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Research Article

An Optimal Homotopy Asymptotic Approach Applied to Nonlinear MHD Jeffery-Hamel Flow

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A simple and effective procedure is employed to propose a new analytic approximate solution for nonlinear MHD Jeffery-Hamel flow. This technique called the Optimal Homotopy Asymptotic Method (OHAM) does not depend upon any small/large parameters and provides us with a convenient way to control the convergence of the solution. The examples given in this paper lead to the conclusion that the accuracy of the obtained results is growing along with increasing the number of constants in the auxiliary function, which are determined using a computer technique. The results obtained through the proposed method are in very good agreement with the numerical results.

1. Introduction

In various fields of science and engineering, nonlinear evolution equations, as well as their analytic and numerical solutions, are fundamentally important. The problem of an incompressible, viscous fluid between nonparallel walls with a sink or source at the vertex was pioneered by Jeffery [1] and Hamel [2]. Hamel mentioned an example of an exact nonsteady solution of the Navier-Stokes equations which describes the process of decay of a vortex through the action of the viscosity and considered the distribution of the tangential velocity component with respect to the radial distance and time and a particular case of the flow through a divergent channel was discussed and exactly solved. Jeffery-Hamel flows are exact similarity solution as the Navier-Stokes equations in the special case of two-dimensional flow through a channel with inclined plane walls meeting at a vertex with a source or sink at the vertex and have been studied by several authors and discussed in many text books and articles [3–5]. Sadri [6] denoted that Jeffery-Hamel flow used an asymptotic boundary condition to examine steady two-dimensional flow of a viscous fluid in a channel by means of certain symmetric solution of the flow although asymmetric solution are both possible and of physical interest [7].

The classical Jeffery-Hamel problem was extended in [8] to include the effects of external magnetic field in conducted fluid. The magnetic field acts as a control parameter, along with the flow, Reynolds number, and the angle of the walls.

Most scientific problems such as Jeffery-Hamel flows and other fluid mechanics problems are inherently nonlinear. Excepting a limited number of these problems, most do not have analytical solutions. Therefore, these nonlinear equations should be solved using other methods [9].

The aim of the present work is to propose an accurate approach to the Jeffery-Hamel flow problem using an analytical technique, namely, OHAM [10, 11].

The efficiency of our procedure, which does not require a small parameter in the equation, is based on the construction and determination of the auxiliary functions combined with a convenient way to optimally control the convergence of the solution.

2. Problem Statement and Governing Equation

We consider a system of cylindrical polar coordinates (r, θ, z) with a steady two-dimensional flow of an incompressible conducting viscous fluid from a source or sink at channel walls lying in planes, with angle 2α, as shown in Figure 1.

Assuming that the velocity is only along the radial direction and depends on r and θ, $V(u(r,\ \theta), 0)$ [3–5], using the continuity Navier-Stokes equations in polar coordinates, the governing equations are

$$\frac{\rho}{r}\frac{\partial}{\partial r}(ru(r,\theta)) = 0, \tag{2.1}$$

$$u(r,\theta)\frac{\partial u(r,\theta)}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + v\left[\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2} - \frac{u(r,\theta)}{r^2}\right], \tag{2.2}$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \rho} + \frac{2v}{r^2}\frac{\partial u(r,\theta)}{\partial \theta} = 0, \tag{2.3}$$

where ρ is the fluid density, p is the pressure, and v is the kinematic viscosity. From (2.1) and using dimensionless parameters we get

$$f(\theta) = ru(r,\theta), \tag{2.4}$$

$$F(x) = \frac{f(\theta)}{f_{\max}}, \qquad x = \frac{\theta}{\alpha}. \tag{2.5}$$

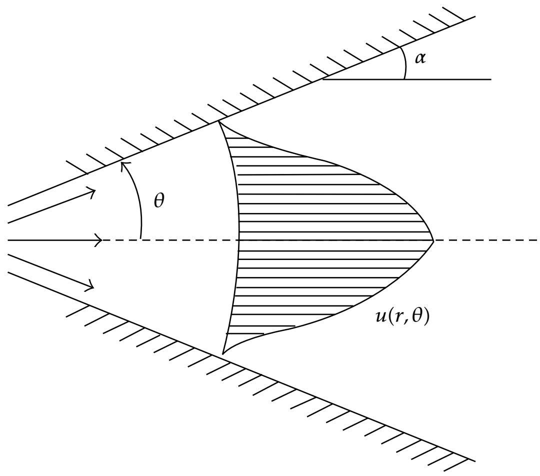

Figure 1: Geometry of the Jeffery-Hamel flow problem.

Substituting (2.5) into (2.2) and (2.3) and eliminating the pressure, we obtain an ordinary differential equation for the normalized function profile $F(x)$:

$$F'''(x) + 2\alpha \operatorname{Re} F(x)F'(x) + 4\alpha^2 F'(x) = 0, \tag{2.6}$$

where prime denotes derivative with respect to x and the Reynolds number is

$$\operatorname{Re} = \frac{\alpha f_{\max}}{v} = \frac{u_{\max}}{v}\begin{pmatrix}\text{divergent channel}: \ \alpha > 0, \ u_{\max} > 0 \\ \text{convergent channel}: \ \alpha < 0, \ \mathrm{u}_{\max} < 0\end{pmatrix} \tag{2.7}$$

and $u_{\max}$ is the maximum velocity at the centre of the channel.

The boundary conditions for (2.6) are

$$F(0) = 1, \qquad F'(0) = 0, \qquad F(1) = 0. \tag{2.8}$$

3. Fundamentals of the OHAM

We consider the following nonlinear differential equation [10, 11]:

$$L(F(x)) + f(x) + N(F(x)) = 0 \tag{3.1}$$

subject to a boundary condition

$$B(F) = 0, \tag{3.2}$$

where L is a linear operator, $f(x)$ is a known analytical function, N is a nonlinear operator, and B is a boundary operator. By means of the OHAM one constructs a family of equations:

$$(1-p)\left[L(\phi(x,p)) + f(x)\right] = h(x,p)\left[L(\phi(x,p)) + f(x) + N(\phi(x,p))\right], \tag{3.3}$$

and the boundary condition is

$$B(\phi(x,p)) = 0. \tag{3.4}$$

In (3.3), $\phi(x,p)$ is an unknown function, $p \in [0,1]$ is an embedding parameter, and $h(x,p)$ is an auxiliary function such that $h(x,0) = 0$ and $h(x,p) \neq 0$ for $p \neq 0$. When p increases from 0 to 1, the solution $\phi(x,p)$, changes from the initial approximation $F_0(x)$ to the solution $F(x)$. Obviously, when $p = 0$ and $p = 1$ it holds that

$$\phi(x,0) = F_0(x), \qquad \phi(x,1) = F(x). \tag{3.5}$$

Expanding $\phi(x,p)$ in series with respect to the parameter p, one has

$$\phi(x,p) = F_0(x) + pF_1(x) + p^2F_2(x) + \cdots . \tag{3.6}$$

If the initial approximation $F_0(x)$ and the auxiliary function $h(x,p)$ are properly chosen so that the series (3.6) converges at $p = 1$, one has

$$F(x) = F_0(x) + F_1(x) + F_2(x) + \ldots . \tag{3.7}$$

Notice that the series (3.6) contains the auxiliary function $h(x,p)$ which determines their convergence regions. The results of the mth-order approximations are given by

$$\overline{F}(x) \approx F_0(x) + F_1(x) + F_2(x) + \cdots + F_m(x). \tag{3.8}$$

We propose an auxiliary function $h(x,p)$ of the form

$$h(x,p) = pK_1(x) + p^2K_2(x) + \cdots + p^mK_m(x), \tag{3.9}$$

where $K_i(x)$, $i = 1,2,\ldots,m$ can be functions on the variable x.

Substituting (3.6) into (3.1) we obtain

$$\begin{aligned} L(F(x)) + f(x) + N(F(x)) = N_0(F_0(x)) &+ pN_1(F_0(x), F_1(x)) \\ &+ p^2N_2(F_0(x), F_1(x), F_2(x)) + \ldots . \end{aligned} \tag{3.10}$$

If we substitute (3.9) and (3.10) into (3.3) and we equate to zero the coefficients of various powers of p, we obtain the following linear equations:

$$L(F_0(x)) + f(x) = 0, \qquad B(F_0(x)) = 0, \tag{3.11}$$

$$L(F_i) - L(F_{i-1}) - \sum_{j=1}^{i} K_j N_{i-j}(F_0, F_1, \ldots, F_{i-j}) = 0, \qquad B(F_i) = 0, \quad i = 1, 2, \ldots, m-1 \tag{3.12}$$

$$L(F_m) - L(F_{m-1}) - \sum_{j=1}^{m-1} K_j N_{m-1-j} - K_m N_0 = 0, \qquad B(F_m) = 0. \tag{3.13}$$

At this moment, the mth-order approximate solution given by (3.8) depends on the functions K_1, K_2, …, K_m. The constants C_1, C_2, …, C_q which appear in the expression of $K_i(x)$ can be identified via various methodologies such as the least square method, the Galerkin method, and the collocation method.

The constants C_1, C_2,…,C_q could be determined, for example, if we substitute (3.8) into (3.1) resulting in the following residual:

$$R(x, C_i) = L\left(\overline{F}(x, C_i)\right) + f(x) + N\left(\overline{F}(x, C_i)\right), \quad i = 1, 2, \ldots. \tag{3.14}$$

For $x_i \in (a, b)$ where a and b are two values depending on the given problem and we substitute x_i into (3.14), we obtain the system of equations

$$R(x_1, C_i) = R(x_2, C_i) = \cdots = R(x_q, C_i), \quad i = 1, 2, \ldots, q, \tag{3.15}$$

where q is the number of constants C_i which appear in the expression of the functions $K_1(x)$, $K_2(x)$, …, $K_m(x)$.

One can observe that our procedure contains the auxiliary function $h(x, p)$ which provides us with a simple but rigorous way to adjust and control the convergence of the solution. It must be underlined that it is very important to properly choose the functions K_1, …, $K_m(x)$ which appear in the approximation (3.8).

4. Application of the Jeffery-Hamel Flow Problem

We introduce the basic ideas of the proposed method by considering (2.6) and (2.8). We choose $f(x) = 0$ and the linear operator

$$L(\phi(x, p)) = \frac{\partial^3 \phi(x, p)}{\partial x^3}. \tag{4.1}$$

The nonlinear operator is

$$N(\phi(x, p)) = 2\alpha \operatorname{Re}\ \phi(x, p)\frac{\partial \phi(x, p)}{\partial x} + 4\alpha^2 \frac{\partial \phi(x, p)}{\partial x}, \tag{4.2}$$

and the boundary conditions are

$$\phi(0, p) = 1, \qquad \frac{\partial \phi(0, p)}{\partial x} = 0, \quad \phi(1, p) = 0. \tag{4.3}$$

Equation (3.11) becomes

$$F_0''''(x) = 0,$$
$$F_0(0) = 1, \quad F_0'(0) = 0, \quad F_0(1) = 0. \tag{4.4}$$

It is obtained that

$$F_0(x) = 1 - x^2. \tag{4.5}$$

From (4.2) and (3.10), we obtain the following expression:

$$N_0(x) = F_0''''(x) + 2\alpha \operatorname{Re} \, F_0(x)F_0'(x) + 4\alpha^2 F_0'(x). \tag{4.6}$$

If we substitute (4.5) into (4.6), we obtain

$$N_0(x) = 4\alpha \operatorname{Re} x^3 - 4\left(\alpha \operatorname{Re} + 2\alpha^2\right)x. \tag{4.7}$$

There are many possibilities to choose the functions K_i, $i = 1, 2, \ldots$. The convergence of the solutions F_i, $i = 1, 2, \ldots m$ and consequently the convergence of the approximate solution $\overline{F}(x)$ given by (3.8) depend on the auxiliary functions K_i. Basically, the shape of $K_i(x)$ should follow the terms appearing in (4.7), (3.12), and (3.13) which are polynomial functions. We consider the following cases ($m = 2$).

Case 1. If $K_1(x)$ is of the form

$$K_1(x) = C_1, \tag{4.8}$$

where C_1 is an unknown constant at this moment, then (3.12) for $i = 1$ becomes

$$F_1''''(x) - F_0''''(x) - K_1 N_0(F_0) = 0. \tag{4.9}$$

Substituting (4.5), (4.7), and (4.8) into (4.9), we obtain the equation in F_1:

$$F_1''''(x) - 4C_1\alpha \operatorname{Re} \, x^3 - 4C_1\left(\alpha \operatorname{Re} \, + 2\alpha^2\right)x = 0, \qquad F_1(0) = F_1'(0) = F_1(1) = 0. \tag{4.10}$$

The solution of (4.10) is given by

$$F_1(x) = \frac{C_1\alpha \operatorname{Re}}{30}x^6 - \frac{C_1\left(\alpha \operatorname{Re} + 2\alpha^2\right)}{6}x^4 + \frac{2\alpha \operatorname{Re} + 5\alpha^2}{15}C_1 x^2. \tag{4.11}$$

Equation (3.13) for $m = 2$ can be written in the form

$$F_2''''(x) - F_1''''(x) - C_1 N_1(F_0, F_1) - K_2(x) N_0(F_0) = 0, \tag{4.12}$$

where N_1 is obtained from (3.10):

$$N_1(F_0, F_1) = F_1''' + 2\alpha \operatorname{Re}\left(F_0 F_1' + F_0' F_1\right) + 4\alpha^2 F_1'. \tag{4.13}$$

If we consider

$$K_2(x) = C_2, \tag{4.14}$$

where C_2 is an unknown constant, then from (4.5), (4.11), (4.12), (4.13), and (4.14) we obtain the following equation in F_2:

$$\begin{aligned} &F_2''' + \frac{8\alpha^2\operatorname{Re}^2 C_1^2}{15}x^7 - \frac{12\alpha^2\operatorname{Re}^2 + 24\alpha^2\operatorname{Re}}{5}C_1^2 x^5 \\ &\quad - \left[4\alpha\operatorname{Re}(2C_1 + C_2) + \frac{60\alpha\operatorname{Re} + 36\alpha^2\operatorname{Re}^2 + 40\alpha^3\operatorname{Re} - 16\alpha^4}{15}C_1^2\right]x^3 \\ &\quad + \left[4\left(\alpha\operatorname{Re} + 2\alpha^2\right)(C_1 + C_2) + \frac{60\alpha\operatorname{Re} + 120\alpha^2 - 8\alpha^2\operatorname{Re}^2 - 36\alpha^3\operatorname{Re} - 40\alpha^4}{15}C_1^2\right]x = 0. \end{aligned} \tag{4.15}$$

So, the solution of (4.15) is given by

$$\begin{aligned} F_2(x) = &-\frac{\alpha^2\operatorname{Re}^2 C_1^2}{1350}x^{10} + \frac{\alpha^2\operatorname{Re}^2 + 2\alpha^2\operatorname{Re}}{140}C_1^2 x^8 \\ &+ \left[\frac{\alpha\operatorname{Re}(2C_1 + C_2)}{30} + \frac{15\alpha\operatorname{Re} + 9\alpha^2\operatorname{Re}^2 + 10\alpha^3\operatorname{Re} - 4\alpha^4}{450}C_1^2\right]x^6 \\ &+ \left[-\frac{\left(\alpha\operatorname{Re} + 2\alpha^2\right)(C_1 + C_2)}{6} + \frac{2\alpha^2\operatorname{Re}^2 + 9\alpha^3\operatorname{Re} + 10\alpha^4 - 15\alpha\operatorname{Re} - 3\alpha^2}{90}C_1^2\right]x^4 \\ &+ \left[\frac{2\alpha\operatorname{Re} + 5\alpha^2}{15}(C_1 + C_2) + \frac{2520\alpha\operatorname{Re} - 1932\alpha^4 + 360\alpha^2 - 919\alpha^2\operatorname{Re}^2 - 2580\alpha^3\operatorname{Re}}{18900}\right]x^2. \end{aligned} \tag{4.16}$$

The second-order approximate solution ($m = 2$) is obtained from (3.8)

$$\overline{F}(x) = F_0(x) + F_1(x) + F_2(x), \tag{4.17}$$

where F_0, F_1, and F_2 are given by (4.5), (4.11), and (4.16), respectively.

Case 2. In this case we consider

$$K_1(x) = C_1, \tag{4.18}$$

$$K_2(x) = C_2 x + C_3, \tag{4.19}$$

where C_1, C_2, and C_3 are unknown constants.

It is clear that the function F_1 is given by (4.11). Equations (3.13) or (4.12) becomes

$$F_2''' + \frac{8\alpha^2\mathrm{Re}^2C_1^2}{15}x^7 - \frac{12\alpha^2\mathrm{Re}^2 + 24\alpha^2\,\mathrm{Re}}{5}C_1^2x^5 - 4\alpha\,\mathrm{Re}\,C_3x^4$$
$$- \left[4\alpha\,\mathrm{Re}(C_1 + C_2) + \frac{60\alpha\,\mathrm{Re} - 36\alpha^2\mathrm{Re}^2 - 120\alpha^3\,\mathrm{Re} - 80\alpha^4}{15}C_1^2\right]x^3 + 4\left(\alpha\,\mathrm{Re} + 2\alpha^2\right)C_3x^2$$
$$+ \left[4\left(\alpha\,\mathrm{Re} + 2\alpha^2\right)(C_1 + C_2) + \frac{60\alpha\,\mathrm{Re} + 12\alpha^2 - 8\alpha^2\mathrm{Re}^2 - 36\alpha^3\,\mathrm{Re} - 40\alpha^4}{15}C_1^2\right]x = 0,$$
$$F_2(0) = F_2'(0) = F_2(1) = 0 \tag{4.20}$$

and has the solution

$$F_2(x) = -\frac{\alpha^2\mathrm{Re}^2C_1^2}{1350}x^{10} + \frac{\alpha^2\mathrm{Re}^2 + 2\alpha^2\,\mathrm{Re}}{140}C_1^2x^8 + \frac{2\alpha\,\mathrm{Re}}{105}C_3x^7$$
$$+ \left[\frac{\alpha\,\mathrm{Re}(2C_1 + C_2)}{30} + \frac{15\alpha\,\mathrm{Re} + 9\alpha^2\mathrm{Re}^2 + 10\alpha^3\,\mathrm{Re} - 4\alpha^4}{450}C_1^2\right]x^6 + \frac{\alpha\,\mathrm{Re} + 2\alpha^2}{15}C_3x^5$$
$$+ \left[-\frac{(\alpha\,\mathrm{Re} + 2\alpha^2)(C_1 + C_2)}{6} + \frac{2\alpha^2\mathrm{Re}^2 + 9\alpha^3\,\mathrm{Re} + 10\alpha^4 - 15\alpha\,\mathrm{Re} - 3\alpha^2}{90}C_1^2\right]x^4 \tag{4.21}$$
$$+ \left[\frac{2\alpha\,\mathrm{Re} + 5\alpha^2}{15}(C_1 + C_2) + \frac{5\alpha\,\mathrm{Re} + 14\alpha^2}{105}C_3\right.$$
$$\left. + \frac{2520\alpha\,\mathrm{Re} - 1932\alpha^4 + 360\alpha^2 - 919\alpha^2\,\mathrm{Re} - 2580\alpha^3\,\mathrm{Re}}{18900}C_1^2\right]x^2.$$

The second-order approximate solution becomes

$$\overline{F}(x) = F_0(x) + F_1(x) + F_2(x), \tag{4.22}$$

where F_0, F_1, and F_2 are given by (4.5), (4.11), and (4.21), respectively.

Case 3. In the third case we consider

$$K_1(x) = C_1 + C_2x,$$
$$K_2(x) = C_3 + C_4x + C_5x^2. \tag{4.23}$$

Equation (3.12) for $i = 1$ or (4.9) can be written as

$$F_1'''' - 4\alpha \,\mathrm{Re}\, C_2 x^4 - 4\alpha \,\mathrm{Re}\, C_1 x^3 + 4\left(\alpha \,\mathrm{Re} + 2\alpha^2\right) C_2 x^2 + 4\left(\alpha \,\mathrm{Re} + 2\alpha^2\right) C_1 x = 0,$$
$$F_1(0) = F_1'(0) = F_1(1) = 0. \tag{4.24}$$

From (4.24) we have

$$F_1(x) = \frac{2\alpha \,\mathrm{Re}\, C_2}{105} x^7 + \frac{\alpha \,\mathrm{Re}\, C_1}{30} x^6 - \frac{(\alpha \,\mathrm{Re} + 2\alpha^2) C_2}{15} x^5 - \frac{(\alpha \,\mathrm{Re} + 2\alpha^2) C_1}{6} x^4$$
$$+ \left[\frac{(\alpha \,\mathrm{Re} + 2\alpha^2)(5C_1 + 2C_2)}{30} - \frac{\alpha \,\mathrm{Re}(7C_1 + 4C_2)}{210} \right] x^2. \tag{4.25}$$

Equation (3.13) becomes

$$F_2''''(x) + \frac{12\alpha^2 \mathrm{Re}^2}{35} C_2^2 x^9 + \frac{92\alpha^2 \mathrm{Re}^2 C_1 C_2}{105} x^8 + \left(\frac{8\alpha^2 \mathrm{Re}^2 C_1^2}{15} - \frac{6\alpha^2 \mathrm{Re}^2 + 12\alpha^3 \,\mathrm{Re}}{5} C_2^2 \right) x^7$$
$$- \frac{18\alpha^2 \mathrm{Re}^2 + 36\alpha^3 \,\mathrm{Re}}{5} C_1 C_2 x^6$$
$$- \left[4\alpha \,\mathrm{Re}\, C_5 + \frac{12\alpha^2 \mathrm{Re}^2 + 24\alpha^3 \,\mathrm{Re}}{5} C_1^2 + \frac{12\alpha \,\mathrm{Re} - 2\alpha^2 \mathrm{Re}^2 - 8\alpha^3 \,\mathrm{Re} - 8\alpha^4}{3} C_2^2 \right] x^5$$
$$- \left[4\alpha \,\mathrm{Re}(C_2 + C_4) + \frac{120\alpha \,\mathrm{Re} - 46\alpha^2 \mathrm{Re}^2 - 160\alpha^3 \,\mathrm{Re} - 120\alpha^4}{15} C_1 C_2 \right.$$
$$\left. - \frac{40\alpha^2 \mathrm{Re}^2 + 112\alpha^3 \,\mathrm{Re}}{105} C_2^2 \right] x^4$$
$$- \left[4\alpha \,\mathrm{Re}(C_1 + C_3) - 4\left(\alpha \,\mathrm{Re} + 2\alpha^2\right) C_3 + \frac{60\alpha \,\mathrm{Re} - 36\alpha^2 \mathrm{Re}^2 - 120\alpha^3 \,\mathrm{Re} - 80\alpha^4}{15} C_1^2 \right.$$
$$\left. - \frac{40\alpha^2 \mathrm{Re}^2 + 112\alpha^3 \,\mathrm{Re}}{105} C_1 C_2 - 4\left(\alpha \,\mathrm{Re} + 2\alpha^2\right) C_2^2 \right] x^3$$
$$+ \left[4\left(\alpha \,\mathrm{Re} + 2\alpha^2\right)(C_2 + C_4) + \frac{120\alpha \,\mathrm{Re} + 240\alpha^2 - 8\alpha^2 \mathrm{Re}^2 - 36\alpha^3 \,\mathrm{Re} - 40\alpha^4}{15} C_1 C_2 \right.$$
$$\left. + \frac{20\alpha^2 \mathrm{Re}^2 + 96\alpha^3 \,\mathrm{Re} + 112\alpha^4}{105} C_2^2 \right] x^2$$
$$+ \left[4\left(\alpha \,\mathrm{Re} + 2\alpha^2\right)(C_1 + C_3) + \frac{60\alpha \,\mathrm{Re} + 120\alpha^2 - 8\alpha^2 \mathrm{Re}^2 - 36\alpha^3 \,\mathrm{Re} - 40\alpha^4}{15} C_1^2 \right.$$
$$\left. - \frac{20\alpha^2 \mathrm{Re}^2 + 96\alpha^3 \,\mathrm{Re} + 112\alpha^4}{105} C_1 C_2 \right] x, \quad F_2(0) = F_2'(0) = F_2(1) = 0. \tag{4.26}$$

The solution of (4.26) is

$$
\begin{aligned}
F_2(x) = &-\frac{\alpha^2\mathrm{Re}^2C_2^2}{3850}x^{12} - \frac{46\alpha^2\mathrm{Re}^2C_1C_2}{51975}x^{11} + \left[-\frac{\alpha^2\mathrm{Re}^2C_1^2}{1350} + \frac{\alpha^2\mathrm{Re}^2 + 2\alpha^3\,\mathrm{Re}}{600}C_2^2\right]x^{10}\\
&+\frac{\alpha^2\mathrm{Re}^2 + 2\alpha^3\,\mathrm{Re}}{420}C_1C_2x^9\\
&+\left[\frac{\alpha\,\mathrm{Re}\,C_5}{84} + \frac{\alpha^2\mathrm{Re}^2 + 2\alpha^3\,\mathrm{Re}}{140}C_1^2 + \frac{6\alpha\,\mathrm{Re} - \alpha^2\mathrm{Re}^2 - 4\alpha^3\,\mathrm{Re} - 4\alpha^4}{504}C_2^2\right]x^8\\
&+\left[\frac{2\alpha\,\mathrm{Re}(C_2+C_4)}{105} + \frac{60\,\mathrm{Re} - 23\alpha^2\mathrm{Re}^2 - 80\alpha^3\,\mathrm{Re} - 60\alpha^4}{1575}C_1C_2\right.\\
&\qquad\left.-\frac{20\alpha^2\mathrm{Re}^2 + 56\alpha^3\,\mathrm{Re}}{11025}C_2^2\right]x^7\\
&+\left[\frac{\alpha\,\mathrm{Re}(C_1+C_3)}{30} - \frac{(\alpha\,\mathrm{Re} + 2\alpha^2)C_5}{30} + \frac{15\alpha\,\mathrm{Re} - 9\alpha^2\mathrm{Re}^2 - 30\alpha^3\,\mathrm{Re} - 20\alpha^4}{450}C_1^2\right.\\
&\qquad\left.+\frac{5\alpha^2\mathrm{Re}^2 + 14\alpha^3\,\mathrm{Re}}{1575}C_1C_2 - \frac{\alpha\,\mathrm{Re} + 2\alpha^2}{30}C_2^2\right]x^6\\
&-\left[\frac{(\alpha\,\mathrm{Re} + 2\alpha^2)(C_2+C_4)}{15} + \frac{30\alpha\,\mathrm{Re} + 60\alpha^2 - 2\alpha^2\mathrm{Re}^2 - 9\alpha^3\,\mathrm{Re} - 10\alpha^4}{225}C_1C_2\right.\\
&\qquad\left.-\frac{5\alpha^2\mathrm{Re}^2 + 24\alpha^3\,\mathrm{Re} + 28\alpha^4}{1575}C_2^2\right]x^5\\
&-\left[\frac{(\alpha\,\mathrm{Re} + 2\alpha^2)(C_1+C_3)}{6} + \frac{15\alpha\,\mathrm{Re} + 30\alpha^2 - 2\alpha^2\mathrm{Re}^2 - 9\alpha^3\,\mathrm{Re} - 10\alpha^4}{90}C_1^2\right.\\
&\qquad\left.-\frac{5\alpha^2\mathrm{Re}^2 + 24\alpha^3\,\mathrm{Re} + 28\alpha^4}{630}C_1C_2\right]x^4\\
&+\left[\frac{(2\alpha\,\mathrm{Re} + 5\alpha^2)(C_1+C_3)}{15} + \frac{(5\alpha\,\mathrm{Re} + 14\alpha^2)(C_2+C_4)}{105} + \frac{9\alpha\,\mathrm{Re} + 28\alpha^2}{420}C_5\right.\\
&\qquad+\frac{2520\alpha\,\mathrm{Re} - 1260\alpha^4 + 6300\alpha^2 - 163\alpha^2\mathrm{Re}^2 - 900\alpha^3\,\mathrm{Re}}{18900}C_1^2\\
&\qquad+\frac{19800\alpha\,\mathrm{Re} + 55440\alpha^2 - 1433\alpha^2\mathrm{Re}^2 - 8514\alpha^3\,\mathrm{Re} - 10560\alpha^4}{207900}C_1C_2\\
&\qquad\left.-\frac{4774\alpha^4 - 32340\alpha^2 - 10395\alpha\,\mathrm{Re} + 2695\alpha^3\,\mathrm{Re} + 380\alpha^2\mathrm{Re}^2}{485100}C_2^2\right]x^2.
\end{aligned}
\tag{4.27}
$$

The second-order approximate solution is

$$\overline{F}(x) = F_0(x) + F_1(x) + F_2(x), \tag{4.28}$$

where F_0, F_1, and F_2 are given by (4.5), (4.24), and (4.26), respectively.

Case 4. In the last case, we consider

$$K_1(x) = C_1 + C_2 x, \tag{4.29}$$

$$K_2(x) = C_3 + C_4 x + C_5 x^2 + C_6 x^3 + C_7 x^4 + C_8 x^5 + C_9 x^6. \tag{4.30}$$

The solution of $F_1(x)$ is given by (4.24). On the other hand, (3.13) has the solution

$$\begin{aligned}
F_2(x) = {}& \left(\frac{\alpha \operatorname{Re}}{330} C_9 - \frac{\alpha^2 \operatorname{Re}^2 C_2^2}{3850}\right) x^{12} + \left(\frac{2\alpha \operatorname{Re}}{495} C_8 - \frac{46\alpha^2 \operatorname{Re}^2}{51975} C_1 C_2\right) x^{11} \\
& + \left[\frac{\alpha \operatorname{Re} C_7 - (\operatorname{Re} + 2\alpha^2) C_9}{180} - \frac{\alpha^2 \operatorname{Re}^2}{1350} C_1^2 + \frac{\alpha^2 \operatorname{Re}^2 + 2\alpha^3 \operatorname{Re}}{600} C_2^2\right] x^{10} \\
& + \left[\frac{\alpha \operatorname{Re} C_6 - (\alpha \operatorname{Re} + 2\alpha^2) C_8}{126} + \frac{\alpha^2 \operatorname{Re}^2 + 2\alpha^3 \operatorname{Re}}{420} C_1 C_2\right] x^9 \\
& + \left[\frac{\alpha \operatorname{Re} C_5 - (\alpha \operatorname{Re} + 2\alpha^2) C_7}{84} + \frac{\alpha^2 \operatorname{Re}^2 + 2\alpha^3 \operatorname{Re}}{140} C_1^2 \right. \\
& \qquad \left. + \frac{6\alpha \operatorname{Re} - \alpha^2 \operatorname{Re}^2 - 4\alpha^3 \operatorname{Re} - 4\alpha^4}{504} C_2^2\right] x^8 \\
& + \left[\frac{2\alpha \operatorname{Re} C_4 - 2(\alpha \operatorname{Re} + 2\alpha^2) C_6}{105} + \frac{2\alpha \operatorname{Re} C_2}{105} + \frac{60\alpha \operatorname{Re} - 23\alpha^2 \operatorname{Re}^2 - 80\alpha^3 \operatorname{Re} - 60\alpha^4}{1575} C_1 C_2 \right. \\
& \qquad \left. - \frac{20\alpha^2 \operatorname{Re}^2 + 56\alpha^3 \operatorname{Re}}{11025} C_2^2\right] x^7 \\
& + \left[\frac{\alpha \operatorname{Re}(C_1 + C_3) - (\alpha \operatorname{Re} + 2\alpha^2) C_5}{30} + \frac{15\alpha \operatorname{Re} - 9\alpha^2 \operatorname{Re}^2 - 30\alpha^3 \operatorname{Re} - 20\alpha^4}{450} C_1^2 \right. \\
& \qquad \left. - \frac{5\alpha^2 \operatorname{Re}^2 + 14\alpha^3 \operatorname{Re}}{1575} C_1 C_2 - \frac{(\alpha \operatorname{Re} + 2\alpha^2) C_2^2}{30}\right] x^6 \\
& + \left[\frac{2\alpha^2 \operatorname{Re}^2 + 9\alpha^3 \operatorname{Re} + 10\alpha^4 - 30\alpha \operatorname{Re} - 60\alpha^2}{225} C_1 C_2 - \frac{5\alpha^2 \operatorname{Re}^2 + 24\alpha^3 \operatorname{Re} + 28\alpha^4}{1575} C_2^2 \right. \\
& \qquad \left. - \frac{(\alpha \operatorname{Re} + 2\alpha^2)(C_2 + C_4)}{15}\right] x^5
\end{aligned}$$

$$+\left[\frac{2\alpha^2\mathrm{Re}^2+9\alpha^3\,\mathrm{Re}+10\alpha^4-15\alpha\,\mathrm{Re}-30\alpha^2}{90}C_1^2+\frac{5\alpha^2\mathrm{Re}^2+24\alpha^3\,\mathrm{Re}+28\alpha^4}{630}C_1C_2\right.$$
$$\left.-\frac{(\alpha\,\mathrm{Re}+2\alpha^2)(C_1+C_3)}{6}\right]x^4$$
$$+\left[\frac{5\alpha\,\mathrm{Re}+14\alpha^2}{105}(C_2+C_4)+\frac{2\alpha\,\mathrm{Re}+5\alpha^2}{15}C_1+\frac{2\alpha\,\mathrm{Re}+15\alpha^2}{15}C_3+\frac{9\alpha\,\mathrm{Re}+28\alpha^2}{420}C_5\right.$$
$$+\frac{7\alpha\,\mathrm{Re}+24\alpha^2}{630}C_6+\frac{4\alpha\,\mathrm{Re}+15\alpha^2}{630}C_7+\frac{27\alpha\,\mathrm{Re}+110\alpha^2}{6930}C_8$$
$$+\frac{2520\alpha\,\mathrm{Re}-1260\alpha^4+6300\alpha^2-163\alpha^2\mathrm{Re}^2-900\alpha^3\,\mathrm{Re}}{18900}C_1^2$$
$$+\frac{5\alpha\,\mathrm{Re}+22\alpha^2}{1980}C_9+\frac{10395\alpha\,\mathrm{Re}-380\alpha^2\mathrm{Re}^2+32340\alpha^2-2685\alpha^3\,\mathrm{Re}-4774\alpha^4}{485100}C_2^2$$
$$\left.+\frac{19800\alpha\,\mathrm{Re}-1443\alpha^2\mathrm{Re}^2+55440\alpha^2-8514\alpha^3\,\mathrm{Re}-10560\alpha^4}{207900}C_1C_2\right]x^2. \tag{4.31}$$

The second-order approximate solution in this case is given by

$$\overline{F}(x)=F_0(x)+F_1(x)+F_2(x), \tag{4.32}$$

where F_0, F_1, and F_2 are given by (4.5), (4.24), and (4.30), respectively.

5. Numerical Examples

In the following, using the algorithm described in Section 3, with the help of a computer program which implement the procedure presented above, we will obtain the convergence-control constants C_i and we will show that the error of the solution decreases when the number of terms in the auxiliary function $h(x,p)$ increases. Obviously, the computational effort increases along with increasing the number of convergence-control constants, but a significant improvement of the accuracy of results is observed.

Example 5.1. For Re = 50 and $\alpha = 5$ in Case 1 it is obtained two solutions for the constants C_1 and C_2:

(a) $C_1 = 0.017506079 \quad C_2 = -0.047286881$,

(b) $C_1 = -0.017506079 \quad C_2 = 0.022737435$

but the second-order approximate solution (4.16) is the same in both cases:

$$\overline{F}(x)\approx 1-1.767845893x^2+1.236877181x^4-0.619019693x^6+0.164176501x^8-0.014188096x^{10} \tag{5.1}$$

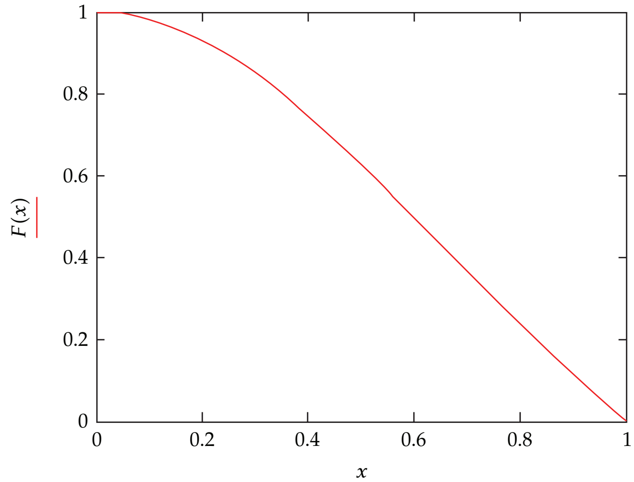

Figure 2: Sample profile of the $F(x)$ function for Re = 50 and $\alpha = 5$, given by (5.1).

Example 5.2. For Re = 50 and $\alpha = 5$ in the Case 2, we obtain

(a) $C_1 = 0.007977212 \quad C_2 = -0.041469373 \quad C_3 = 0.022761122$,

(b) $C_1 = -0.007977212 \quad C_2 = -0.009560525 \quad C_3 = 0.022761122$.

The second-order approximate solution (4.18) becomes

$$\overline{F}(x) \approx 1 - 1.769527092x^2 + 1.40514047x^4 - 0.45522244x^5 - 0.319921792x^6 + 0.108386295x^7 + 0.03409066x^8 - 0.002946101x^{10}. \tag{5.2}$$

Example 5.3. For Re = 50 and $\alpha = 5$ in Case 3, the second-order approximate solution (4.27) can be written in the form

$$\overline{F}(x) \approx 1 - 1.769777647x^2 + 1.295738478x^4 - 0.010406134x^5 - 0.871743272x^6 + 0.178832164x^7 + 0.299266794x^8 - 0.090606944x^9 - 0.040442155x^{10} + 0.00935599x^{11} - 0.0002178x^{12}. \tag{5.3}$$

Example 5.4. For Re = 80 and $\alpha = -5$ in Case 4, the second-order approximate solution (4.31) becomes

$$\overline{F}(x) \approx 1 - 0.399291819x^2 - 0.461970063x^4 - 0.014703786x^5 - 0.12415397x^6 - 0.07325724x^7 - 0.08278982x^8 + 0.45101379x^9 - 0.648015234x^{10} + 0.47466473x^{11} - 0.121496588x^{12}. \tag{5.4}$$

The profile of the $F(x)$ function is presented in Figure 2 for Re = 50 and $\alpha = 5$.

It is easy to verify the accuracy of the obtained solutions if we compare these analytical solutions with the numerical ones or with results obtained by other procedures.

Table 1: The results of the second-order approximate solutions (5.1), (5.2), and (5.3) and numerical solution of $F(x)$ for Re = 50, $\alpha = 5$.

x	$\overline{F}(x)$, (5.1)	$\overline{F}(x)$, (5.2)	$\overline{F}(x)$, (5.3)	Numerical solution
0	1	1	1	1
0.1	0.98244611	0.982440382	0.982430842	0.98243124
0.2	0.931225969	0.931302469	0.931225959	0.93122597
0.3	0.850471997	0.850810709	0.850611445	0.85061063
0.4	0.746379315	0.747074996	0.746790784	0.74679081
0.5	0.626298626	0.627192084	0.626947253	0.62694818
0.6	0.497665923	0.498340984	0.498235028	0.49823446
0.7	0.366966345	0.366966353	0.366970088	0.36696635
0.8	0.238952034	0.238148782	0.238142322	0.23812375
0.9	0.116313019	0.115260361	0.115219025	0.11515193
1	0	0	0	0

Table 2: Comparison between the OHAM and numerical solutions for Re = 50 and $\alpha = 5$ (error = $|F(x)_{\text{num}} - F(x)_{\text{app}}|$).

x	Error of the solution (5.1)	Error of the solution (5.2)	Error of the solution (5.3)
0	0	0	0
0.1	0.00001487	0.000009142	0.000000398
0.2	0.000000001	0.000076499	0.000000011
0.3	0.000138633	0.0002	0.000000815
0.4	0.000411495	0.000284186	0.000000026
0.5	0.000649554	0.000243904	0.000000927
0.6	0.000568537	0.000106524	0.000000568
0.7	0.000000005	0.000000003	0.000003738
0.8	0.000828284	0.000024962	0.000018572
0.9	0.0001161089	0.000108431	0.000067095
1	0	0	0

It can be seen from Tables 1, 2, 3, and 4 that the analytical solutions of Jeffery-Hamel flows obtained by OHAM are very accurate.

6. Conclusions

In this paper the Optimal Homotopy Asymptotic Method (OHAM) is employed to propose a new analytic approximate solution for the nonlinear MHD Jeffery-Hamel flow problems. The proposed procedure is valid even if the nonlinear equation does not contain any small or large parameters.

OHAM provides us with a simple and rigorous way to control and adjust the convergence of the solution through the auxiliary functions $h(x, p)$ involving several constants C_i which are optimally determined.

From the results presented above, we can conclude that the following.

(1) When $\alpha > 0$ and steep of the channel is divergent, stream in value of Reynolds number is caused by decreasing in velocity.

Table 3: Comparison between Differential Transformation Method (DTM) [4], Homotopy Perturbation Method (HPM) [4], Homotopy Analysis Method [4], and OHAM-(5.4) for Re = 80, $\alpha = -5$.

x	$\overline{F}(x)$ (DTM)	$\overline{F}(x)$ (HPM)	$\overline{F}(x)$ (HAM)	$\overline{F}(x)$ (OHAM)	Numerical
0	1	1	1	1	1
0.1	0.9959603887	0.9960671874	0.9995960242	0.995960605	0.9959606278
0.2	0.9832745481	0.9836959424	0.9832755258	0.983275548	0.9832755383
0.3	0.9601775551	0.9610758773	0.9601798911	0.960179914	0.96017991139
0.4	0.9235170706	0.9249245156	0.9235215737	0.923521643	0.9235215894
0.5	0.8684511349	0.8701997697	0.8684588997	0.868458963	0.86845887772
0.6	0.7880785402	0.7898325937	0.7880910186	0.788090923	0.78809092032
0.7	0.673248448	0.6745334968	0.6731437690	0.673143633	0.6731436346
0.8	0.5119644061	0.5128373095	0.5119909939	0.511991107	0.5119910891
0.9	0.2915280122	0.2918936991	0.2915580178	0.291558742	0.29155874261
1	0	0	−0.000001149	0	0

Table 4: Comparison between OHAM (5.4) and numerical solutions [4] for Re = 80, $\alpha = -5$.

x	$\overline{F}(x)$, (5.4)	Numerical	Error
0	1	1	0
0.1	0.995960605	0.9959606278	0.000000022
0.2	0.983275548	0.9832755383	0.000000009
0.3	0.960179914	0.96017991139	0.000000002
0.4	0.923521643	0.9235215894	0.000000053
0.5	0.868458963	0.86845887772	0.000000085
0.6	0.788090923	0.78809092032	0.000000002
0.7	0.673143633	0.6731436346	0.000000001
0.8	0.511991107	0.5119910891	0.000000017
0.9	0.291558742	0.29155874261	0.000000006
1	0	0	0

(2) When $\alpha < 0$ and steep of the channel is convergent, the results are inverse. Increase in value of Reynolds number is caused by increasing in velocity.

The examples related to the Jeffery-Hamel flow problem presented in this paper lead to the very important conclusion that the accuracy of the obtained results is growing along with increasing the number of constants in the auxiliary function. This paper confirmed that DTM, HPM, or HAM gives a good accuracy, but OHAM is by far the best method delivering faster convergence and better accuracy. In the proposed procedure, iterations are performed in a very simple manner by identifying some coefficients, and therefore very good approximations are obtained in few terms. Actually the capital strength of the proposed procedure is its fast convergence, since after only two iterations it converges to the exact solution, which proves that this method is very effective in practice. This version of the method proves to be very rapid and effective, and this is proved by comparing the analytic solutions obtained through the proposed method with the solutions obtained via numerical simulations or other known procedures.

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