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Research Article

Approximate Method for Studying the Waves Propagating along the Interface between Air-Water

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This paper is devoted to consider the approximate solutions of the nonlinear water wave problem for a fluid layer of finite depth in the presence of gravity. The method of multiple-scale expansion is employed to obtain the Korteweg-de Vries (KdV) equations for solitons, which describes the behavior of the system for free surface between air and water in a nonlinear approach. The solutions of the water wave problem split up into two wave packets, one moving to the right and one to the left, where each of these wave packets evolves independently as the solutions of KdV equations. The solution of KdV equations is obtained analytically by using a reliable modification of Laplace decomposition method (LDM), namely, the modified Laplace decomposition method (MLDM) is presented. This procedure is a powerful tool for solving large amount of nonlinear problems. The proposed method provides the solution as a series which may converge to the exact solution of the problem. Also, the convergence analysis of the proposed method is given. Finally, we observe that the elevation of the water waves is in form of traveling solitary waves. The horizontal and vertical of the velocity components have nonlinear characters.

1. Introduction

We are concerned with a two-dimensional, irrotational flow of an incompressible ideal fluid with a free surface under the gravitational field. The domain occupied by the fluid is bounded from below by a solid bottom. The upper surface is a free boundary and we take the influence of the gravitational field into account on the free surface. Our main interest is motion of the free surface, which is called a gravity wave.

The KdV equation was originally derived by Korteweg and de Vries [1] from the model surface waves in a canal. The key to a soliton's behavior is a robust balance between the effects of dispersion and nonlinearity. When one grafts these two effects onto the wave equation for shallow water waves; then at leading order in the strengths of the dispersion and nonlinearity one gets the KdV equation for solitons [2].

The KdV equation was obtained by Benjamin [2] and Benney [3], for example, who assume that the waves are weakly nonlinear and weakly dispersive. In other words, the wave amplitude is much smaller than, for example, the upper layer thickness, and the water depth is much smaller than the typical wavelength. Solitons are among the most interesting structures in nature. Being configurations of continuous fields, they retain their localized shape even after interactions and collisions. Observed originally long ago as stable moving humps in shallow water channels, they have been established since then in various physical systems including optical wave guides, crystal lattices, Josephson junctions, plasmas, and spiral galaxies [4]. Long lasting efforts to theoretically describe their intriguing properties have culminated in the development of the inverse scattering technique [5] which is among the most powerful methods to obtain exact solutions of nonlinear partial differential equations [6, 7]. Particularly popular examples for solitons in hydrodynamic systems are the solutions of KdV equation:

$$Z_T(X,T) + Z(X,T)Z_X(X,T) + Z_{XXX}(X,T) = 0, \tag{1.1}$$

where X stands for a space coordinate, T denotes time, and Z represents the surface elevation of a liquid in a shallow duct. This equation can be derived perturbatively from the Euler equation for the motion of an incompressible and inviscid fluid [1, 8]. The one-soliton solution of (1.1) is given by

$$Z(X,T) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(X - cT)\right), \tag{1.2}$$

which for all values of $c > 0$ describes a hump of invariable shape moving to the right with velocity c. The amplitude of the hump is given by $c/2$.

Concerning the KdV approximation for water waves, we refer to Craig [9], Schneider and Wayne [10, 11], and Iguchi [12, 13] for the approximation of the solution, and Craig et al. [14] for the approximation of the Hamiltonian. We remark that the KdV equation is a model of water waves in the long wave regime. The dynamics of the free surface is approximately translation of two waves without change of the shape, one moving to the right and the other to the left, for a short time interval. The dynamics of each wave is very slow so that it is invisible for the short time interval. Craig [9] gave the justification in the framework of Sobolev spaces. Schneider and Wayne gave the justification without assuming the one-directional motion of the wave in [10] and extended it to the capillary-gravity waves in [11]. They showed that the interactions between two waves are negligible so that the solution of the full water wave problem is approximated by a sum of the solutions, which are appropriately scaled, of the decoupled KdV equations for the long time interval. However, they treated the problem in unscaled variables, whereas Craig treated it in the scaled variables called Boussinesq ones.

The nonlinear surface water waves in perturbed problem are discussed by [15, 16] in the presence of the effect of surface tension. Based on the method of multiple-scale expansion for a small amplitude, they derived two KdV equations and discussed the two-soliton solution for KdV equations by using analytical methods.

Many different methods have been recently introduced to solve nonlinear problems, such as, Adomian decomposition method [17–23], variational iteration method [24–29], and homotopy perturbation method [26, 30].

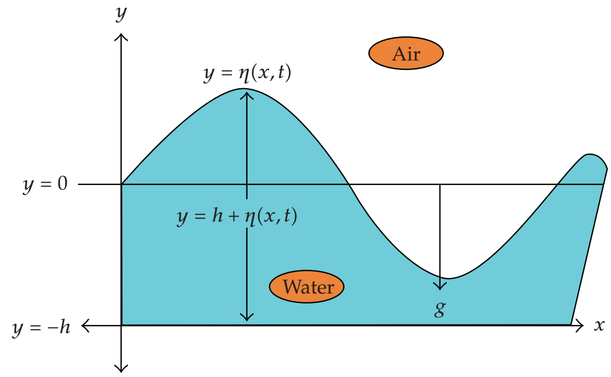

Figure 1: The setup of the shallow water wave problem.

While dealing with nonlinear ordinary differential or partial differential equations, the nonlinearity is replaced by a series of what are called Adomian polynomials [31]. The evaluation of these polynomials is necessary, as they contribute to the solutions series components. Recently, Adomian [18] introduced the phenomena of the so-called "noise terms". The "noise terms" were defined in [31, 32] as the identical terms with opposite signs that appear in the components of the series solution of $u(x)$. In [18], it is concluded that if terms in the component u_0 are cancelled by terms in the component u_1, even through u_1 contains further terms, then the remaining noncancelled terms of u_0 provide the exact solution. It was suggested in [18] that the noise terms appear always for inhomogeneous equations.

The main aim in this work is to effectively derive the KdV equations and employ the modified form of Laplace decomposition method introduced by Khuri [33] to establish approximate solutions of waves propagating along the interface between air-water. The proposed modification will accelerate the rapid convergence of a series solution when compared with Laplace decomposition method and therefore provides major progress. This numerical technique basically illustrates how the Laplace transform may be used to approximate the solutions of the nonlinear partial differential equations (PDEs) by manipulating the decomposition method. Hussain and Khan [34] applied Laplace decomposition method to solve the nonlinear PDEs.

The structure of the paper is organized as follows: in Section 2, we collect the basic equations and boundary conditions. In Section 3, we derive the KdV equations by multiple scale perturbation theory. In Section 4, we introduce the modified Laplace decomposition method. In Section 5, we present, solution procedure using MLDM. In Section 6 we discuss the convergence analysis of the standard Adomian decomposition method. In Section 7, numerical study using MLDM is presented to investigate the two-soliton solution for KdV equations. The last section gives a discussion of our results.

2. The Physical Problem and Basic Equations

We consider the unsteady two-dimensional flow of inviscid, incompressible fluid in a constant gravitational field. The space coordinates are (x, y) and the gravitational acceleration g is in the negative y direction. Let h be the undisturbed depth of the fluid. The bottom of the fluid is assumed to have no topography at $y = -h$. This problem describes the interface dynamics, between air and water waves, under the gravity g, see Figure 1.

The equation for the surface of water is $y = \eta(x,t)$ where $y = 0$ represents the equilibrium situation. We assume that the motion is irrotational within the wave. Therefore, we can describe the wave inside the water by a velocity potential $\Phi(x,y,t)$ whose gradient is the velocity field:

$$\underline{v} = (u,v) = \nabla\Phi = \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}\right). \tag{2.1}$$

The divergence-free condition on the velocity field implies that the velocity potential Φ satisfies the Laplace's equation:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0, \quad \text{for } -h < y < \eta(x,t). \tag{2.2}$$

On a solid fixed boundary, the normal velocity of the fluid must vanish:

$$\frac{\partial\Phi}{\partial y} = 0, \quad \text{at } y = -h, \tag{2.3}$$

which dictates that there is no flow perpendicular to the bottom.

The boundary conditions at the free surface $y = \eta(x,t)$ are given by

$$\begin{gathered}\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x}\frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial y}, \\ \frac{\partial\Phi}{\partial t} + g\eta + \frac{1}{2}|\nabla\Phi|^2 = 0.\end{gathered} \tag{2.4}$$

3. Derivation of Korteweg-de Vries Equations

In this section, we show that it is indeed possible to derive KdV equations [35] for the free surface $\eta(x,t)$ by using a multiple-scale perturbation theory. We introduce a small parameter, ϵ, and consider the following dimensionless forms:

$$y = hy^\star, \qquad x = \frac{h}{\sqrt{\epsilon}}x^\star, \qquad t = \sqrt{\frac{h}{\epsilon g}}t^\star, \qquad \eta = \epsilon h\eta^\star, \qquad \Phi = h\sqrt{\epsilon gh}\Phi^\star. \tag{3.1}$$

The superposed stars refer to the dimensionless quantities, from now on, it will be omitted for simplicity.

To derive the KdV equations, we will substitute the new variables from (3.1) into (2.2)–(2.4), we get the system equations up to order ϵ^2

$$\varepsilon\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0,\quad \text{for } -1<y<\epsilon\eta, \tag{I}$$

$$\frac{\partial\Phi}{\partial y}=0,\quad \text{at } y=-1, \tag{II}$$

$$\epsilon\frac{\partial\eta}{\partial t}+\epsilon^2\frac{\partial\eta}{\partial x}\frac{\partial\Phi}{\partial x}=\frac{\partial\Phi}{\partial y},\quad \text{at } y=\epsilon\eta, \tag{III}$$

$$\frac{\partial\Phi}{\partial t}+\eta+\frac{1}{2}\frac{\partial^2\Phi}{\partial y^2}+\frac{\epsilon}{2}\frac{\partial^2\Phi}{\partial x^2}=0,\quad \text{at } y=\epsilon\eta. \tag{IV}$$

To get a suitable expansion for the velocity potential Φ, we note that from the Laplace equation (I) and the boundary condition (II), we may derive the following representation for $\Phi(x,y,t)$ as a series in $y+1$. Since $\partial\Phi/\partial y=0$ at $y=-1$,

$$\Phi(x,y,t)=\sum_{n=0}^{\infty}(y+1)^n\Phi_n(x,t). \tag{3.2}$$

Substituting (3.2) into (I), we find the Laplace equation for the velocity potential Φ:

$$\sum_{n=0}^{\infty}(y+1)^n\left[\epsilon\partial_x^2\Phi_n(x,t)+(n+1)(n+2)\Phi_{n+2}(x,t)\right]=0, \tag{3.3}$$

leading to the recursion relation:

$$\Phi_{n+2}=\frac{-\epsilon\partial_x^2\Phi_n}{(n+1)(n+2)}. \tag{3.4}$$

From the boundary condition (II), we get

$$\sum_{n=1}^{\infty}n(y+1)^{n-1}\Phi_n=0, \tag{3.5}$$

implying $\Phi_1(x,t)=0$. From the recursion relation (3.4), we hence find $\Phi_{2n+1}=0$ for all n. The velocity potential may therefore be expressed entirely in terms of Φ_0 and its derivatives

$$\Phi(x,y,t)=\sum_{n=0}^{\infty}(y+1)^{2n}\epsilon^n\frac{(-1)^n}{(2n)!}\partial_x^{2n}\Phi_0(x,t), \tag{3.6}$$

with the so far undetermined function $\Phi_0(x,t)$.

Using this expansion for Φ in (III) and (IV) and observing that both equations hold at the interface, that is, for $y = \epsilon\eta$, we get the desired order in ϵ

$$\partial_t \eta + \partial_x^2 \Phi_0 = \frac{\epsilon}{6}\partial_x^4 \Phi_0 - \epsilon \partial_x \Phi_0 \partial_x \eta - \epsilon \eta \partial_x^2 \Phi_0, \tag{V}$$

$$\partial_t \Phi_0 + \eta = \frac{\epsilon}{2}\partial_t \partial_x^2 \Phi_0 - \frac{\epsilon}{2}(\partial_x \Phi_0)^2. \tag{VI}$$

It is convenient to differentiate (VI) with respect to x and to introduce the x-component of the velocity potential $u = \partial_x \Phi_0$. We then find the final set of equations to determine η and u

$$\begin{aligned} \partial_t \eta + \partial_x u &= \epsilon\left(\frac{1}{6}\partial_x^3 u - u\partial_x \eta - \eta \partial_x u\right), \\ \partial_t u + \partial_x \eta &= \epsilon\left(\frac{1}{2}\partial_t \partial_x^2 u - u \partial_x u\right). \end{aligned} \tag{3.7}$$

Now we will solve these equations perturbatively, by using the method of multiple scale [15, 16, 36]. To employ this method, we introduce the variable

$$t_m = \epsilon^m t, \quad (m = 0, 1), \tag{3.8}$$

and let

$$\begin{aligned} u(x,t) &= u_0 + \epsilon u_1 + O\left(\epsilon^2\right), \\ \eta(x,t) &= \eta_0 + \epsilon \eta_1 + O\left(\epsilon^2\right), \end{aligned} \tag{3.9}$$

where ϵ represents a small parameter characterizing the steepness ratio of the wave. Also, for the derivative, we write

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + O\left(\epsilon^2\right). \tag{3.10}$$

Therefore, referring to t_0 as variable appropriate for fast variation, t_1 corresponds to the slow one. On substituting (3.8)–(3.10) into (3.7) and equating the coefficients of equal powers of ϵ, we obtain the linear and the successive nonlinear partial differential equations of various orders.

(i) The Zeroth Order of ϵ

$$\frac{\partial^2 \eta_0}{\partial t_0^2} - \frac{\partial^2 \eta_0}{\partial x^2} = 0, \qquad \frac{\partial^2 u_0}{\partial t_0^2} - \frac{\partial^2 u_0}{\partial x^2} = 0. \tag{3.11}$$

Both η_0 and u_0 satisfy the wave equations as arbitrary functions of x and t_0. The solutions are traveling waves of d'Alembert form:

$$\begin{aligned}\eta_0 &= p(x+t_0,t_1)+q(x-t_0,t_1),\\ u_0 &= p(x+t_0,t_1)-q(x-t_0,t_1).\end{aligned}\tag{3.12}$$

Thus, on the fastest time scale t_0, all initial data splits into two directions, with one part p propagating to the right and the other one q propagating to the left. On this time scale, no solitons are observed, as the main phenomenon is wave propagation, according to the wave equation.

(ii) The First Order of ϵ

$$\begin{aligned}\frac{\partial\eta_1}{\partial t_0}+\frac{\partial u_1}{\partial x} &= -\left[\frac{\partial\eta_0}{\partial t_1}+u_0\frac{\partial\eta_0}{\partial x}+\eta_0\frac{\partial u_0}{\partial x}-\frac{1}{6}\frac{\partial^3 u_0}{\partial x^3}\right],\\ \frac{\partial u_1}{\partial t_0}+\frac{\partial\eta_1}{\partial x} &= -\left[\frac{\partial u_0}{\partial t_1}+u_0\frac{\partial u_0}{\partial x}-\frac{1}{2}\frac{\partial^3 u_0}{\partial t_0\partial x^2}\right].\end{aligned}\tag{3.13}$$

These inhomogeneous equations involve again the set of unknown functions η_1 and u_1. Let as change variables in these two equations, to express them in characteristic variables:

$$\ell = x-t_0,\qquad r = x+t_0,\tag{3.14}$$

where ℓ is the characteristic variable for a left-moving wave and r is the characteristic variable for a right-moving wave. On substituting (3.14) into (3.13), we obtain

$$\frac{\partial\eta_1}{\partial\ell}-\frac{\partial\eta_1}{\partial r}+\frac{\partial u_1}{\partial r}+\frac{\partial u_1}{\partial\ell} = -\left[\frac{\partial p}{\partial t_1}+\frac{\partial q}{\partial t_1}+2p\frac{\partial p}{\partial r}-2q\frac{\partial q}{\partial\ell}-\frac{1}{6}\left(\frac{\partial^3 p}{\partial r^3}-\frac{\partial^3 q}{\partial\ell^3}\right)\right],\tag{3.15}$$

$$\frac{\partial\eta_1}{\partial\ell}+\frac{\partial\eta_1}{\partial r}+\frac{\partial u_1}{\partial\ell}-\frac{\partial u_1}{\partial r} = -\left[\frac{\partial p}{\partial t_1}-\frac{\partial q}{\partial t_1}+p\left(\frac{\partial p}{\partial r}-\frac{\partial q}{\partial\ell}\right)+q\left(\frac{\partial q}{\partial\ell}-\frac{\partial p}{\partial r}\right)+\frac{1}{2}\left(\frac{\partial^3 p}{\partial r^3}+\frac{\partial^3 q}{\partial\ell^3}\right)\right].\tag{3.16}$$

Adding and subtracting (3.15) and (3.16) and integrating the result of two equations with respect to ℓ and r, respectively, we obtain

$$2(\eta_1+u_1) = -\left[2\frac{\partial p}{\partial t_1}+3p\frac{\partial p}{\partial r}+\frac{1}{3}\frac{\partial^3 p}{\partial r^3}\right]\ell+pq+\frac{\partial p}{\partial r}\int q\,d\ell+\frac{1}{2}q^2-\frac{2}{3}\frac{\partial^2 q}{\partial\ell^2}+c_1,\tag{3.17}$$

$$2(\eta_1-u_1) = \left[2\frac{\partial q}{\partial t_1}-3q\frac{\partial q}{\partial\ell}-\frac{1}{3}\frac{\partial^3 q}{\partial\ell^3}\right]r+pq+\frac{\partial q}{\partial\ell}\int p\,dr+\frac{1}{2}p^2-\frac{2}{3}\frac{\partial^2 p}{\partial r^2}+c_2,\tag{3.18}$$

where c_1 and c_2 may depend on all variables but not ℓ and r. All the terms on the right-hand sides are bounded, except for the first terms, which are secular. This growth is unphysical, as the quantities on the left-hand side are bounded. Thus we impose that the secular terms vanish. This gives

$$2\frac{\partial p}{\partial t_1} + 3p\frac{\partial p}{\partial r} + \frac{1}{3}\frac{\partial^3 p}{\partial r^3} = 0, \tag{3.19}$$

$$2\frac{\partial q}{\partial t_1} - 3q\frac{\partial q}{\partial \ell} - \frac{1}{3}\frac{\partial^3 q}{\partial \ell^3} = 0. \tag{3.20}$$

Equation (3.19) is for the right-going wave and (3.20) for the left-going wave. These are KdV-type equations. Therefore, to obtain η_1 and u_1 from (3.17) and (3.18), we get after some algebra:

$$\begin{aligned} \eta_1 &= \frac{1}{4}\left[2pq + \frac{\partial p}{\partial r}\int q\,d\ell + \frac{\partial q}{\partial \ell}\int p\,dr + \frac{1}{2}\left(p^2 + q^2\right) - \frac{2}{3}\left(\frac{\partial^2 q}{\partial \ell^2} + \frac{\partial^2 p}{\partial r^2}\right) + c_3\right], \\ u_1 &= \frac{1}{4}\left[\frac{\partial p}{\partial r}\int q\,d\ell - \frac{\partial q}{\partial \ell}\int p\,dr + \frac{1}{2}\left(q^2 - p^2\right) - \frac{2}{3}\left(\frac{\partial^2 q}{\partial \ell^2} - \frac{\partial^2 p}{\partial r^2}\right) + c_4\right], \end{aligned} \tag{3.21}$$

where $c_3 = c_1 + c_2$ and $c_4 = c_1 - c_2$.

Now we try to find the solutions of (3.19) and (3.20) by using the following approximate method.

4. Fundamentals of Modified Laplace Decomposition Method

In this section, a brief outline of MLDM is explained. For this, we consider the general nonlinear partial differential equation of first order (without loss of generality) in the following form:

$$Lu(x,t) + Ru(x,t) + N(u(x,t)) = h(x,t), \tag{4.1}$$

with the following initial condition:

$$u(x,0) = f(x), \tag{4.2}$$

where L is the first-order differential operator, $L = \partial/\partial t$, R is linear differential operator, $N(u)$ presents the nonlinear term, and $h(x,t)$ is the source term. The methodology consists of applying Laplace transform first on both sides of (4.1)

$$£[Lu(x,t)] + £[Ru(x,t)] + £[N(u(x,t))] = £[h(x,t)]. \tag{4.3}$$

Using the differentiation property of Laplace transform, we get

$$s£[u(x,t)] - f(x) + £[Ru(x,t)] + £[N(u(x,t))] = £[h(x,t)]. \tag{4.4}$$

Now, we will define the solution $u(x,t)$ by the series in the following form:

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \tag{4.5}$$

and the nonlinear operator $N(u)$ represented by an infinite series of the so-called Adomian's polynomials:

$$N(u)=\sum_{n=0}^{\infty}A_n, \tag{4.6}$$

where $u_n(x,t),\ n\geq 0$ are the components of $u(x,t)$ that will be elegantly determined and A_n are called Adomian's polynomials and defined by

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n\geq 0. \tag{4.7}$$

Using (4.5) and (4.6) in (4.4), we get

$$£\left[\sum_{n=0}^{\infty}u_n(x,t)\right]=\frac{1}{s}f(x)+\frac{1}{s}£[h(x,t)]-\frac{1}{s}£\left[R\sum_{n=0}^{\infty}u_n(x,t)\right]+\frac{1}{s}£\left[\sum_{n=0}^{\infty}A_n\right]. \tag{4.8}$$

On comparing both sides of (4.8), we have

$$\begin{aligned}
£[u_0(x,t)]&=\frac{1}{s}f(x)+\frac{1}{s}£[h(x,t)],\\
£[u_1(x,t)]&=-\frac{1}{s}£[Ru_0(x,t)]+\frac{1}{s}£[A_0],\\
£[u_2(x,t)]&=-\frac{1}{s}£[Ru_1(x,t)]+\frac{1}{s}£[A_1],
\end{aligned} \tag{4.9}$$

in general, the recursive relation is given by

$$£[u_{n+1}(x,t)]=-\frac{1}{s}£[Ru_n(x,t)]+\frac{1}{s}£[A_n], \quad n\geq 0. \tag{4.10}$$

From the above considerations, and applying inverse Laplace transform, the components $u_n(x,t)$ for $n\geq 0$ can be obtained by the following recursive relationship:

$$u_0(x,t)=K(x,t), \tag{4.11}$$

$$u_{n+1}(x,t)=-£^{-1}\left[\frac{1}{s}£[Ru_n(x,t)]+\frac{1}{s}£[A_n]\right], \quad n\geq 0, \tag{4.12}$$

where $K(x,t)$ represents the term arising from source term and prescribe initial conditions. This will enable us to determine the components $u_n(x,t)$ recurrently. However, in many cases the exact solution in a closed form may be obtained. For numerical comparisons purpose, we construct the solution $u(x,t)$ such that

$$\lim_{n\to\infty} U_n(x,t) = u(x,t), \quad \text{where } U_n(x,t) = \sum_{i=0}^{n-1} u_i(x,t), \quad n \geq 0. \tag{4.13}$$

For more details about LDM and its convergence, see for example, [33, 34].

Now, to modify this method, we assume that $K(x,t)$ can be divided into the sum of two parts namely, $K_0(x,t)$ and $K_1(x,t)$; therefore, we get

$$K(x,t) = K_0(x,t) + K_1(x,t). \tag{4.14}$$

Under this assumption, we propose a slight variation only in the components u_0, u_1. The variation we propose is that only the part $K_0(x,t)$ be assigned to u_0, whereas the remaining part $K_1(x,t)$ be combined with the other terms given in (4.12) to define u_1. In view of these suggestion, the components $u_n(x,t)$ for $n \geq 0$, can obtain by the following recursive relationship:

$$\begin{aligned} u_0(x,t) &= K_0(x,t), \\ u_1(x,t) &= K_1(x,t) - £^{-1}\left[\frac{1}{s}£[Ru_0(x,t)] + \frac{1}{s}£[A_0]\right], \\ u_{n+1}(x,t) &= -£^{-1}\left[\frac{1}{s}£[Ru_n(x,t)] + \frac{1}{s}£[A_n]\right], \quad n \geq 1. \end{aligned} \tag{4.15}$$

The solution through the modified Adomian decomposition method is highly dependent upon the choice of $K_0(x,t)$ and $K_1(x,t)$.

5. Solution Procedure Using MLDM

In this section, we will implement MLDM to the problem of the nonlinear water wave, KdV equations (3.19)-(3.20), with the initial conditions:

$$p(r,0) = \frac{1}{9}k^2\left[7 - 12\tanh^2 kr\right], \qquad q(\ell,0) = \frac{1}{9}k^2\left[7 - 12\tanh^2 k\ell\right]. \tag{5.1}$$

First, we apply the Laplace transform of (3.19)-(3.20) and use the given initial condition as follows:

$$\begin{aligned} sp(r,s) - \frac{1}{9}k^2\left[7 - 12\tanh^2 kr\right] + \frac{1}{6}£\left[\frac{\partial^3 p}{\partial r^3}\right] + £\left[\frac{3}{2}p\frac{\partial p}{\partial r}£\right] &= 0, \\ sq(\ell,s) - \frac{1}{9}k^2\left[7 - 12\tanh^2 k\ell\right] - \frac{1}{6}£\left[\frac{\partial^3 q}{\partial \ell^3}\right] - £\left[\frac{3}{2}q\frac{\partial q}{\partial \ell}£\right] &= 0. \end{aligned} \tag{5.2}$$

Applying inverse Laplace transform, we get

$$p(r,t_1) = \frac{1}{9}k^2\left[7 - 12\tanh^2 kr\right] - £^{-1}\left[\frac{1}{6s}£\left[\frac{\partial^3 p}{\partial r^3}\right]\right] - £^{-1}\left[\frac{3}{2s}£\left[p\frac{\partial p}{\partial r}\right]\right],$$
$$q(\ell,t_1) = \frac{1}{9}k^2\left[7 - 12\tanh^2 k\ell\right] + £^{-1}\left[\frac{1}{6s}£\left[\frac{\partial^3 q}{\partial \ell^3}\right]\right] + £^{-1}\left[\frac{3}{2s}£\left[q\frac{\partial q}{\partial \ell}\right]\right]. \quad (5.3)$$

The second step of the proposed method suggests that the solution can be decomposed by an infinite series of components in the following form:

$$p(r,t_1) = \sum_{n=0}^{\infty} p_n(r,t_1), \qquad q(\ell,t_1) = \sum_{n=0}^{\infty} q_n(\ell,t_1), \quad (5.4)$$

and the nonlinear terms which appeared in (5.3), $N_1(p) = p(\partial p/\partial r)$, $N_2(q) = q(\partial q/\partial \ell)$, decomposed by the infinite series:

$$N_1(p) = \sum_{n=0}^{\infty} A_n, \qquad N_2(q) = \sum_{n=0}^{\infty} B_n, \quad (5.5)$$

where $p_n(r,t_1)$ and $q_n(\ell,t_1)$, $n \geq 0$, are the components of that will be elegantly determined and A_n and B_n, $n \geq 0$ are called Adomian's polynomials defined by (4.7). Now, by substituting from (5.4), (5.5) in (5.3), we can obtain the following recurrence relations:

$$p_0(r,t_1) = \frac{7}{9}k^2,$$
$$q_0(\ell,t_1) = \frac{7}{9}k^2,$$
$$p_1(r,t_1) = \frac{-12}{9}k^2\left[\tanh^2 kr\right] - £^{-1}\left[\frac{1}{6s}£\left[\frac{\partial^3 p_0}{\partial r^3}\right]\right] - £^{-1}\left[\frac{3}{2s}£[A_0]\right],$$
$$q_1(\ell,t_1) = \frac{-12}{9}k^2\left[\tanh^2 k\ell\right] + £^{-1}\left[\frac{1}{6s}£\left[\frac{\partial^3 q_0}{\partial \ell^3}\right]\right] + £^{-1}\left[\frac{3}{2s}£[B_0]\right], \quad (5.6)$$
$$p_{n+1}(r,t_1) = -£^{-1}\left[\frac{1}{6s}£\left[\frac{\partial^3 p_n}{\partial r^3}\right]\right] - £^{-1}\left[\frac{3}{2s}£[A_n]\right], \quad n \geq 1,$$
$$q_{n+1}(\ell,t_1) = £^{-1}\left[\frac{1}{6s}£\left[\frac{\partial^3 q_n}{\partial \ell^3}\right]\right] + £^{-1}\left[\frac{3}{2s}£[B_n]\right], \quad n \geq 1.$$

This will enable us to determine the components recurrently. However, in many cases the exact solution in a closed form may be obtained. For numerical comparison purpose, we construct the solutions such that:

$$\lim_{n\to\infty}\Theta_n(r,t_1)=p(r,t_1),\qquad \lim_{n\to\infty}\Omega_n(\ell,t_1)=q(\ell,t_1),\tag{5.7}$$

where $\Theta_n(r,t_1)\simeq\sum_{i=0}^{n-1}p_i(r,t_1)$, $\Omega_n(\ell,t_1)\simeq\sum_{i=0}^{n-1}q_i(\ell,t_1), n\geq 0$.

To find the first components of the solution of (3.19)-(3.20) using MLDM, we calculate the first Adomian's polynomials using (4.7) as follows:

$$\begin{aligned}
A_0&=p_0\frac{\partial p_0}{\partial r}, & B_0&=q_0\frac{\partial q_0}{\partial \ell},\\
A_1&=p_1\frac{\partial p_0}{\partial r}+p_0\frac{\partial p_1}{\partial r}, & B_1&=q_1\frac{\partial q_0}{\partial \ell}+q_0\frac{\partial q_1}{\partial \ell},\\
A_2&=p_2\frac{\partial p_0}{\partial r}+p_0\frac{\partial p_2}{\partial r}+p_1\frac{\partial p_1}{\partial r}, & B_2&=q_2\frac{\partial q_0}{\partial \ell}+q_0\frac{\partial q_2}{\partial \ell}+q_1\frac{\partial q_1}{\partial \ell}.
\end{aligned}\tag{5.8}$$

Other polynomials can be obtained in a like manner. Then by using the recurrence relations (5.6), we can obtain directly the first components of the solution in the following form:

$$\begin{aligned}
p_0(r,t_1)&=\frac{7}{9}k^2,\\
q_0(\ell,t_1)&=\frac{7}{9}k^2,\\
p_1(r,t_1)&=\frac{-12}{9}k^2\left[\tanh^2 kr\right],\\
q_1(\ell,t_1)&=\frac{-12}{9}k^2\left[\tanh^2 k\ell\right],\\
p_2(r,t_1)&=t_1\left[\frac{4}{9}k^5\operatorname{sech}^2(kr)\tanh(kr)\left(7-12\tanh^2(kr)\right)\right.\\
&\qquad\left.+\frac{2}{9}k^2\left(-16k^3\operatorname{sech}^4(kr)\tanh(kr)+8k^3\operatorname{sech}^2(kr)\tanh^3(kr)\right)\right],\\
q_2(\ell,t_1)&=t_1\left[-\frac{4}{9}k^5\operatorname{sech}^2(k\ell)\tanh(k\ell)\left(7-12\tanh^2(k\ell)\right)\right.\\
&\qquad\left.-\frac{2}{9}k^2\left(-16k^3\operatorname{sech}^4(k\ell)\tanh(k\ell)+8k^3\operatorname{sech}^2(k\ell)\tanh^3(kr)\right)\right].
\end{aligned}\tag{5.9}$$

Other solutions can be obtained in a like manner.

6. Convergence Analysis of Standard ADM

In this section, we study the convergence analysis of ADM to the solution when applied to the problem (3.19). Let us define the Hilbert space $H = L^2((\alpha,\beta) \times [0,T])$ as a set of all applications:

$$p : (\alpha,\beta) \times [0,T] \longrightarrow \Re \qquad \text{with} \int_{(\alpha,\beta)\times[0,T]} \left|p(r,s)\right|^2 ds\, d\tau < \infty. \tag{6.1}$$

Let us consider $L(p) = \partial p/\partial t_1$; then we can rewrite (3.19) in the following operator form:

$$\frac{\partial p}{\partial t_1} = -\frac{3}{2} p \frac{\partial p}{\partial r} - \frac{1}{6}\frac{\partial^3 p}{\partial r^3}. \tag{6.2}$$

Theorem 6.1. *The ADM applied to the nonlinear equation* (3.19) *converges towards a particular solution if the following two hypotheses are satisfied.*

(H1) $(L(p) - L(u), p-u) \geq m\|p-u\|^2, m > 0$, *for all* $p, u \in H$;

(H2) *there exist* $C(K) > 0$, $K > 0$, *such that for all* $p, u \in H$ *with* $\|p\| \leq K$, $\|u\| \leq K$, *we have* $(L(p) - L(u), w) \leq C(K)\|p-u\|\|w\|$ *for all* $w \in H$.

Proof. To verify (H1) for the operator $L(p)$, we have

$$L(p) - L(u) = -\frac{3}{2}\left(p\frac{\partial p}{\partial r} - u\frac{\partial u}{\partial r}\right) - \frac{1}{6}\frac{\partial^3}{\partial r^3}(p-u) = -\frac{3}{4}\frac{\partial}{\partial r}\left(p^2 - u^2\right) - \frac{1}{6}\frac{\partial^3}{\partial r^3}(p-u). \tag{6.3}$$

Then we claim that

$$(L(p) - L(u), p-u) = \frac{3}{4}\left(-\frac{\partial}{\partial r}\left(p^2-u^2\right), p-u\right) + \frac{1}{6}\left(-\frac{\partial^3}{\partial r^3}(p-u), p-u\right). \tag{6.4}$$

Since $\partial^3/\partial r^3$ is differential operator in H, then there exist constant c_1 such that

$$\left(-\frac{\partial^3}{\partial r^3}(p-u), p-u\right) \geq c_1\|p-u\|^2, \tag{6.5}$$

and according to the Schwartz inequality, we can obtain

$$\left(\frac{\partial}{\partial r}\left(p^2-u^2\right), p-u\right) \leq c_2\left\|p^2-u^2\right\| \cdot \|p-u\|. \tag{6.6}$$

Now, using the mean value theorem, we have

$$\begin{aligned}\left(\frac{\partial}{\partial r}\left(p^2-u^2\right), p-u\right) &\leq c_2\left\|p^2-u^2\right\| \cdot \|p-u\| \\ &= 2c_2\mu\|p-u\|^2 \leq 2c_2K\|p-u\|^2,\end{aligned} \tag{6.7}$$

where $p<\mu<u$ and $\|p\|<K,\ \|u\|<K$. Therefore,

$$\left(-\frac{\partial}{\partial r}\left(p^2-u^2\right), p-u\right) \geq 2c_2K\|p-u\|^2. \tag{6.8}$$

Substituting from (6.5)–(6.8) into (6.4), we get

$$(L(p)-L(u), p-u) \geq \left(\frac{3}{2}c_2\mu K+\frac{1}{6}c_1\right)\|p-u\|^2 = m\|p-u\|^2, \tag{6.9}$$

where $m=(3/2)c_2\mu K+(1/6)c_1$. Hence, we verified (H1).

To verify (H2) for the operator $L(p)$, we have

$$(L(p)-L(u), w) = \frac{3}{4}\left(-\frac{\partial}{\partial r}\left(p^2-u^2\right), w\right)+\frac{1}{6}\left(-\frac{\partial^3}{\partial r^3}(p-u), w\right), \tag{6.10}$$

therefore,

$$(L(p)-L(u), w) \leq \left(\frac{3}{2}K\right)\|p-u\|\|w\|+\|p-u\|\|w\| = C(K)\|p-u\|\|w\|, \tag{6.11}$$

where $C(K)=1+(3/2)K$. Hence, we verified (H2) and the end of the proof. □

7. Numerical Study

Returning to dimensional variables, we obtain the components of the solution of MLDM as follows:

$$
\begin{aligned}
p_0(x,t) &= \frac{7}{9}k^2,\\
q_0(x,t) &= \frac{7}{9}k^2,\\
p_1(x,t) &= \frac{-12}{9}k^2\left[\tanh^2 k\left(\frac{\sqrt{\epsilon}}{h}x+\sqrt{\frac{g\epsilon}{h}}t\right)\right],\\
q_1(x,t) &= \frac{-12}{9}k^2\left[\tanh^2 k\left(\frac{\sqrt{\epsilon}}{h}x-\sqrt{\frac{g\epsilon}{h}}t\right)\right],\\
p_2(x,t) &= \epsilon\sqrt{\frac{g\epsilon}{h}}t\left[\frac{4}{9}k^5\operatorname{sech}^2\left(k\left(\frac{\sqrt{\epsilon}}{h}x+\sqrt{\frac{g\epsilon}{h}}t\right)\right)\tanh\left(k\left(\frac{\sqrt{\epsilon}}{h}x+\sqrt{\frac{g\epsilon}{h}}t\right)\right)\right.\\
&\quad\times\left(7-12\tanh^2\left(k\left(\frac{\sqrt{\epsilon}}{h}x+\sqrt{\frac{g\epsilon}{h}}t\right)\right)\right)\\
&\quad+\frac{2}{9}k^2\left(-16k^3\operatorname{sech}^4\left(k\left(\frac{\sqrt{\epsilon}}{h}x+\sqrt{\frac{g\epsilon}{h}}t\right)\right)\tanh\left(k\left(\frac{\sqrt{\epsilon}}{h}x+\sqrt{\frac{g\epsilon}{h}}t\right)\right)\right.\\
&\quad+8k^3\operatorname{sech}^2\left(k\left(\frac{\sqrt{\epsilon}}{h}x+\sqrt{\frac{g\epsilon}{h}}t\right)\right)\\
&\quad\left.\left.\times\tanh^3\left(k\left(\frac{\sqrt{\epsilon}}{h}x+\sqrt{\frac{g\epsilon}{h}}t\right)\right)\right)\right],\\
q_2(x,t) &= \epsilon\sqrt{\frac{g\epsilon}{h}}t\left[-\frac{4}{9}k^5\operatorname{sech}^2\left(k\left(\frac{\sqrt{\epsilon}}{h}x-\sqrt{\frac{g\epsilon}{h}}t\right)\right)\tanh\left(k\left(\frac{\sqrt{\epsilon}}{h}x+\sqrt{\frac{g\epsilon}{h}}t\right)\right)\right.\\
&\quad\times\left(7-12\tanh^2\left(k\left(\frac{\sqrt{\epsilon}}{h}x-\sqrt{\frac{g\epsilon}{h}}t\right)\right)\right)\\
&\quad-\frac{2}{9}k^2\left(-16k^3\operatorname{sech}^4\left(k\left(\frac{\sqrt{\epsilon}}{h}x-\sqrt{\frac{g\epsilon}{h}}t\right)\right)\tanh\left(k\left(\frac{\sqrt{\epsilon}}{h}x-\sqrt{\frac{g\epsilon}{h}}t\right)\right)\right.\\
&\quad+8k^3\operatorname{sech}^2\left(k\left(\frac{\sqrt{\epsilon}}{h}x-\sqrt{\frac{g\epsilon}{h}}t\right)\right)\\
&\quad\left.\left.\times\tanh^3\left(k\left(\frac{\sqrt{\epsilon}}{h}x-\sqrt{\frac{g\epsilon}{h}}t\right)\right)\right)\right].
\end{aligned}
\tag{7.1}
$$

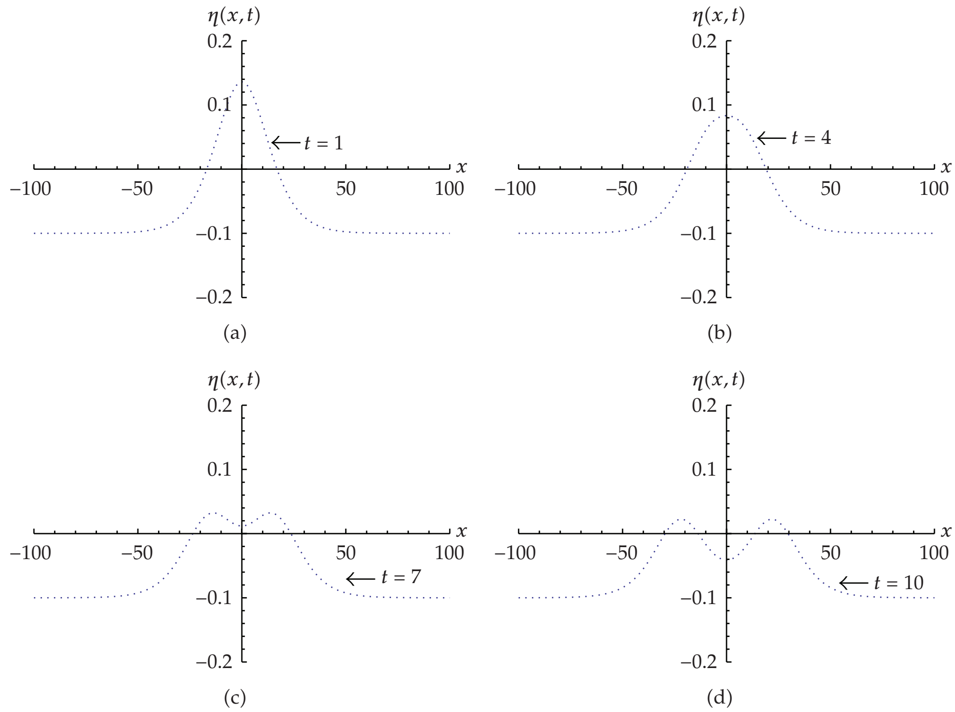

Figure 2: The water wave at $t = 1$. The water wave at $t = 4$. The water wave at $t = 7$. The water wave at $t = 10$.

In the same manner, we can obtain other components of the solution. In order to verify numerically whether the proposed methodology leads to higher accuracy, we can evaluate the numerical solutions using $n = 2$ term approximation.

$$p(x,t) \simeq p_0(x,t) + p_1(x,t) + p_2(x,t),$$
$$q(x,t) \simeq q_0(x,t) + q_1(x,t) + q_2(x,t). \tag{7.2}$$

We achieved a very good approximation with the actual solution of (3.19)-(3.20) by using two terms only of the iteration equations derived above. It is evident that the overall errors can be made smaller by adding new terms of the iteration formula. The numerical approximation shows a high degree of accuracy and in most cases $p_n(x,t), q_n(x,t)$, the n-term approximations are accurate for quite low values of n, and the solutions are very rapidly convergent by utilizing MLDM. The numerical results we obtained justify the advantage of this method, even in the few terms approximation is accurate. It must be noted that MLDM used here gives the possibility of obtaining an analytical satisfactory solution for which the other techniques of calculation are more laborious and the results contain a great complexity.

From the above solution process, we can see clearly that the approximate solutions converge to its exact solution. After returning to dimensional variables and substitution from (7.2) into (3.21), (3.12), (3.9), (3.6), and (2.1), we get the elevation of the water surface, the horizontal velocity, and the vertical velocity, which describes the physical situation of the system, where $k = 0.4$, $\sigma = 0.072\,\mathrm{N/m}$, $g = 9.8\,\mathrm{m/sec^2}$, $h = 0.5\,\mathrm{m}$, $\epsilon = 0.01$, and $y = h$. The water wave gradually splits into two solitary waves with increasing $t > 1$ in Figures 2(a)–2(d), which are in excellent agreement with the exact solution which were represented in Figures 5(a)–5(d).

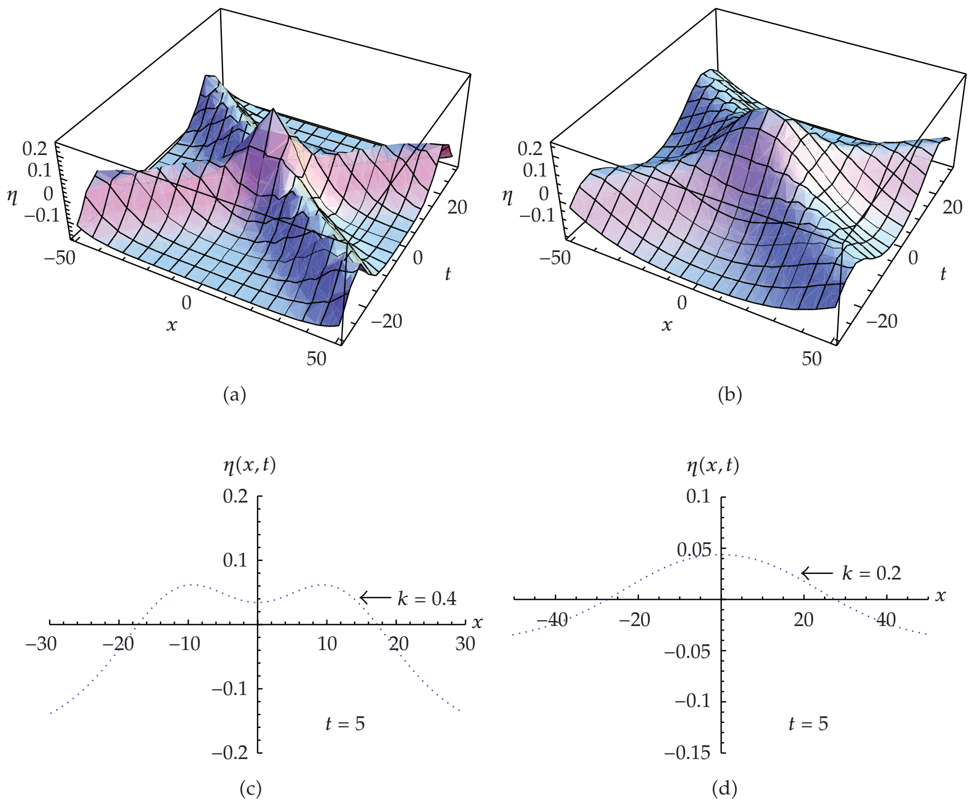

Figure 3: The water wave at $k = 0.4$. The water wave at $k = 0.2$. The water wave at $k = 0.4$. The water wave at $k = 0.2$.

In Figures 3(a) and 3(b), the elevation of the water waves $\eta(x,t)$ is always less than depth and acquires nonlinear solitary characters. We observe that the elevation of the water waves is in the form of traveling solitary waves; it increases in amplitude as the wave number increases k, as shown in Figures 3(a)–3(d), also the interaction of two equal-amplitude solitary waves by head-on collision is illustrated in Figures 3(a)–3(d) at different wave numbers, which are in excellent agreement with the exact solution which was represented in Figures 6(a)–6(d). The parts of the velocity components u and v also bring a nonlinear solitary characters as shown in Figures 4(a)–4(b), which are in excellent agreement with the exact solution which represented in Figures 7(a)–7(b).

8. Conclusions and Discussion

In this study, we present model equations for surface water waves by using a new method of multiple-scale technique. Multiple-scale technique is used to estimate the KdV equations for the nonlinear theory, describing the behaviour of the perturbed system. We observe that the method of multiple scale is one of the modern methods which were used to obtain the KdV equations because it is relatively short in mathematical calculation, more effective, and more enlightening. While the Hamiltonian expansions and the Dirichlet-Neumann operator

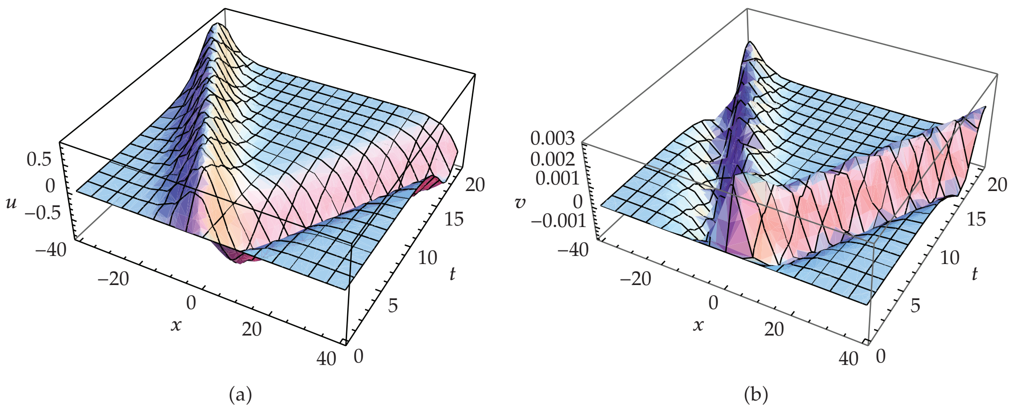

Figure 4: The horizontal velocity. The vertical velocity.

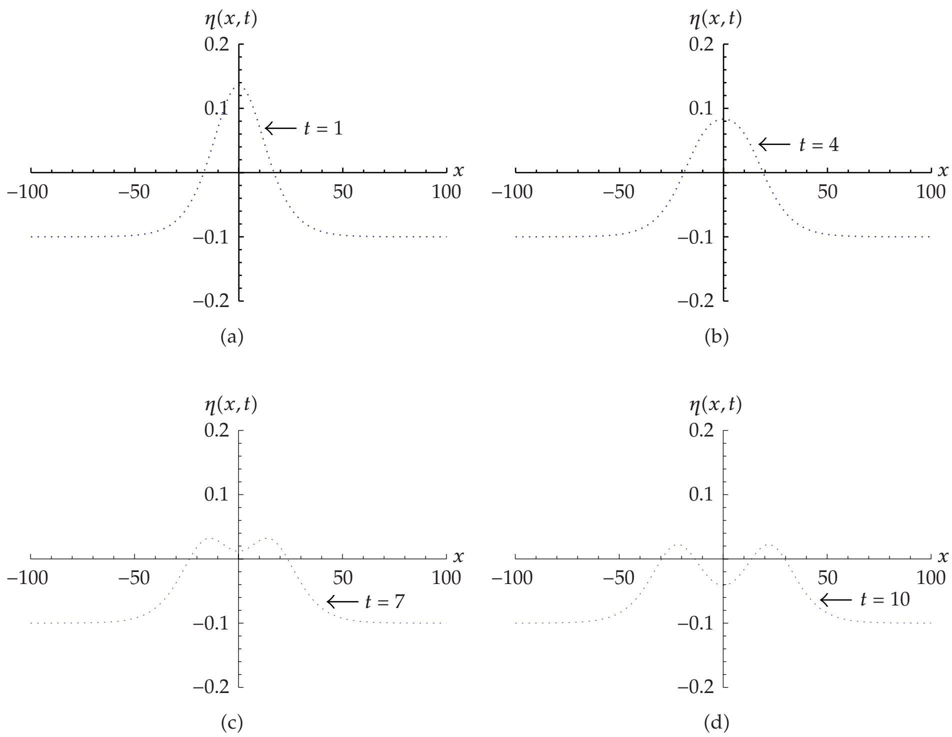

Figure 5: The water wave at $t = 1$. The water wave at $t = 4$. The water wave at $t = 7$. The water wave at $t = 10$.

expansions are complex in mathematical calculation and relatively a long method when compared with the method of multiple scale.

Special attention is given to derive the solutions for the KdV equations, which describe the water waves propagation by using the MLDM, and then analyzed and discussed theoretically and computationally.

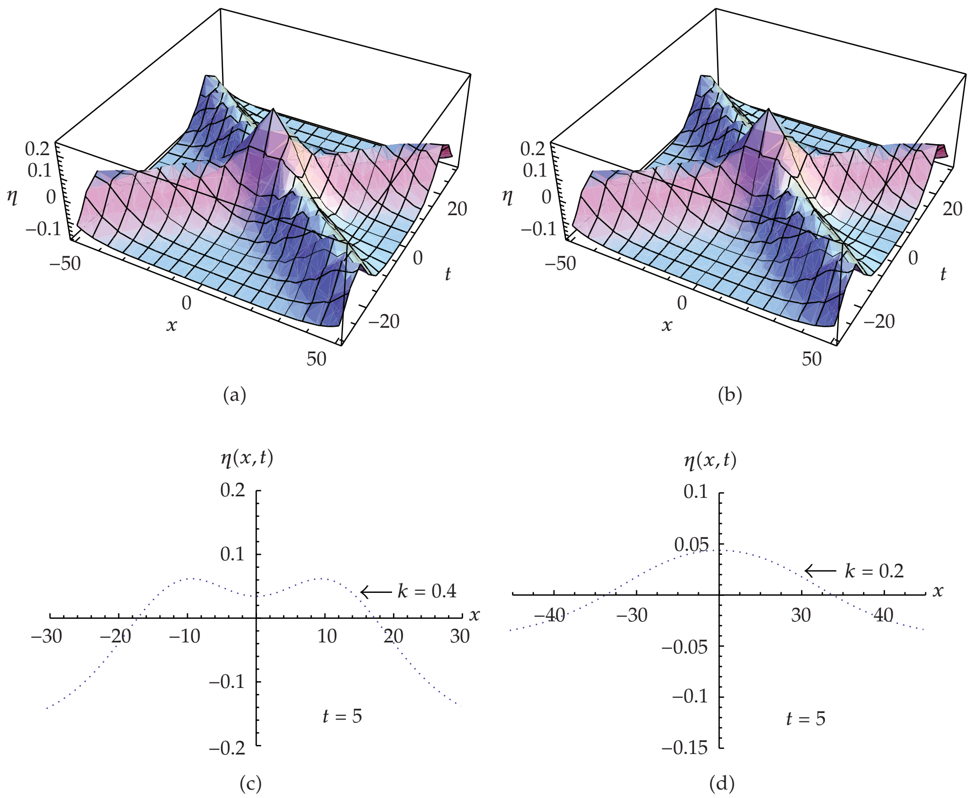

Figure 6: The water wave at $k = 0.4$. The water wave at $k = 0.2$. The water wave at $k = 0.4$. The water wave at $k = 0.2$.

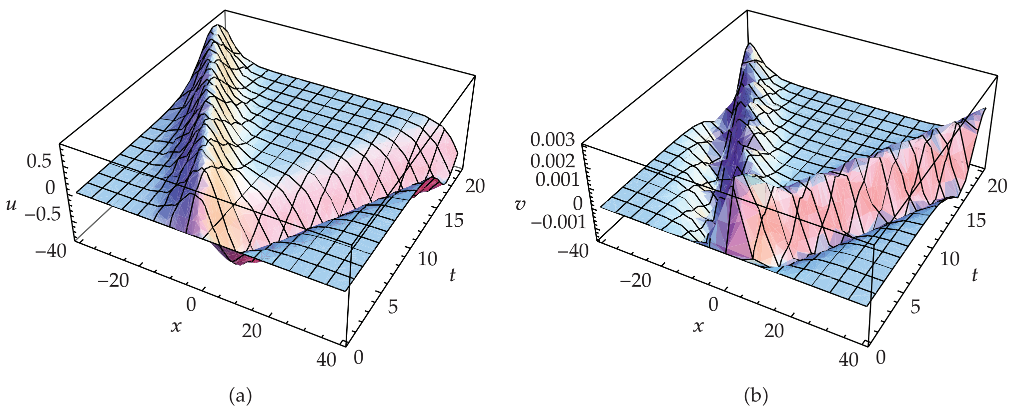

Figure 7: The horizontal velocity. The vertical velocity.

The diagrams that are drawn to illustrate the elevation of the water waves show a solitary character. We observe that the elevation of the water waves is in form of traveling solitary waves, which increases in amplitude as the wave number increases.

Finally, the horizontal and vertical velocities have a nonlinear characters, which describe the physical situation of the system for free surface between air and water.

The presented examples show that the results of the proposed method MLDM are in excellent agreement with exact solution. An interesting point about MLDM is that only few iterations or, even in some special cases, one iteration, lead to exact solutions or solutions with high accuracy. Special attention to prove the convergence analysis of the method is given. The main merits of MLDM are as follows:

(1) MLDM does not require small parameters which are needed in perturbation method;

(2) it is fast convergent;

(3) no linearization is needed; the method is very promising for solving wide application in nonlinear differential equations.

In our work, we use the Mathematica Package.

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