D}\cdot\frac{\partial v}{\partial \tilde{r}}(1,\tilde{t}) + v(1,\tilde{t}) = 0. \tag{D.2c}$$

Now the separation ansatz and (D.2a), (D.2c) imply

$$v(\tilde{r},\tilde{t}) = \sum_{j=1}^{\infty} A_j e^{-k_j^2\tilde{t}} J_0(k_j\tilde{r}), \tag{D.3}$$

where $0 < k_1 < k_2 < \cdots$ are the positive roots of (2.7), as in Section 2.3.

The coefficients A_j must be determined from the initial condition (D.2b)

$$\sum_{j=1}^{\infty} A_j J_0(k_j\tilde{r}) = \frac{\tilde{q}}{4}(1+2\tilde{D}-\tilde{r}^2). \tag{D.4}$$

Using the orthogonality relation (A.12), we get

$$A_j = \frac{\tilde{q}}{2(\tilde{D}^2k_j^2+1)J_1(k_j)^2}\int_0^1 rJ_0(k_jr)(1+2\tilde{D}-r^2)\,dr \quad (j=1,2,\ldots). \tag{D.5}$$

For the integral, we can use (A.14) and (B.8) to get

$$A_j = \frac{\tilde{q}}{2(\tilde{D}^2k_j^2+1)J_1(k_j)^2}\left\{\frac{2}{k_j^2}\left[\tilde{D}k_jJ_1(k_j) - J_0(k_j)\right] + \frac{4}{k_j^3}J_1(k_j)\right\}. \tag{D.6}$$

The term in brackets vanishes by the definition of k_j, hence

$$A_j = \frac{2\tilde{q}}{(\tilde{D}^2k_j^2+1)k_j^3J_1(k_j)} \quad (j=1,2,\ldots). \tag{D.7}$$

Finally, (D.1), (D.3), and (D.7), yield (5.10) for $0 \le \tilde{t} \le \tilde{t}_0$.

We now consider the *case* $\tilde{t} \geq \tilde{t}_0$. Using (5.4a), (5.4c), and (5.10) for $\tilde{t} = \tilde{t}_0$, we have the problem

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}} \frac{\partial \tilde{u}}{\partial \tilde{r}}, \tag{D.8a}$$

$$\tilde{u}(\tilde{r}, \tilde{t}_0) = \frac{\tilde{q}}{4}(1 + 2\tilde{D} - \tilde{r}^2) - \sum_{j=1}^{\infty} \frac{2\tilde{q}e^{-k_j^2 \tilde{t}_0}}{(\tilde{D}^2 k_j^2 + 1) k_j^3 J_1(k_j)} J_0(k_j \tilde{r}), \tag{D.8b}$$

$$\tilde{D} \cdot \frac{\partial \tilde{u}}{\partial \tilde{r}}(1, \tilde{t}) + \tilde{u}(1, \tilde{t}) = 0. \tag{D.8c}$$

From (D.8a), (D.8c), we get

$$\tilde{u}(\tilde{r}, \tilde{t}) = \sum_{j=1}^{\infty} A_j e^{-k_j^2 (\tilde{t} - \tilde{t}_0)} J_0(k_j \tilde{r}) \tag{D.9}$$

with k_j as before (cf. (D.3)). The coefficients A_j are now determined by (D.8b)

$$\sum_{j=1}^{\infty} A_j J_0(k_j \tilde{r}) = \frac{\tilde{q}}{4}(1 + 2\tilde{D} - \tilde{r}^2) - \sum_{j=1}^{\infty} \frac{2\tilde{q}e^{-k_j^2 \tilde{t}_0}}{(\tilde{D}^2 k_j^2 + 1) k_j^3 J_1(k_j)} J_0(k_j \tilde{r}). \tag{D.10}$$

Comparison with (D.4) and (D.7) yields

$$A_j = \frac{2\tilde{q}}{(\tilde{D}^2 k_j^2 + 1) k_j^3 J_1(k_j)} \left[1 - e^{-k_j^2 \tilde{t}_0}\right], \tag{D.11}$$

verifying (5.10) for $\tilde{t} \geq \tilde{t}_0$.

Now we derive (5.23). We must solve (5.21) with the boundary condition (5.22). We set

$$v(x) := \tilde{u}'(x) \tag{D.12}$$

and get the linear first order equation

$$x \cdot v'(x) + v(x) + x \cdot \tilde{q}(x) = 0, \tag{D.13}$$

which can be integrated (using $x \cdot v' + v = (x \cdot v)'$) to get

$$v(x) = -\frac{g(x)}{x}, \tag{D.14}$$

where g is an antiderivative of the function $x \cdot \tilde{q}(x)$. By symmetry we must have $v(0) = \tilde{u}'(0) = 0$, hence we can fix the integration constant in the definition of g as in (5.23). Using (D.12), we thus get

$$\tilde{u}(\tilde{r}) = \tilde{u}(1) - \int_1^{\tilde{r}} \frac{g(x)}{x} dx. \tag{D.15}$$

The boundary condition (5.22) now reads $-\tilde{D} \cdot g(1) + \tilde{u}(1) = 0$ and (5.23) follows.

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