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Research Article

Consecutive k-within-m-out-of-n:F System with Nonidentical Components

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As a generalisation of consecutive k-out-of-n:F and k-out-of-n:F system models, a consecutive k-within-m-out-of-n:F system consists of n linearly ordered components and fails if and only if there are m consecutive components which include among them at least k failed components. In this paper, we study the survival function of a consecutive k-within-m-out-of-n:F system consisting of independent but nonidentical components. We obtain exact expressions for the survival function when $2m \geq n$. A detailed analysis for consecutive 2-within-m-out-of-n:F systems is presented and the asymptotic behaviour of hazard rate of these systems is investigated using mixture representations.

1. Introduction

It is a well-accepted fact that all components in an engineered system are not created equal. This in turn implies that different components may have different survival probabilities. The study of systems consisting of nonidentical components is a difficult task especially when the system has a complex structure. Some recent contributions on systems with independent but nonidentical components appear in Navarro [1], Zhao et al. [2], Kochar and Xu [3], and Navarro et al. [4].

Consecutive type systems have been extensively studied in the literature. One of the most widely studied consecutive type system is a linear consecutive k-out-of-n:F system which consists of n linearly ordered components and fails if and only if at least k consecutive components fail. This type of systems is potentially useful for modeling transportation and transmission systems. Much of the previous research has concentrated on the optimal design or reliability computation of such systems. There are several papers which study the dynamic reliability properties of consecutive k-out-of-n systems. Boland and Samaniego [5] obtained some stochastic ordering results on lifetimes of consecutive k-out-of-n systems consisting of independent components. Triantafyllou and Koutras [6] studied the lifetime distribution of consecutive k-out-of-n:F systems consisting of independent and identical components. A review of recent developments on consecutive k-out-of-n and related systems is presented in Eryilmaz [7].

In this paper we study the dynamic reliability of consecutive k-within-m-out-of-n:F systems consisting of independent but nonidentical (inid) components. A consecutive k-within-m-out-of-n:F system consists of n linearly ordered components and fails if and only if there are m consecutive components which include among them at least k-failed components ($1 < k \leq m \leq n$). There are numerous applications for such systems in practice, for example, quality control, inspection procedures, radar detection, transportation, and transmission systems (see, e.g., Chang et al. [8]). A consecutive k-within-m-out-of-n:F system involves consecutive k-out-of-n:F and k-out-of-n:F (a system which fails if and only if at least k components fail) systems for $m = k$ and $m = n$, respectively. The dynamic reliability properties of consecutive k-within-m-out-of-n:F systems with identical components have been studied in several papers (Papastavridis [9], Iyer [10], Eryilmaz et al. [11], Eryilmaz and Kan [12], Triantafyllou and Koutras [13]).

Consecutive k-within-m-out-of-n:F system can be represented as a series system of $n-m+1$-dependent k-out-of-m:F systems. That is, the lifetime of this system can be expressed as

$$T_{k,m:n} = \min\left(T_{k:m}^{[1:m]}, T_{k:m}^{[2:m+1]}, \ldots, T_{k:m}^{[n-m+1:n]}\right), \tag{1.1}$$

where $T_{k:m}^{[i:i+m-1]}$ shows the lifetime of k-out-of-m:F subsystem of components with the lifetimes $T_i, T_{i+1}, \ldots, T_{i+m-1}$, $1 \leq i \leq n-m+1$.

The evaluation of the survival function associated with $T_{k,m:n}$ is of special importance for understanding the dynamic behaviour of the system since the reliability characteristics such as hazard rate and mean residual life function can be obtained from this function. In the present paper, we obtain expressions for the survival functions of consecutive k-within-m-out-of-n:F systems for $2m \geq n$ when the components are independent but not necessarily identically distributed. In Section 2, a detailed analysis for consecutive 2-within-m-out-of-n:F systems is presented. Section 3 contains results for $2m \geq n$.

In the following, we provide the notations that will be used throughout the paper. n is the number of components; T_i is the lifetime of component i; $X_i(t)$ is the state of component i at time t: $X_i(t) = 1(0)$ if $T_i \leq t (T_i > t)$; $T_{k,m:n}$ is the lifetime of consecutive k-within-m-out-of-n:F system; $T_{k:b-a+1}^{[a:b]}$ is kth smallest among $T_a, T_{a+1}, \ldots, T_b$; $R_{k,m:n}(t) = P\{T_{k,m:n} > t\}$ is the survival function of consecutive k-within-m-out-of-n:F system; $h_{k,m:n}(t)$ is the hazard rate of consecutive k-within-m-out-of-n:F system.

Throughout the paper the components are assumed to be independent and the survival function associated with the ith component is $\overline{F}_i(t) = P\{T_i > t\} = 1 - F_i(t)$, $i = 1, 2, \ldots, n$.

2. Results for Consecutive 2-within-m-out-of-n:F Systems

If $T_1, T_2, \ldots, T_n$ represent the lifetimes of n components in a coherent system, then the system lifetime can be represented as

$$T = \max_{1 \leq j \leq s} \min_{i \in P_j} T_i, \tag{2.1}$$

where $P_1, P_2, \ldots, P_s$ are the minimal path sets. If $T_1, T_2, \ldots, T_n$ are independent, then the system survival function can be computed from the series survival functions as

$$S(t) = P\{T > t\} = \sum_{A \subseteq \{1,2,\ldots,s\}} (-1)^{|A|+1} \prod_{i \in P_A} \overline{F}_i(t), \tag{2.2}$$

where $\overline{F}_i(t) = P\{T_i > t\}$, $i = 1, 2, \ldots, n$, and $P_A = \bigcup_{j \in A} P_j$.

The hazard rate associated with the subset A of (2.2) is

$$h_A(t) = \sum_{i \in P_A} r_i(t), \tag{2.3}$$

where $r_i(t)$ is the hazard rate associated with $\overline{F}_i(t)$.

Example 2.1. Let $n = 4$, $m = 3$, and $k = 2$. Then the path sets of consecutive 2-within-3-out-of-4:F system are $\{1,2,3,4\}$, $\{2,3,4\}$, $\{1,3,4\}$, $\{1,2,4\}$, $\{1,2,3\}$, and $\{2,3\}$. The minimal path sets are $P_1 = \{2,3\}$, $P_2 = \{1,2,4\}$, and $P_3 = \{1,3,4\}$. Therefore

$$T_{2,3:4} = \max(\min(T_2, T_3), \min(T_1, T_2, T_4), \min(T_1, T_3, T_4)), \tag{2.4}$$

and hence the survival function of consecutive 2-within-3-out-of-4:F system is

$$\begin{aligned} R_{2,3:4}(t) = {} & \overline{F}_2(t)\overline{F}_3(t) + \overline{F}_1(t)\overline{F}_2(t)\overline{F}_4(t) + \overline{F}_1(t)\overline{F}_3(t)\overline{F}_4(t) \\ & - 2\overline{F}_1(t)\overline{F}_2(t)\overline{F}_3(t)\overline{F}_4(t) \end{aligned} \tag{2.5}$$

which is a mixture of series survival functions with the weight vector $(1, 1, 1, -2)$. The hazard rates of each element of (2.5) are

$$\begin{aligned} h_1(t) &= r_2(t) + r_3(t), \\ h_2(t) &= r_1(t) + r_2(t) + r_4(t), \\ h_3(t) &= r_1(t) + r_3(t) + r_4(t), \\ h_4(t) &= r_1(t) + r_2(t) + r_3(t) + r_4(t). \end{aligned} \tag{2.6}$$

Lemma 2.2. *For $2m \geq n$ the consecutive 2-within-m-out-of-n:F system has $n + \binom{n-m+1}{2} + 1$ path sets, and these sets are*

$$\begin{gathered} C = \{1, 2, \ldots, n\}, \\ C \setminus \{i\}, \quad i = 1, 2, \ldots, n, \\ C \setminus \{i, m+i+j\}, \quad i = 1, 2, \ldots, n-m, \ j = 0, 1, \ldots, n-i-m. \end{gathered} \tag{2.7}$$

Proof. For $2m \geq n$, the consecutive 2-within-m-out-of-n:F system works if and only if there is no failed component or there is only one failed component or there are at most two failed components separated by at least $m-1$ working components. Thus the proof is complete. □

The following results are direct consequences of Lemma 2.2.

Lemma 2.3. *For $2m \geq n$, the consecutive 2-within-m-out-of-n:F system has $\binom{n-m+1}{2}$ minimal path sets with $n-2$ elements. These minimal path sets are*

$$\left\{l_1^{(s)},\ldots,l_{n-2}^{(s)}\right\} \equiv \{1,2,\ldots,n\} \setminus \{i,m+i+j\},$$
$$i=1,2,\ldots,n-m,\ \ j=0,1,\ldots,n-i-m,\ \ s=1,\ 2,\ldots,\binom{n-m+1}{2}. \tag{2.8}$$

Lemma 2.4. *Let $T_1,T_2,\ldots,T_n$ be inid lifetimes of components with $F_i(t)=P\{T_i \leq t\}$, $i=1,2,\ldots,n$. For $2m \geq n$,*

$$R_{2,m:n}(t) = \prod_{j=1}^{n}\overline{F}_j(t) + \sum_{i=1}^{n} F_i(t)\prod_{\substack{j=1\\ j\neq i}}^{n}\overline{F}_j(t) + \sum_{i=1}^{n-m}\sum_{j=0}^{n-i-m} F_i(t)F_{m+i+j}(t)\prod_{\substack{l=1\\ l\neq i,m+i+j}}^{n}\overline{F}_l(t). \tag{2.9}$$

Theorem 2.5. [1] *Let S be a survival function such that*

$$S(t) = \sum_{i=1}^{n}\omega_i S_i(t), \tag{2.10}$$

for all $t \geq 0$, where $\omega_1,\ldots,\omega_n$ are real numbers such that $\sum_{i=1}^{n}\omega_i = 1$. Let $h_i(t)$ be the failure rate function corresponding to $S_i(t)$, $i=1,\ldots,n$. If

$$\lim_{t\to\infty}\inf\frac{h_i(t)}{h_1(t)} > 1, \qquad \lim_{t\to\infty}\sup\frac{h_i(t)}{h_1(t)} < \infty, \tag{2.11}$$

for $i=2,3,\ldots,n$, then $\lim_{t\to\infty}(h(t)/h_1(t))=1$, where $h(t)$ is the failure rate function corresponding to $S(t)$.

In the following, one will study the limiting behaviour of the hazard rate of a consecutive 2-within-m-out-of-n:F system.

Theorem 2.6. *Let $T_1,T_2,\ldots,T_n$ be independent and the hazard rate of T_i is $r_i(t)$. For $2m \geq n$, if $\lim_{t\to\infty} r_i(t) = \lambda_i$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, then*

$$\lim_{t\to\infty} h_{2,m:n}(t) = \min\left(\lambda_{l_1^{(1)}}+\cdots+\lambda_{l_{n-2}^{(1)}},\lambda_{l_1^{(2)}}+\cdots+\lambda_{l_{n-2}^{(2)}},\ldots,\lambda_{l_1^{(s)}}+\cdots+\lambda_{l_{n-2}^{(s)}}\right), \tag{2.12}$$

where $s=\binom{n-m+1}{2}$.

Proof. From Lemma 2.3, the minimum number of elements in the minimal path sets of consecutive 2-within-m-out-of-n system is $n-2$ (for $2m \geq n$) and the total number of these minimal path sets is $\binom{n-m+1}{2}$. Thus the proof follows from Theorem 2.5 and the conditions of Theorem 2.6. □

Example 2.7. Let $n = 4, m = 3$, and $k = 2$. Suppose that $\lim_{t\to\infty} r_i(t) = \lambda_i$, $i = 1,2,3,4$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. Then using the hazard rates given in (2.6), we have

$$\lim_{t\to\infty} \inf \frac{h_2(t)}{h_1(t)} = \frac{\lambda_1+\lambda_2+\lambda_4}{\lambda_2+\lambda_3} > 1,$$
$$\lim_{t\to\infty} \inf \frac{h_3(t)}{h_1(t)} = \frac{\lambda_1+\lambda_3+\lambda_4}{\lambda_2+\lambda_3} > 1,$$
$$\lim_{t\to\infty} \inf \frac{h_4(t)}{h_1(t)} = \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{\lambda_2+\lambda_3} > 1, \tag{2.13}$$
$$\lim_{t\to\infty} \sup \frac{h_i(t)}{h_1(t)} < \infty,$$

for $i = 2,3,4$. Therefore

$$\lim_{t\to\infty} h_{2,3:4}(t) = \lambda_2 + \lambda_3. \tag{2.14}$$

3. General Results

The reliability of consecutive k-within-m-out-of-n:F system is closely related to the discrete scan statistic defined by

$$S_{n,m}(t) = \max\left\{ \sum_{j=i}^{i+m-1} X_j(t) : 1 \leq i \leq n-m+1 \right\}. \tag{3.1}$$

A consecutive k-within-m-out-of-n:F system survives at time t if and only if less than k components are failed among any consecutive m components. Thus its survival function can be expressed as

$$R_{k,m:n}(t) = P\{T_{k,m:n} > t\} = P\{S_{n,m}(t) < k\}, \tag{3.2}$$

or equivalently,

$$R_{k,m:n}(t) = P\left\{ T_{k:m}^{[1:m]} > t, T_{k:m}^{[2:m+1]} > t, \ldots, T_{k:m}^{[n-m+1:n]} > t \right\}, \tag{3.3}$$

for $t \geq 0$ and $1 \leq k \leq m \leq n$.

The proof of the following result is easy and hence is omitted.

Lemma 3.1. *For $k = 1$ and $b - a \geq 0$,*

$$P\left\{T_{k:b-a+1}^{[a:b]} > t\right\} = \prod_{i=a}^{b} \overline{F}_i(t), \tag{3.4}$$

for $k > 1$ and $b - a = k - 1$,

$$P\left\{T_{k:b-a+1}^{[a:b]} > t\right\} = 1 - \prod_{i=a}^{b} F_i(t), \tag{3.5}$$

and for $k > 1$ and $b - a \geq k$,

$$P\left\{T_{k:b-a+1}^{[a:b]} > t\right\} = P\left\{T_{k-1:b-a}^{[a:b-1]} > t\right\} F_b(t) + P\left\{T_{k:b-a}^{[a:b-1]} > t\right\} \overline{F}_b(t). \tag{3.6}$$

Theorem 3.2. *For $2m \geq n$,*

$$R_{k,m:n}(t) = \sum_{s=0}^{\min(n-m,\ k-1)} P\left\{T_{k-s:2m-n}^{[n-m+1:m]} > t\right\} \left[R_{s+1,\ n-m:2(n-m)}^{*}(t) - R_{s,n-m:2(n-m)}^{*}(t)\right], \tag{3.7}$$

where $R_{s,n-m:2(n-m)}^{*}(t)$ is the reliability of consecutive s-within-$(n-m)$-out-of-$2(n-m)$:F system with components $1, \ldots, n-m,\ m+1, \ldots, n$.

Proof. By the definition of $S_{n,m}(t)$,

$$P\{S_{n,m}(t) < k\} = P\left\{\sum_{j=1}^{m} X_j(t) < k,\ \sum_{j=2}^{m+1} X_j(t) < k, \ldots, \sum_{j=n-m+1}^{n} X_j(t) < k\right\}. \tag{3.8}$$

For $2m \geq n$,

$$\begin{aligned} P\{S_{n,m}(t) < k\} = \sum_{x_1,\ldots,x_{n-m},\ x_{m+1},\ldots,x_n \in \{0,1\}} P\Bigg\{ &\sum_{i=n-m+1}^{m} X_i(t) < m^{*}, \\ X_1(t) = x_1, \ldots, X_{n-m}(t) = x_{n-m}, X_{m+1}(t) &= x_{m+1}, \ldots, X_n(t) = x_n \Bigg\}, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} m^{*} &= \min(k - x_1 - \cdots - x_{n-m}, k - x_2 - \cdots - x_{n-m} - x_{m+1}, \ldots, k - x_{m+1} - \cdots - x_n) \\ &= k - \max(x_1 + \cdots + x_{n-m}, x_2 + \cdots + x_{n-m} + x_{m+1}, \ldots, x_{m+1} + \cdots + x_n). \end{aligned} \tag{3.10}$$

If $S^*_{2(n-m),n-m}(t)$ denotes the scan statistic based on $X_1(t),\ldots,X_{n-m}(t),X_{m+1}(t),\ldots,X_n(t)$, then

$$P\{S_{n,m}(t)<k\}=\sum_{s=0}^{\min(n-m,k-1)}P\left\{\sum_{i=n-m+1}^{m}X_i(t)<k-s,S^*_{2(n-m),n-m}(t)=s\right\}. \tag{3.11}$$

Thus the proof is completed by the independence of $\sum_{i=n-m+1}^{n}X_i(t)$ and $S^*_{2(n-m),n-m}(t)$ and

$$\begin{gathered}P\left\{\sum_{i=n-m+1}^{m}X_i(t)<k-s\right\}=P\left\{T^{[n-m+1:m]}_{k-s:2m-n}>t\right\},\\ P\left\{S^*_{2(n-m),n-m}(t)=s\right\}=R^*_{s+1,n-m:2(n-m)}(t)-R^*_{s,n-m:2(n-m)}(t).\end{gathered} \tag{3.12}$$

□

Theorem 2.6 can be extended to any consecutive k-within-m-out-of-n:F system in the following way.

Theorem 3.3. *Let $T_1,T_2,\ldots,T_n$ be independent and the hazard rate of T_i is $r_i(t)$. For $1<k\leq m\leq n$, if $\lim_{t\to\infty}r_i(t)=\lambda_i$ and $\lambda_1>\lambda_2>\cdots>\lambda_n$, then*

$$\lim_{t\to\infty}h_{k,m:n}(t)=\min\left(\lambda_{l_1^{(1)}}+\cdots+\lambda_{l^{(1)}_{n-z(n,m,k)}},\lambda_{l_1^{(2)}}+\cdots+\lambda_{l^{(2)}_{n-z(n,m,k)}},\ldots,\lambda_{l_1^{(s)}}+\cdots+\lambda_{l^{(s)}_{n-z(n,m,k)}}\right), \tag{3.13}$$

where s is the number of minimal path sets with $n-z(n,m,k)$ elements and $z(n,m,k)$ is the maximum number of failed components such that the system can still work.

The number $z(n,m,k)$ has been derived in Eryilmaz and Kan [12] as

$$z(n,m,k)=\begin{cases}n-\left[\dfrac{n}{m}\right](m-k+1) & \text{if } n-m\left[\dfrac{n}{m}\right]<k,\\ (k-1)\left(1+\left[\dfrac{n}{m}\right]\right) & \text{if } n-m\left[\dfrac{n}{m}\right]\geq k.\end{cases} \tag{3.14}$$

Example 3.4. Let $n=7,m=3$, and $k=2$. Then $z(n,m,k)=3$ and there is only one minimal path set with $n-z(n,m,k)=7-3=4$ elements, that is, $s=1$ and the corresponding minimal path set is $\{2,3,5,6\}$. Thus under the conditions of Theorem 3.3, we have

$$\lim_{t\to\infty}h_{2,3:7}(t)=\lambda_2+\lambda_3+\lambda_5+\lambda_6. \tag{3.15}$$

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