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Research Article

Exact and Numerical Solutions of Poisson Equation for Electrostatic Potential Problems

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Homotopy perturbation method (HPM) and boundary element method (BEM) for calculating the exact and numerical solutions of Poisson equation with appropriate boundary and initial conditions are presented. Exact solutions of electrostatic potential problems defined by Poisson equation are found using HPM given boundary and initial conditions. The same problems are also solved using the BEM. The cell integration approach is used for solving Poisson equation by BEM. The problem region containing the charge density is subdivided into triangular elements. In addition, this paper presents a numerical comparison with the HPM and BEM.

1. Introduction

It is well known that there are many linear and nonlinear partial equations in various fields of science and engineering. The solution of these equations can be obtained by many different methods. In recent years, the studies of the analytical solutions for the linear or nonlinear evolution equations have captivated the attention of many authors. The numerical and seminumerical/analytic solution of linear or nonlinear, ordinary differential equation or partial differential equation has been extensively studied in the resent years. There are several methods have been developed and used in different problems [1–3]. The homotopy perturbation method is relatively new and useful for obtaining both analytical and numerical approximations of linear or nonlinear differential equations [4–7]. This method yields a very rapid convergence of the solution series. The applications of homotopy perturbation method among scientists received more attention recently [8–10]. In this study, we will first concentrate on analytical solution of Poisson equation, using frequently in electrical engineering, in the form of Taylor series by homotopy perturbation method [11–13].

The boundary element method is a numerical technique to solve boundary value problems represented by linear partial differential equations [14] and has some important advantages. The main advantage of the BEM is that it replaces the original problem with an integral equation defined on the boundary of the solution domain. For the case of a homogeneous partial differential equation, the BEM requires only the discretization on the boundary of the domain [15]. If the simulation domain is free from the electric charge, the governing equation is known as Laplace equation. The BEM computes an approximate solution for the boundary integral formulation of Laplace's equation by discretizing the problem boundary into separate elements, each containing a number of collocation nodes.

The distribution of the electrostatic potential can be determined by solving Poisson equation, if there is charge density in problem domain. In this case, the boundary integral equation obtained from Poisson equation has a domain integral. In the BEM, several methods had been developed for solving this integral. These methods are commonly known as cell integration approach, dual reciprocity method (DRM) and multiple reciprocity method (MRM) [16].

The electric field is related to the charge density by the divergence relationship

$$\begin{gathered} E = \text{electric field}, \\ \nabla E = \frac{\rho}{\varepsilon_0}, \quad \rho = \text{charge density}, \\ \varepsilon_0 = \text{permittivity}, \end{gathered} \tag{1.1}$$

and the electric field is related to the electric potential by a gradient relationship

$$E = -\nabla V. \tag{1.2}$$

Therefore the potential is related to the charge density by Poisson equation:

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{-\rho}{\varepsilon_0}, \quad V = \text{electric potential}. \tag{1.3}$$

2. Theory of the numerical methods

2.1. Homotopy perturbation method

Homotopy perturbation method has been suggested to solve boundary value problems in [17–19]. According to this method, a homotopy with an imbedding parameter $p \in [0,1]$ is constructed and the imbedding parameter is considered as a "small parameter". Here, homotopy perturbation method is used to solve analytic solution of Poisson equation with given boundary conditions.

To illustrate this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2.1}$$

with boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{2.2}$$

where $A(u)$ is written as follows:

$$A(u) = L(u) + N(u). \tag{2.3}$$

A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A can be generally divided into two parts L and N, where L is linear operator and N is nonlinear operator. Thus, (2.1) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{2.4}$$

By the homotopy technique [20], we obtain a homotopy $v(r,p) : \Omega \times [0,1] \rightarrow \Re$ satisfying

$$H(v,p) = (1-p)\left[L(v) - L\left(u_0\right)\right] + p\left[A(v) - f(r)\right] = 0, \quad p \in [0,1], \; r \in \Omega, \tag{2.5}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of (2.1), which satisfies the boundary conditions. Clearly, from (2.5), we have

$$\begin{aligned} H(v,0) &= L(v) = L\left(u_0\right) = 0, \\ H(v,1) &= A(v) = f(r) = 0, \end{aligned} \tag{2.6}$$

the changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology this is called deformation and $L(v) - L(u_0)$, $A(v) - f(r)$ are called homotopic.

We consider v as follows:

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots = \sum_{n=0}^{\infty} p^n v_n. \tag{2.7}$$

According to homotopy perturbation method, an acceptable approximation solution of (2.4) can be explained as a series of the power of p,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots = \sum_{n=0}^{\infty} v_n. \tag{2.8}$$

Convergence of the series (2.8) is given in [20, 21]. Besides, the same results have been discussed in [22–24].

2.2. Boundary element method

Consider the Poisson equation

$$\nabla^2 u = b_0, \tag{2.9}$$

where b_0 is a known function (for the electrostatic problems, according to Gauss law is $b_0 = -\rho/\varepsilon_0$).

We can develop the boundary element method for the solution of $\nabla^2 u = b_0$ in a two-dimensional domain Ω. We must first form an integral equation from the Poisson equation by using a weighted integral equation and then use the Green-Gauss theorem:

$$\int_\Omega (\nabla^2 u - b_0) w_0 d\Omega = \int_\Gamma \frac{\partial u}{\partial n} w_0 d\Gamma - \int_\Omega \nabla u \cdot \nabla w_0 d\Omega. \tag{2.10}$$

To derive the starting equation for the boundary element method, we use the Green-Gauss theorem again on the second integral. This gives

$$\int_\Omega u(\nabla^2 w_0) d\Omega - \int_\Omega b w_0 d\Omega = \int_\Gamma u \frac{\partial w_0}{\partial n} d\Gamma - \int_\Gamma w_0 \frac{\partial u}{\partial n} d\Gamma, \tag{2.11}$$

and thus, the boundary integral equations are obtained for a domain Ω with boundary Γ, where u potential, $\partial u/\partial n$, is derivative with respect to normal of u and w_0 is the known fundamental solution to Laplace's equation applied at point $\xi(w_0 = -(1/2\pi)\ell nr)$. $r = \sqrt{(\xi - x)^2 + (\eta - y)^2}$ (singular at the point $(\xi, \eta) \in \Omega$).

Then, using the property of the Dirac delta from (2.11),

$$\int_\Omega u(\nabla^2 w_0) d\Omega = -\int_\Omega u\delta(\xi - x, \eta - y) d\Omega = -u(\xi, \eta), \quad (\xi, \eta) \in \Omega, \tag{2.12}$$

that is, the domain integral has been replaced by a point value [25].

Thus, from Poisson equation the boundary integral equation is obtained on the boundary:

$$c(\xi)u(\xi) + \int_\Gamma u \frac{\partial w_0}{\partial n} d\Gamma + \int_\Omega b_0 w_0 d\Omega = \int_\Gamma w_0 \frac{\partial u}{\partial n} d\Gamma, \tag{2.13}$$

where

$$c(\xi) = \begin{cases} 1 & \text{in } \Omega, \\ \dfrac{1}{2} & \text{on } \Gamma. \end{cases} \tag{2.14}$$

The boundary integral equation for the internal points is

$$u(\xi) = \int_\Gamma w_0 \frac{\partial u}{\partial n} d\Gamma - \int_\Gamma u \frac{\partial w_0}{\partial n} d\Gamma - \int_\Omega b_0 w_0 d\Omega. \tag{2.15}$$

2.2.1. Cell integration approach

One of solution of domain integral in the BEM is cell integration approach which is the problem region subdivided to triangular elements as done in the finite element method (Figure 1). Domain integral is solved with respect to relationship between all cells and each boundary node by Gauss quadraturemethod.

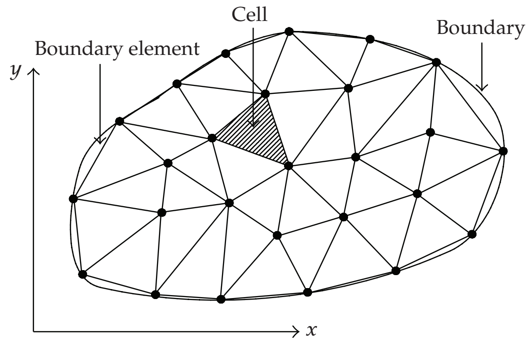

Figure 1: Subdivided regions.

The domain integral in (2.15) for each boundary point i can be written as

$$d_i = \int_{\Omega} b_0 w_0 d\Omega = \sum_{e=1}^{M} \left[\sum_{k=1}^{R} \omega_k (b_0 w_0)_k \right] \Omega_e, \tag{2.16}$$

where the integral approximated by a summation over different cells. In (2.16), M is the total number of cells describing the domain Ω, ω_k is the Gauss integration weights and Ω_e is the area of cell e. Besides, the function $(b_0 w_0)$ needs to be evaluated at integration point's k on each cell by 1 to R, see [26].

In this study, a Matlab program has been developed to solve the Poisson equation with BEM by using cell integration approach. This program calculates the potentials in the problem domain.

3. Implementation of homotopy perturbation method to Poisson equation

3.1. Case 1

First, let us investigate exact solution in the y-direction of Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\rho}{\varepsilon_0} = 0, \tag{3.1}$$

with the initial condition

$$u(0, y) = \frac{\rho a^2}{2\varepsilon_0} \left(1 - \frac{32}{\pi^3} \frac{\operatorname{Cosh}(\pi y / 2a)}{\operatorname{Cosh}(\pi b / 2a)} \right), \tag{3.2}$$

and with the Dirichlet boundary conditions (Figure 2); $u = 0$, on $x = \mp 1$ and $y = \mp 1$ (coordinates; $-1 \le x \le 1$ and $-1 \le y \le 1$).

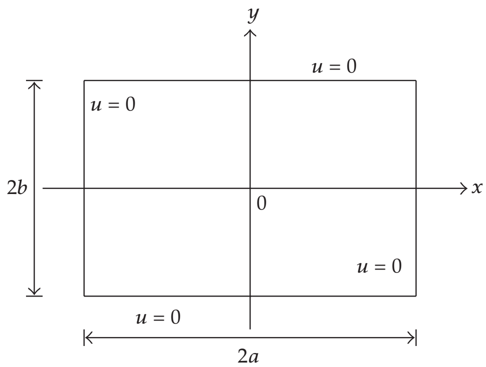

Figure 2: The problem domain and boundary conditions.

To investigate the solution of (3.1), we can construct a homotopy as follows:

$$(1-p)\left[Y''-y_0''\right]+p\left[Y''+\overset{''}{Y}+\frac{\rho}{\varepsilon_0}\right]=0, \tag{3.3}$$

where $\overset{''}{Y}=\partial^2 Y/\partial y^2$, $Y''=\partial^2 Y/\partial x^2$, and $p\in[0,1]$, with initial approximation $Y_0=u_0=(\rho a^2/2\varepsilon_0)(1-(32/\pi^3)(\mathrm{Cosh}(\pi y/2a)/\mathrm{Cosh}(\pi b/2a)))$. The solution of (3.1) can be expressed in a series in p:

$$Y=Y_0+pY_1+p^2Y_2+p^3Y_3+\cdots. \tag{3.4}$$

Then, substituting (3.4) into (3.3), and arranging the coefficients of "p" powers, we have

$$Y_0''+pY_1''+p^2Y_2''+p^3Y_3''-y_0''+py_0''+p\overset{''}{Y_0}+p^2\overset{''}{Y_1}+p^3\overset{''}{Y_2}+p^4\overset{''}{Y_3}+p\frac{\rho}{\varepsilon_0}+\cdots=0, \tag{3.5}$$

where the $Y_i(x,t)$, $i=1,2,3,\ldots$, are functions to be determined. We have to solve the following system which includes four equations with four unknowns:

$$\begin{aligned}
&p^0:\ Y_0''-y_0''=0,\\
&p^1:\ Y_1=-\iint\left(\frac{\partial^2 y_0}{\partial x^2}\right)dx\,dx-\iint\left(\frac{\partial^2 Y_0}{\partial y^2}\right)dx\,dx-\iint\left(\frac{\rho}{\varepsilon_0}\right)dx\ dx,\\
&p^2:\ Y_2=-\iint\left(\frac{\partial^2 Y_1}{\partial y^2}\right)dx\,dx,\\
&p^3:\ Y_3=-\iint\left(\frac{\partial^2 Y_2}{\partial y^2}\right)dx\,dx.
\end{aligned} \tag{3.6}$$

To found unknowns $Y_1, Y_2, Y_3, \ldots$, we must use the initial condition (3.2) for the above system, then we obtain

$$
\begin{aligned}
Y_0 &= \frac{\rho a^2}{2\varepsilon_0}\left(1 - \frac{32}{\pi^3}\frac{\operatorname{Cosh}(\pi y/2a)}{\operatorname{Cosh}(\pi b/2a)}\right),\\
Y_1 &= \frac{16\rho a^2}{\pi^3\varepsilon_0}\frac{x^2}{2!}\left(\frac{\pi}{2a}\right)^2\frac{\operatorname{Cosh}(\pi y/2a)}{\operatorname{Cosh}(\pi b/2a)} - \frac{x^2}{2}\frac{\rho}{\varepsilon_0},\\
Y_2 &= \frac{16\rho a^2}{\pi^3\varepsilon_0}\frac{x^4}{4!}\left(\frac{\pi}{2a}\right)^4\frac{\operatorname{Cosh}(\pi y/2a)}{\operatorname{Cosh}(\pi b/2a)},\\
Y_3 &= \frac{16\rho a^2}{\pi^3\varepsilon_0}\frac{x^6}{6!}\left(\frac{\pi}{2a}\right)^6\frac{\operatorname{Cosh}(\pi y/2a)}{\operatorname{Cosh}(\pi b/2a)},\\
&\vdots
\end{aligned}
\tag{3.7}
$$

Thus, as considering (3.4) with (3.7) and using Taylor series, we obtain the analytical solutions as

$$
u = \frac{\rho a^2}{2\varepsilon_0} - \frac{16\rho a^2}{\pi^3\varepsilon_0}\frac{\operatorname{Cosh}(\pi y/2a)}{\operatorname{Cosh}(\pi b/2a)}\left[1 - \frac{1}{2!}\left(\frac{\pi x}{2a}\right)^2 + \frac{1}{4!}\left(\frac{\pi x}{2a}\right)^4 - \frac{1}{6!}\left(\frac{\pi x}{2a}\right)^6 + \cdots\right] - \frac{x^2}{2}\frac{\rho}{\varepsilon_0}. \tag{3.8}
$$

Therefore, the exact solution of $u(x, y)$ in closed form is

$$
u(x,y) = \frac{\rho a^2}{2\varepsilon_0}\left(1 - \frac{x^2}{a^2}\right) - \frac{16\rho a^2}{\pi^3\varepsilon_0}\frac{\operatorname{Cosh}(\pi y/2a)}{\operatorname{Cosh}(\pi b/2a)}\operatorname{Cos}\frac{\pi x}{2a}. \tag{3.9}
$$

3.2. Case 2

Let us investigate exact solution in the x-direction of Poisson equation

$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\rho}{\varepsilon_0} = 0, \tag{3.10}
$$

with the initial condition

$$
u(x,0) = \frac{\rho b^2}{2\varepsilon_0}\left(1 - \frac{32}{\pi^3}\frac{\operatorname{Cosh}(\pi x/2b)}{\operatorname{Cosh}(\pi a/2b)}\right), \tag{3.11}
$$

and with the Dirichlet boundary conditions. To investigate the solution of (3.10), we can construct a homotopy as follows:

$$
(1-p)\left[\overset{''}{Y} - \overset{''}{y_0}\right] + p\left[Y'' + \overset{''}{Y} + \frac{\rho}{\varepsilon_0}\right] = 0. \tag{3.12}
$$

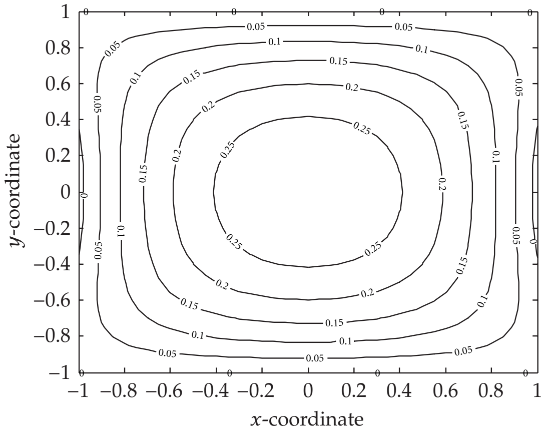

Figure 3: Equipotential lines for $\rho/\varepsilon_0 = 1$ (using HPM, x-direction).

After that, substituting (3.4) into (3.12), and arranging the coefficients of "p" powers, we have to solve the following system including four equations with four unknowns:

$$p^0: \; Y_0'' - y_0'' = 0,$$

$$p^1: \; Y_1 = -\iint\left(\frac{\partial^2 y_0}{\partial x^2}\right)dy\,dy - \iint\left(\frac{\partial^2 Y_0}{\partial y^2}\right)dy\,dy - \iint\left(\frac{\rho}{\varepsilon_0}\right)dy\,dy,$$

$$p^2: \; Y_2 = -\iint\left(\frac{\partial^2 Y_1}{\partial x^2}\right)dy\,dy, \tag{3.13}$$

$$p^3: \; Y_3 = -\iint\left(\frac{\partial^2 Y_2}{\partial x^2}\right)dy\,dy.$$

As found unknowns $Y_1, Y_2, Y_3, \ldots$, we have exact solution of (3.10):

$$u(x,y) = \frac{\rho b^2}{2\varepsilon_0} - \frac{16\,\rho b^2}{\pi^3\varepsilon_0}\frac{\mathrm{Cosh}(\pi x/2b)}{\mathrm{Cosh}(\pi a/2b)}\left[1 - \frac{1}{2!}\left(\frac{\pi y}{2b}\right)^2 + \frac{1}{4!}\left(\frac{\pi y}{2b}\right)^4 - \frac{1}{6!}\left(\frac{\pi y}{2b}\right)^6 + \cdots\right] - \frac{y^2}{2}\frac{\rho}{\varepsilon_0}. \tag{3.14}$$

Therefore, the exact solution of $u(x,y)$ in closed form is

$$u(x,y) = \frac{\rho b^2}{2\varepsilon_0}\left(1 - \frac{y^2}{b^2}\right) - \frac{16\rho b^2}{\pi^3\varepsilon_0}\frac{\mathrm{Cosh}(\pi x/2b)}{\mathrm{Cosh}(\pi a/2b)}\mathrm{Cos}\frac{\pi y}{2b}. \tag{3.15}$$

The equipotential lines obtained using exact solution and numerical results have been shown in Figures 3–5 (for $\rho/\varepsilon_0 = 1$). These results then are compared in Tables 1 and 2 (for $\rho/\varepsilon_0 = 1$ and $\rho/\varepsilon_0 = 50$).

Tables 1 and 2 compare the exact HPM and approximate BEM of the Poission equation for $\rho/\varepsilon_0 = 1$ and $\rho/\varepsilon_0 = 50$, respectively. Tables 1 and 2 show that the differences between HPM and BEM for both directions x and y. The differences clearly show that the results of the approximate BEM introduced in this study are acceptable.

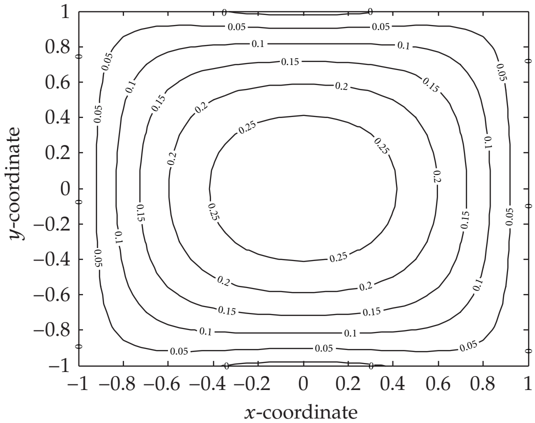

Figure 4: Equipotential lines for $\rho/\varepsilon_0 = 1$ (using HPM, y-direction).

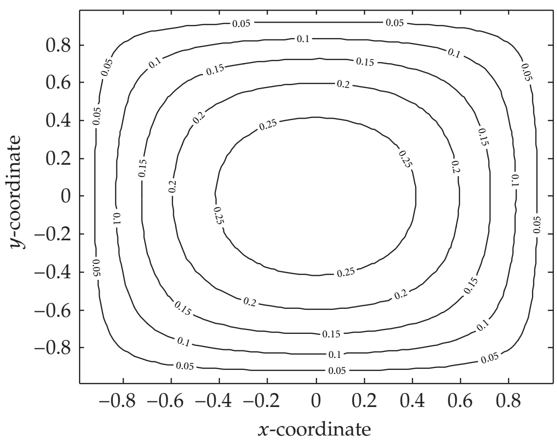

Figure 5: Equipotential lines for $\rho/\varepsilon_0 = 1$ (using BEM).

Table 1: The comparison of potentials (*Volts*) for $\rho/\varepsilon_0 = 1$ at $y = 0$ $(a = 1, b = 1)$.

x	HPM (y-direction)	HPM (x-direction)	BEM	|HPM(y)-BEM|	|HPM(x)-BEM|
1.0	0.0	– 0.01602455	0.0	0.0	0.01602455
0.9	0.062828504	0.052243813	0.063177929	0.000349425	0.010934116
0.8	0.116449177	0.109441502	0.116356508	0.000092669	0.006915006
0.7	0.161634683	0.156982714	0.161274602	0.000360081	0.004291888
0.6	0.199119152	0.196042898	0.198740117	0.000379035	0.002697219
0.5	0.229580109	0.227587807	0.229420039	0.000160070	0.001832232
0.4	0.253621787	0.252397383	0.253604246	0.000013624	0.001206863
0.3	0.271760248	0.271085037	0.271809901	0.000049653	0.000724864
0.2	0.284410680	0.284112817	0.284797109	0.000386429	0.000684292
0.1	0.291877170	0.291802833	0.292117395	0.000240225	0.000314562
0.0	0.294345218	0.294345218	0.294569673	0.000224455	0.000224455

Table 2: The comparison of potentials $(Volts)$ for $\rho/\varepsilon_0 = 50$ at $x = 0$ $(a = 1,\ b = 1)$.

y	HPM (y-direction)	HPM (x-direction)	BEM	|HPM(y)-BEM|	|HPM(x)-BEM|
1.0	− 0.80122754	0.0	0.0	0.80122754	0.0
0.9	2.612190681	3.141425217	3.158722460	0.546531779	0.017297243
0.8	5.472075101	5.822458882	5.815100354	0.343025253	0.007358528
0.7	7.849135743	8.081734156	8.083558316	0.234422573	0.001824160
0.6	9.802144924	9.955957630	9.930100041	0.127955117	0.025857589
0.5	11.37939038	11.47900548	11.46598698	0.08659660	0.01301850
0.4	12.61986917	12.68108935	12.67698049	0.05711132	0.00410886
0.3	13.55425187	13.58801241	13.60251771	0.04826584	0.01450530
0.2	14.20564089	14.22053401	14.22555053	0.01990964	0.00501652
0.1	14.59014168	14.59385852	14.62567229	0.03553061	0.03181377
0.0	14.71726094	14.71726094	14.72848367	0.01122273	0.01122273

4. Conclusions

In this paper, we proposed homotopy perturbation method to find exact solution in the x- and y-directions of Poisson equation with appropriate boundary and initial conditions. The numerical results of this electrostatic potential problem have been calculated at the same boundary conditions by BEM. These results are compared with those of HPM in Tables 1 and 2. The obtained numerical results by using BEM are in agreement with the exact solutions obtained by HPM. This adjustment is clearly seen in Figures 3, 4, and 5. It is shown that these methods are acceptable and very efficient for solving electrostatic field problems with charge density.

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