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Research Article

Computing Exact Solutions to a Generalized Lax-Sawada-Kotera-Ito Seventh-Order KdV Equation

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The Cole-Hopf transform is used to construct exact solutions to a generalization of both the seventh-order Lax KdV equation (Lax KdV7) and the seventh-order Sawada-Kotera-Ito KdV equation (Sawada-Kotera-Ito KdV7).

1. Introduction

Many direct and computational methods have been used to handle nonlinear partial differential equations (NLPDE's). Some methods used in a satisfactory way to obtain exact solutions to NLPDE's are inverse scattering method [1], Hirota bilinear method [2, 3], Backlund transformations [4], Painlevé analysis [5], Lie groups [6], the tanh method [7], the generalized tanh method [8, 9], the extended tanh method [10–12], the improved tanh-coth method [13, 14], the Exp-function method [15–17], the projective Riccati equation method [18], the generalized projective Riccati equations method [19–24], the extended hyperbolic function method [25], variational iteration method [26, 27], He's polynomials [28], homotopy perturbation method [29], and many other methods [30]. However, there is not a unified method that could be used to handle all NLPDE's; in this sense, the implementation of new methods or variants of the some well-known methods is relevant. The principal objective of this paper consists in obtaining exact traveling wave solutions which include periodic and soliton solutions to a particular case of the general seventh-order KdV (KdV7), which is a generalization of the seventh-order Sawada-Kotera-Ito (SKI-KdV7) equation, by using a variant of the exp-function method. The general seventh-order KdV (KdV7) equation [31] reads

$$u_t + au^3u_x + bu_x^3 + cuu_xu_{xx} + du^2u_{xxx} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0. \tag{1.1}$$

The (KdV7) was introduced initially by Pomeau et al. [32] for discussing the structural stability of KdV equation under a singular perturbation. Some particular cases of (1.1) are

(i) seventh-order Lax equation [1, 6] ($a = 140, b = 70, c = 280, d = 70, e = 70, f = 42, g = 14$):

$$u_t + 140u^3u_x + 70u_x^3 + 280uu_xu_{xx} + 70u^2u_{xxx} + 70u_{2x}u_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x} = 0; \tag{1.2}$$

(ii) seventh-order Sawada-Kotera-Ito equation [1, 8–10] ($a = 252, b = 63, c = 378, d = 126, e = 63, f = 42, g = 21$):

$$u_t + 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{2x}u_{3x} + 42u_xu_{4x} + 21uu_{5x} + u_{7x} = 0; \tag{1.3}$$

(iii) seventh-order Kaup-Kupershmidt equation [1, 7] ($a = 2016, b = 630, c = 2268, d = 504, e = 252, f = 147, g = 42$):

$$u_t + 2016u^3u_x + 630u_x^3 + 2268uu_xu_{xx} + 504u^2u_{xxx} + 252u_{2x}u_{3x} + 147u_xu_{4x} + 42uu_{5x} + u_{7x} = 0. \tag{1.4}$$

2. Generalization of the Lax KdV7 and the Sawada-Kotera-Ito KdV7

Observe that (1.2) and (1.3) satisfy the relation

$$a = \frac{d}{63}(e + f + g). \tag{2.1}$$

For this reason we will study equation

$$u_t + \frac{d}{63}(e + f + g)u^3u_x + bu_x^3 + cuu_xu_{xx} + du^2u_{xxx} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0. \tag{2.2}$$

We seek solutions to (2.2) in the Cole-Hopf form

$$u(t, x) = A\partial_x \tanh(\xi), \tag{2.3}$$

where A is some constant to be determined later and

$$\xi = \xi(t,x) = \mu(x + \lambda t + \delta)\ , \quad \mu, \delta, \lambda = \text{const.} \tag{2.4}$$

Substituting (2.3) into (2.2), we obtain a polynomial equation in the variable $\zeta = \exp(\xi)$. Equating the coefficients of the different powers of ζ to zero, we obtain following algebraic system:

$$\begin{gathered}
\lambda + 64\mu^6 = 0, \\
64\mu^5(A(e+f+g) - 247\mu) + 5\lambda = 0, \\
64\mu^4\left(A^2(b+c+d) - 3A\mu(5e+9f+19g) + 4293\mu^2\right) + 9\lambda = 0, \\
64\mu^3\left(A^3 d(e+f+g) - 63A^2\mu(3b+5c+11d) + 126A\mu^2(28e+46f+151g) - 983997\mu^3\right) + 315\lambda = 0.
\end{gathered} \tag{2.5}$$

Eliminating A, λ, and μ from system (2.5) gives

$$\begin{gathered}
b = d + \frac{1}{126}(e+f+g)(e-5f+10g), \\
c = \frac{5}{21}g(e+f+g) - 2d.
\end{gathered} \tag{2.6}$$

It is easy to verify that (1.2) and (1.3) are particular cases of general KdV7 equation (1.1) subject to (2.1) and (2.6). This motivates us to define the generalized Lax-Sawada-Kotera-Ito seventh-order equation (LSKI KdV7) as follows:

$$\begin{aligned}
&u_t + \frac{1}{63}d(e+f+g)u^3u_x + \left(d + \frac{1}{126}(e+f+g)(e-5f+10g)\right)u_x^3 \\
&\quad + \left(\frac{5}{21}g(e+f+g) - 2d\right)uu_xu_{xx} + du^2u_{xxx} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0.
\end{aligned} \tag{2.7}$$

3. Solutions to Generalized LSKI KdV7

In order to look for solutions to (2.7), we will use the exp ansatz

$$u(\xi) = p + \frac{q}{1 + r\exp(-\xi) + s\exp(\xi)}, \tag{3.1}$$

where p, q, r, and s are some constants. Substituting (3.1) into (2.7) gives an algebraic system. Solving it, we obtain

$$\lambda = -\frac{1}{63}d(e+f+g)p^3 - \mu^2\left(dp^2 + gp\mu^2 + \mu^4\right), \quad q = \frac{126\mu^2}{e+f+g}, \quad s = \frac{1}{4r}, \quad r = r, \quad \mu = \mu. \tag{3.2}$$

From (2.4), (3.1), and (3.2), we obtain following solution to (2.7) subject:

$$u(x,t) = p + \frac{126\mu^2}{(e+f+g)(1+r\exp(\xi)+(1/4r)\exp(-\xi))},$$
$$\xi = \mu(x+\lambda t+\delta),$$
$$\lambda = -\frac{1}{63}d(e+f+g)p^3 - \mu^2\left(dp^2+gp\mu^2+\mu^4\right). \tag{3.3}$$

In particular, if $r = 1/2$, equation (3.3) gives

$$u(x,t) = p + \frac{63\mu^2}{e+f+g}\operatorname{sech}^2\left(\frac{\mu}{2}(x+\lambda t+\delta)\right),$$
$$\lambda = -\frac{1}{63}d(e+f+g)p^3 - \left(dp^2+gp\mu^2+\mu^4\right)\mu^2. \tag{3.4}$$

Replacing μ with $\mu\sqrt{-1}$ gives the following periodic solutions:

$$u(x,t) = p - \frac{63\mu^2}{e+f+g}\sec^2\left(\frac{\mu}{2}(x+\lambda t+\delta)\right),$$
$$\lambda = -\frac{1}{63}d(e+f+g)p^3 + \left(dp^2-gp\mu^2+\mu^4\right)\mu^2. \tag{3.5}$$

On the other hand, if $r = -1/2$, equation (3.3) gives

$$u(x,t) = p - \frac{63\mu^2}{e+f+g}\operatorname{csch}^2\left(\frac{\mu}{2}(x+\lambda t+\delta)\right),$$
$$\lambda = -\frac{1}{63}d(e+f+g)p^3 - \left(dp^2+gp\mu^2+\mu^4\right)\mu^2. \tag{3.6}$$

Replacing μ with $\mu\sqrt{-1}$ gives the following periodic solutions:

$$u(x,t) = p - \frac{63\mu^2}{e+f+g}\csc^2\left(\frac{\mu}{2}(x+\lambda t+\delta)\right),$$
$$\lambda = -\frac{1}{63}d(e+f+g)p^3 + \left(dp^2-gp\mu^2+\mu^4\right)\mu^2. \tag{3.7}$$

4. Solutions to Sawada-Kotera-Ito KdV7 Equation

From (3.3)–(3.7) with $d = 126, e = 63, f = 42$, and $g = 21$, we obtain the following analytic solutions to equation (1.3):

$$\begin{aligned}
u(x,t) &= p + \frac{4r\mu^2 \exp(\mu(x+\lambda t+\delta))}{(1+2r\exp(\mu(x+\lambda t+\delta)))^2}, \quad \lambda = -252p^3 - 126p^2\mu^2 - 21p\mu^4 - \mu^6, \\
u(x,t) &= p + \frac{1}{2}\mu^2 \operatorname{sech}^2\left(\frac{1}{2}\mu(x+\lambda t+\delta)\right), \quad \lambda = -252p^3 - 126p^2\mu^2 - 21p\mu^4 - \mu^6, \\
u(x,t) &= p - \frac{1}{2}\mu^2 \sec^2\left(\frac{1}{2}\mu(x+\lambda t+\delta)\right), \quad \lambda = -252p^3 + 126p^2\mu^2 - 21p\mu^4 + \mu^6, \\
u(x,t) &= p - \frac{1}{2}\mu^2 \operatorname{csch}^2\left(\frac{1}{2}\mu(x+\lambda t+\delta)\right), \quad \lambda = -252p^3 - 126p^2\mu^2 - 21p\mu^4 - \mu^6.
\end{aligned} \tag{4.1}$$

5. Conclusions

We exhibited an equation that generalizes both seventh-order Lax equation and seventh-order Sawada-Kotera-Ito equation. At the same time, we obtained exact solutions to these equations with the aid of a Cole-Hopf ansatz. These same ideas are suitable for the seventh-order Kaup-Kupershmidt equation. We think that some of the solutions in this work are new in the open literature. We may apply other methods to find exact solutions to a variety of nonlinear PDE's. See [3, 12–52].

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