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Research Article

Exp-Function Method for Solving Huxley Equation

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Huxley equation is a core mathematical framework for modern biophysically based neural modeling. It is often useful to obtain a generalized solitary solution for fully understanding its physical meanings. There are many methods to solve the equation, but each method can only lead to a special solution. This paper suggests a relatively new method called the Exp-function method for this purpose. The obtained result includes all solutions in open literature as special cases, and the generalized solution with some free parameters might imply some fascinating meanings hidden in Huxley equation.

1. Introduction

The core mathematical framework for modern biophysically based neural modeling was developed half a century ago by Hodgkin and Huxley [1]. In a series of papers published in 1952, they presented the results of an elegant series of electrophysiological experiments in which they investigated the flow of electric current through the surface membrane of the giant nerve fiber of a squid. Huxley equation [2–6] is a nonlinear partial differential equation of second order of the form

$$u_t = u_{xx} + u(k-u)(u-1). \tag{1.1}$$

This equation is an evolution equation that describes the nerve propagation in biology from which molecular CB properties can be calculated. It also gives a phenomenological description of the behavior of the myosin heads II. This equation has many fascinating phenomena such as bursting oscillation [7], interspike [8], bifurcation, and chaos [9]. A generalized exact solution can gain an insight into these phenomena. There is not a universal method for nonlinear equations. The traditional approaches to this task are the variational iteration method [10–12], the homotopy perturbation method [13, 14], Adomain's decomposition method [15], and the tanh method [16–18]; however, many methods may sometimes fail or the solution procedure is complex as the degree of nonlinearity increases, for example, calculation of Adomian polynomials in Adomian's method is terribly tedious. Recently, the study showed that the homotopy perturbation method [19–21] and the variational iteration method [22, 23] can completely overcome the difficulty. If we are really determined to extract physical meanings from analytic formulations of biological processes, we must resort to amelioration of the classical methods using modern mathematical tools. Exp-function method [24–29] is at this moment the most promising candidate theory for this purpose.

2. Basic idea of Exp-function method

Rational approximation for soliton and soliton-like solutions was first proposed by Hirota [30] and further developed by many authors [31, 32]. In this paper, we will apply the Exp-function method to the discussed problem. The basic idea of the Exp-function was proposed in He's monograph [30]. Some illustrative examples in [24–27] showed that this method is very effective to search for various solitary and periodic solutions of nonlinear equations. Zhu applied the method to some difference-differential equations [28, 29].

Consider the following general partial differential equation:

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, \dots) = 0. \tag{2.1}$$

We first unite the independent variables x and t into one wave variable $\eta = \omega x + \vartheta t$, leading (2.1) to an ordinary differential equation,

$$Q(u, u', u'', u''', \dots) = 0. \tag{2.2}$$

The Exp-function method is based on the assumption that traveling wave solutions can be expressed in the following form [30]:

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{n=-p}^{q} b_m \exp(m\eta)} = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + a_{-q} \exp(-q\eta)}, \tag{2.3}$$

where c, d, p, and q are positive integers which are unknown to be further determined, and a_n and b_m are unknown constants. To determine the values of c and p, we balance the linear term of highest order in (2.2) with the highest-order nonlinear term. Similarly to determine the values of d and q, we balance the linear term of lowest order in (2.2) with the lowest-order nonlinear term.

3. Application to Huxley equation

Using the wave variable $\eta = \omega x + \vartheta t$, we have

$$-\vartheta u' + \omega^2 u'' + u(k - u)(u - 1) = 0. \tag{3.1}$$

The highest linear term u'' is now given by

$$\begin{aligned} u'' &= \frac{c_1 \exp[(3p + c)\eta] + \cdots}{c_2 \exp[4p\eta] + \cdots}, \\ u^3 &= \frac{c_3 \exp[3c\eta] + \cdots}{c_4 \exp[3p\eta] + \cdots} = \frac{c_3 \exp[(3c + p)\eta] + \cdots}{c_4 \exp[4p\eta] + \cdots}. \end{aligned} \tag{3.2}$$

Balancing the highest order of Exp-function in (3.2), we have $3p+c = 3c+p$, and this gives $p = c$. Using the same method, we can also obtain that $q = d$. Wu and He [33] systematically studied the choice of the values of the parameters, and revealing the solution very weakly depends upon the values of the parameters. An illustrating example of Dodd–Bullough–Mikhailov equation was given.

Wu and He considered the following three cases.

Case 1. $p = c = 1$, $q = d = 1$.

Case 2. $p = c = 2$, $q = d = 2$.

Case 3. $p = c = 2$, $q = d = 1$.

All cases led to the equivalent result. Bekir and Boz [34] pointed out that $p = c = 1$ and $q = d = 1$ are valid for most nonlinear partial differential equations.

For simplicity, we set $p = c = 1$ and $q = d = 1$, so (2.3) reduces to

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{3.3}$$

Substituting (3.3) into (3.1), and by the help of Mathematica, we have

$$-\frac{1}{A}\left[C_3 e^{3\eta} + C_2 e^{2\eta} + C_1 e^{\eta} + C_0 + C_{-1} e^{-\eta} + C_{-2} e^{-2\eta} + C_{-3} e^{-3\eta}\right] = 0, \tag{3.4}$$

where

$$\begin{aligned}
A &= \left(e^{\eta} + b_0 + b_{-1} e^{-\eta}\right)^3, \\
C_3 &= k a_1 - a_1^2 - k a_1^2 + a_1^3, \\
C_2 &= k a_0 - \omega^2 a_0 - \vartheta a_0 - 2a_0 a_1 - 2k a_0 a_1 + 3a_0 a_1^2 - 2k a_1 b_0 + \vartheta a_1 b_0 + \omega^2 a_1 b_0 - a_1^2 b_0 - k a_1^2 b_0, \\
C_1 &= k a_{-1} - 2\vartheta a_{-1} - 4\omega^2 a_{-1} - a_0^2 - k a_0^2 - 2a_{-1} a_1 - 2k a_{-1} a_1 + 3a_0^2 a_1 + 3a_{-1} a_1^2 + 2k a_1 b_{-1} \\
&\quad + 2\vartheta a_1 b_{-1} + 4\omega^2 a_1 b_{-1} - a_1^2 b_{-1} - k a_1^2 b_{-1} + 2k a_0 b_0 - \omega^2 a_0 b_0 - \vartheta a_0 b_0 - 2a_0 a_1 b_0 \\
&\quad - 2k a_0 a_1 b_0 + k^2 a_1 b_0^2 + \vartheta a_1 b_0^2 - \omega^2 a_1 b_0^2, \\
C_0 &= -2a_{-1} a_0 - 2k a_{-1} a_0 + a_0^3 + 6a_{-1} a_0 a_1 + 2k a_0 b_{-1} + 6\omega^2 a_0 b_{-1} - 2a_0 a_1 b_{-1} - 2k a_0 a_1 b_{-1} \\
&\quad + 2k a_{-1} b_0 + 3\omega^2 a_{-1} b_0 - a_0^2 b_0 - k a_0^2 b_0 - 2a_{-1} a_1 b_0 - 2k a_{-1} a_1 b_0 + 2k a_1 b_{-1} b_0 \\
&\quad - 3\omega^2 a_1 b_{-1} b_0 + 3\vartheta a_1 b_{-1} b_0 + k a_0 b_0^2, \\
C_{-1} &= -a_{-1}^2 - k a_{-1}^2 + 3a_{-1} a_0^2 + 3a_{-1}^2 a_1 + 2k a_{-1} b_{-1} - 2\vartheta a_{-1} b_{-1} + 4\omega^2 a_{-1} b_{-1} - a_0^2 b_{-1} - k a_0^2 b_{-1} \\
&\quad - 2a_{-1} a_1 b_{-1} - 2k a_{-1} a_1 b_{-1} + k a_1 b_{-1}^2 - 4\omega^2 a_1 b_{-1}^2 + 2\vartheta a_1 b_{-1}^2 - 2a_{-1} a_0 b_0 - 2k a_{-1} a_0 b_0 \\
&\quad + 2k a_0 b_{-1} b_0 + \omega^2 a_0 b_{-1} b_0 + \omega a_0 b_{-1} b_0 + k a_{-1} b_0^2 - \omega^2 a_{-1} b_0^2 - \vartheta a_{-1} b_0^2, \\
C_{-2} &= 3a_{-1}^2 a_0 - 2a_{-1} a_0 b_{-1} - 2k a_{-1} a_0 b_{-1} + k a_0 b_{-1}^2 - \omega^2 a_0 b_{-1}^2 + \vartheta a_0 b_{-1}^2 - a_{-1}^2 b_0 - k a_{-1}^2 b_0 \\
&\quad + 2k a_{-1} b_{-1} b_0 + \omega^2 a_{-1} b_{-1} b_0 - \vartheta a_{-1} b_{-1} b_0, \\
C_{-3} &= a_{-1}^2 - a_{-1}^2 b_{-1} - k a_{-1}^2 b_{-1} + k a_{-1} b_{-1}^2.
\end{aligned} \tag{3.5}$$

Solving the system (3.5) simultaneously using Mathematica, we obtain the following results.

Case 1.

$$\omega = \pm\frac{1}{\sqrt{2}}, \qquad \vartheta = -\frac{2k-1}{2}, \qquad a_{-1} = 0, \qquad a_0 = 0, \qquad a_1 = 1, \qquad b_{-1} = 0. \tag{3.6}$$

Case 2.

$$\begin{gathered} \omega = \pm\frac{k}{\sqrt{2}}, \qquad \vartheta = \frac{2k-k^2}{2}, \qquad a_{-1} = kb_{-1}, \qquad a_0 = \frac{1}{2}\left(kb_0 \pm \sqrt{k^2b_0^2 - 4k^2b_{-1}}\right), \\ a_1 = 0, \quad b_{-1} \neq 0, \quad b_0^2 - b_{-1} \geq 0, \end{gathered} \tag{3.7}$$

where b_{-1} and b_0 are free parameters.

Case 3.

$$\omega = \pm\frac{k-1}{\sqrt{2}}, \quad \vartheta = \frac{1-k^2}{2}, \quad a_{-1} = 0, \quad a_0 = kb_0, \quad a_1 = 1, \quad b_{-1} = 0, \quad b_0 \neq 0, \tag{3.8}$$

where b_0 is a free parameter.

For Case 1, we obtain the following solution of (1.1) by substituting (3.6) into (3.3):

$$u_1(x,t) = \frac{1}{1 + b_0 e^{-(1/\sqrt{2})x + ((2k-1)/2)t}}. \tag{3.9}$$

For Case Case 2, we have

$$u_2(x,t) = \frac{(1/2)\left(kb_0 \pm \sqrt{k^2b_0^2 - 4k^2b_{-1}}\right) + kb_{-1}e^{-\eta}}{e^{\eta} + b_0 + b_{-1}e^{-\eta}}, \tag{3.10}$$

where $\eta = (k/\sqrt{2})x + ((2k-k^2)/2)t$ or $\eta = -(k/\sqrt{2})x + ((2k-k^2)/2)t$.

Case Case 3 leads to the following exact solution:

$$u_3(x,t) = \frac{e^{\eta} + kb_0}{e^{\eta} + b_0}, \tag{3.11}$$

where $\eta = ((k-1)/\sqrt{2})x + ((1-k^2)/2)t$ or $\eta = -((k-1)/\sqrt{2})x + ((1-k^2)/2)t$.

To compare our results with those obtained in [16], we set $b_0 = 1$ in (3.9). Equation (3.9) becomes

$$u_{1(1)}(x,t) = \frac{1}{1 + e^{-(1/\sqrt{2})x + ((2k-1)/2)t}} = \frac{1}{2}\left(1 + \tanh\left[\frac{1}{2\sqrt{2}}\left(x - \frac{2k-1}{\sqrt{2}}t\right)\right]\right). \tag{3.12}$$

In (3.10), if we set $b_0 = 2$, $b_{-1} = 1$, and $\eta = (k/\sqrt{2})x + ((2k-k^2)/2)t$, (3.10) becomes

$$u_{2(1)}(x,t) = \frac{k + ke^{-\eta}}{e^{\eta} + 2 + e^{-\eta}} = \frac{k}{1 + e^{\eta}} = \frac{k}{2}\left(1 - \tanh\left[\frac{k}{2\sqrt{2}}\left(x - \frac{k-2}{\sqrt{2}}t\right)\right]\right). \tag{3.13}$$

In (3.11), if we set $b_0 = 1$ and $\eta = -((k-1)/\sqrt{2})x + ((1-k^2)/2)t$, (3.11) becomes

$$u_{3(1)}(x,t) = \frac{k+e^{\eta}}{1+e^{\eta}} = \frac{k+1}{2} + \frac{1-k}{2}\tanh\left[\frac{k-1}{2\sqrt{2}}\left(x - \frac{k+1}{\sqrt{2}}t\right)\right], \tag{3.14}$$

These are kink solutions obtained by the tanh-coth method in [16].

In (3.9), (3.10), and (3.11), if we set parameters as follows: (1) $b_0 = -1$; (2) $b_0 = -2, b_{-1} = 1$, and $\eta = (k/\sqrt{2})x + ((2k-k^2)/2)t$; (3) $b_0 = -1$, and $\eta = -((k-1)/\sqrt{2})x + ((1-k^2)/2)t$, respectively, we have

$$\begin{aligned}
u_{1(2)}(x,t) &= \frac{1}{1-e^{-(1/\sqrt{2})x+(2k-1/2)t}} = \frac{1}{2}\left(1+\coth\left[\frac{1}{2\sqrt{2}}\left(x-\frac{2k-1}{\sqrt{2}}t\right)\right]\right),\\
u_{2(2)}(x,t) &= \frac{-k+ke^{-\eta}}{e^{\eta}-2+e^{-\eta}} = \frac{k}{1-e^{\eta}} = \frac{k}{2}\left(1-\coth\left[\frac{k}{2\sqrt{2}}\left(x-\frac{k-2}{\sqrt{2}}t\right)\right]\right),\\
u_{3(2)}(x,t) &= \frac{k-e^{\eta}}{1-e^{\eta}} = \frac{k+1}{2} + \frac{1-k}{2}\coth\left[\frac{k-1}{2\sqrt{2}}\left(x-\frac{k+1}{\sqrt{2}}t\right)\right].
\end{aligned} \tag{3.15}$$

These are the traveling solutions obtained by the tanh-coth method in [16].

The other three traveling solutions $u_7(x,t)$, $u_8(x,t)$, and $u_9(x,t)$ in [16] are equivalent with $u_{1(2)}(x,t)$, $u_{2(2)}(x,t)$, and $u_{3(2)}(x,t)$, respectively.

He and Wu [26] compared the method with the Sinh-function and the Tanh-function methods, and found a single generalized solitonary solution including all solutions obtained by Yomba using the subequation method.

4. Discussions and conclusions

The Exp-function method leads to generalized solitary solutions with some free parameters involving the known solutions in open literature. The free parameters might imply some physically meaningful results in biological process. Considering the generalized solution $u_1(x,t)$ expressed in (3.9), in case $b_0 = 1$, it turns out to a special solution in [16]; the physical understating of the special solution was given in [16]. Of course we can set b_0 equal to other values, resulting in different solitary shapes. The free parameter b_0 might be also relative to initial conditions.

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