Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 208375, 14 pages
doi:10.1155/2012/208375

Research Article

A General Solution for Troesch's Problem

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Received 23 August 2012; Accepted 6 October 2012

Academic Editor: Farzad Khani

The homotopy perturbation method (HPM) is employed to obtain an approximate solution for the nonlinear differential equation which describes Troesch's problem. In contrast to other reported solutions obtained by using variational iteration method, decomposition method approximation, homotopy analysis method, Laplace transform decomposition method, and HPM method, the proposed solution shows the highest degree of accuracy in the results for a remarkable wide range of values of Troesch's parameter.

1. Introduction

Troesch's equation is a boundary value problem (BVP) expressed as

$$y'' = n \sinh(ny), \qquad y(0) = 0,\ y(1) = 1, \tag{1.1}$$

where prime denotes differentiation with respect to x and n is known as Troesch's parameter.

Equation (1.1) arises in the investigation of confinement of a plasma column by a radiation pressure [1] and also in the theory of gas porous electrodes [2, 3]. This BVP problem has a pole [4] approximately located at

$$x_\infty = \frac{1}{n}\ln\left(\frac{8}{y'(0)}\right), \tag{1.2}$$

which makes the solution of (1.1) a difficult task for numerical methods.

In order to overcome such difficulties, there are several reported numerical solutions for Troesch's problem [4–10]. Recently, after some decades using wrong numerical results, in [8, 9] were reported the most accurate solutions for $n = 0.5$ and $n = 1$. Besides, there are approximated analytical solutions obtained by using different methods like homotopy perturbation method (HPM) [11, 12], decomposition method approximation (DMA) [12, 13], homotopy analysis method (HAM) [14], variational iteration method (VIM) [15], and Laplace transform decomposition method (LTDM) [16]. The main disadvantage of aforementioned approximated solutions is that they are obtained for specific values of n, like $n = 0.5$, 1 or 10. In contrast, we propose a general approximate solution for Troesch's problem, useful for $n \geq 0$, by using HPM method [17–27].

This paper is organized as follows. In Section 2, we provide a brief review of HPM method. In Section 3, we obtain the solution of Troesch's problem employing HPM. Section 4 shows numerical simulations and discuss our findings. Finally, a brief conclusion is given in Section 5.

2. Basic Idea of HPM Method

Basically, the HPM method [17–26, 31–53] introduces a homotopy parameter p, which takes values ranging from 0 up to 1. When parameter $p = 0$, the equation usually reduces to a simple, or trivial, equation to solve. Then, p is gradually increased to 1, producing a sequence of deformations. Eventually, at $p = 1$, the homotopy equation takes the original form of the equation to solve, and the final stage of deformation provides the desired solution. Usually, few iterations are required to obtain good results [17–19].

In the HPM method, it is considered that a nonlinear differential equation can be expressed as

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2.1}$$

with the boundary condition

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, \quad r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, $f(r)$ is a known analytic function, B is a linear initial/boundary operator, Γ is the boundary of domain Ω, and $\partial u/\partial \eta$ denotes differentiation along the normal drawn outwards from Ω. The A operator, generally, can be divided into two operators, L and N, where L is the linear operator, and N is the nonlinear operator. Hence, (2.1) can be rewritten as

$$L(u) + N(u) - f(r) = 0. \tag{2.3}$$

Now, a possible homotopy formulation is

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p\big(L(v) + N(v) - f(r)\big) = 0, \quad p \in [0, 1], \tag{2.4}$$

where u_0 is the initial approximation for the solution of (2.3) which satisfies the boundary conditions, and p is known as the perturbation homotopy parameter.

We assume that the solution of (2.4) can be written as a power series of p as

$$v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots . \tag{2.5}$$

When $p \to 1$, results that the approximate solution for (2.1) is

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{2.6}$$

The series (2.6) is convergent on most cases, nevertheless, the convergence depends on the nonlinear operator $A(v)$ [19, 21, 35, 36, 52].

3. Solution of Troesch's Problem by Using HPM Method

Straight forward application of HPM to solve (1.1) is not possible due to the hyperbolic sin term of dependent variable. However, in [11, 12] were reported HPM solutions, obtained by using a power series expansion of the sinh term of (1.1), which are limited to $n \leq 1$, due to the truncate power series. Nevertheless, the polynomial type nonlinearities are easier to handle by the HPM method. Therefore, in order to apply HPM successfully for a wide range of Troesch's parameter (n), we convert the hyperbolic-type nonlinearity in Troesch's problem into a polynomial type nonlinearity, using the variable transformation reported in [15]. First, we consider that

$$y(x) = \frac{4}{n} \tanh^{-1}(u(x)), \tag{3.1}$$

from which we find

$$y' = \frac{4}{n(1 - u^2)} u', \tag{3.2}$$

$$y'' = \frac{4}{n(1 - u^2)} u'' + \frac{8u}{n(1 - u^2)^2} (u')^2, \tag{3.3}$$

where prime denotes differentiation with respect to x.

Now, by substituting (3.1) into the sinh term of (1.1), we obtain

$$y'' = n\sinh\left(4\tanh^{-1}(u(x))\right) = \frac{4nu(u^2+1)}{(u+1)^2(u-1)^2}. \tag{3.4}$$

Then, equating (3.3) and (3.4), we achieve to the following transformed problem:

$$\left(1-u^2\right)u'' + 2u(u')^2 - n^2u\left(1+u^2\right) = 0, \tag{3.5}$$

where conditions are obtained by using variable transformation (see (3.1))

$$u(x) = \tanh\left(\frac{n}{4}y(x)\right), \tag{3.6}$$

and substituting original boundary conditions $y(0) = 0$ and $y(1) = 1$ into above equation, results

$$u(0) = 0, \qquad u(1) = \tanh\left(\frac{n}{4}\right). \tag{3.7}$$

From (2.4) and (3.5), we can formulate the following homotopy [17–19]:

$$H(v,p) = (1-p)\left(v'' - n^2v\right) + p\left(\left(1-v^2\right)v'' + 2v(v')^2 - n^2v\left(1+v^2\right)\right) = 0, \tag{3.8}$$

where p is the homotopy parameter.

Substituting (2.5) into (3.8) and equating identical powers of p terms, we obtain

$$\begin{aligned}
&p^0: \ v_0'' - n^2v_0 = 0, \quad v_0(0) = 0, \ v_0(1) = \tanh\left(\frac{n}{4}\right),\\
&p^1: \ v_1'' - n^2v_1 - n^2v_0^3 - v_0^2v_0'' + 2v_0(u_0')^2 = 0, \quad v_1(0) = 0, \ v_1(1) = 0,\\
&p^2: \ v_2'' - n^2v_2 - 3n^2v_0^2v_1 + 4v_0v_0'v_1' - 2v_0v_1v_0'' - v_0^2v_1'' + 2v_1(v_0')^2 = 0, \quad v_2(0) = 0, \ v_2(1) = 0.
\end{aligned} \tag{3.9}$$

We solve (3.9) by using Maple software, resulting

$$
\begin{aligned}
v_0 &= w \sinh(nx), \quad w = \frac{\tanh(n/4)}{\sinh(n)}, \\
v_1 &= -\frac{w^3 n \exp(-nx)}{2\exp(2n)-2}\left[(x-1)(\exp(2n(x+1))-1)+(x+1)(\exp(2n)-\exp(2nx))\right], \\
v_2 &= m + \frac{q\exp(-3nx)}{2n} \\
&\quad \times \left[\left(\frac{3}{4}+\left(-x^2+6x\right)n^2+\left(-\frac{3}{2}x-3\right)n\right)\exp((4x+2)n)\right. \\
&\quad -\frac{1}{8}\exp((6x+2)n)+\left(-\frac{3}{4}+\left(6x+x^2\right)n^2+\left(-\frac{3}{2}x+3\right)n\right)\exp(2n(x+1)) \\
&\quad +\left(\frac{3}{4}+\left(-x^2+6x\right)n^2+\left(3+\frac{3}{2}x\right)n\right)\exp(2nx) \\
&\quad +\left(-\frac{3}{4}+\left(6x+x^2\right)n^2+\left(\frac{3}{2}x-3\right)n\right)\exp(4nx) \\
&\quad \left.-\frac{1}{8}+\frac{1}{8}\exp(6nx)+\frac{1}{8}\exp(2n)\right],
\end{aligned}
\tag{3.10}
$$

where m is

$$
\begin{aligned}
m &= \frac{q(\exp(-nx)-\exp(nx))\exp(-3n)}{16n(\exp(n)-\exp(-n))} \\
&\quad \times \left[7\exp(6n)+40n^2\exp(6n)-36n\exp(6n)\right. \\
&\quad -\exp(8n)-12\exp(4n)+112n^2\exp(4n)+7\exp(2n) \\
&\quad \left.+40n^2\exp(2n)+36n\exp(2n)-1\right],
\end{aligned}
\tag{3.11}
$$

and q is

$$
q = -\frac{w^5 n}{2\exp(2n)-2}. \tag{3.12}
$$

Next, calculating the limit when $p \to 1$, we obtain the second-order approximated solution of (3.5)

$$
u_2(x) = \lim_{p\to 1} v = v_0 + v_1 + v_2. \tag{3.13}
$$

Finally, from (3.1) and (3.13), the proposed solution of Troesch's problem is

$$y(x) = \frac{4}{n}\tanh^{-1}(u_2(x)), \quad 0 \le x \le 1, \; n \ge 0. \tag{3.14}$$

3.1. Interval of Solution

The real branch of $\tanh^{-1}(z)$ is restricted to the range $-1 \le z \le 1$. Therefore, (3.14) requires

$$-1 \le u_2(x) \le 1, \tag{3.15}$$

where x is delimited by the boundary conditions as

$$0 \le x \le 1. \tag{3.16}$$

In order to show that (3.14) fullfills the conditions (3.15) and (3.16), we plot (3.13) in Figure 1. From such figure, we can observe that $0 < u_2(x) < 1$ is valid in the intervals of $x \in [0, 1]$ and $n \in [0, 30]$. Now, from (3.13) and (3.14), we calculate the following limits:

$$\lim_{n \to \infty} u_2(1) = 1, \tag{3.17}$$

$$\lim_{x \to 0} u_2(x) = 0, \tag{3.18}$$

$$\lim_{n \to 0} y(x) = x. \tag{3.19}$$

From (3.17), (3.18), (3.19), and Figure 1, we can conclude that the maximum value of (3.13) is 1 in the range of $0 \le n \le \infty$. Therefore, (3.13) fullfills (3.15) in the range given by (3.16). Additionally, limit (3.19) shows that for $n = 0$, the presented solution (3.14) becomes the exact/trivial solution for (1.1).

4. Numerical Simulation and Discussion

In the case of $n = 0.5$ (see Table 1), we can observe that the lowest average absolute relative error (A.A.R.E.) is for LDTM [16], followed closely by the proposed solution (3.14). A possible reason can lie in the fact that (3.14) is a second-order approximation, while LDTM is of third order. For $n = 1$ (see Table 2), there is a change now the lowest A.A.R.E. is for the proposed solution (3.14), followed by LDTM solution. For both cases, the other approximations ADM [13], HPM [12], HPM [11], and HAM [14] have lower accuracy than (3.14). Equation (3.14) did not require an adjustment parameter, unlike LDTM solution, which required a specific adjustment parameter calculated for each value of Troesch's parameter n. Therefore, the proposed solution is easier to use than LDTM solution.

In Table 3, we can observe a comparison of (3.14) with numerical solution [15], and other solutions obtained by HAM [14], VIM [15], and HPM [12] for $n = 10$. Approximation (3.14) has the lowest A.A.R.E. from all above solutions, followed by VIM approximation.

Table 1: Comparison between (3.14), exact solution [8, 9], and other reported approximate solutions, using $n = 0.5$.

x	Exact [8, 9]	This work (3.14)	ADM [13]	HPM [12]	HPM [11]	HAM [14]	LDTM [16]
0.1	0.0959443493	0.0959443155	0.0959383534	0.0959395656	0.095948026	0.0959446190	0.0959443520
0.2	0.1921287477	0.1921286848	0.1921180592	0.1921193244	0.192135797	0.1921292845	0.1921287539
0.3	0.2887944009	0.2887943176	0.2887803297	0.2887806940	0.288804238	0.2887952148	0.2887944107
0.4	0.3861848464	0.3861847539	0.3861687095	0.3861675428	0.386196642	0.3861859313	0.3861848612
0.5	0.4845471647	0.4845470753	0.4845302901	0.4845274183	0.4845599	0.4845485110	0.4845471832
0.6	0.5841332484	0.5841331729	0.5841169798	0.5841127822	0.584145785	0.5841348222	0.5841332650
0.7	0.6852011483	0.6852010943	0.6851868451	0.6851822495	0.685212297	0.6852028604	0.6852011675
0.8	0.7880165227	0.7880164925	0.7880055691	0.7880018367	0.788025104	0.7880181729	0.7880165463
0.9	0.8928542161	0.8928542059	0.8928480234	0.8928462193	0.892859085	0.8928553997	0.8928542363
	Order	2	6	2	2	6	3
	A.A.R.E.	$1.83327e(-07)$	$3.47802e(-05)$	$3.57932e(-05)$	$2.44418e(-05)$	$2.51374e(-06)$	$3.10957e(-08)$

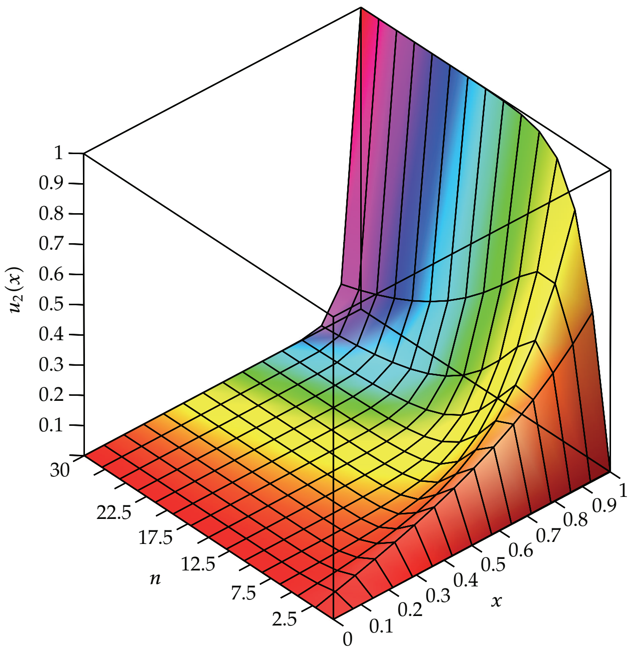

Figure 1: Plot for (3.13) in the range of $x \in [0, 1]$ and $n \in [0, 30]$.

Table 2: Comparison between (3.14), exact solution [8, 9], and other reported approximate solutions, using $n = 1$.

x	Exact [8]	This work (3.14)	ADM [13]	HPM [12]	HPM [11]	HAM [14]	LDTM [16]
0.1	0.0846612565	0.08466075858	0.084248760	0.0843817004	0.084934415	0.0846732692	0.08466308972
0.2	0.1701713582	0.1701704581	0.169430700	0.1696207644	0.170697546	0.1701954538	0.1701750442
0.3	0.2573939080	0.2573927827	0.256414500	0.2565929224	0.258133224	0.2574302342	0.2573994845
0.4	0.3472228551	0.3472217324	0.346085720	0.3462107378	0.348116627	0.3472715981	0.3472303763
0.5	0.4405998351	0.4405989511	0.439401985	0.4394422743	0.44157274	0.4406610140	0.4406093753
0.6	0.5385343980	0.5385339413	0.537365700	0.5373300622	0.539498234	0.5386072529	0.5385460046
0.7	0.6421286091	0.6421286573	0.641083800	0.6410104651	0.642987984	0.7526899495	0.6421421393
0.8	0.7526080939	0.7526085475	0.751788000	0.7517335467	0.753267551	0.7526899495	0.7526226886
0.9	0.8713625196	0.8713630450	0.870908700	0.8708835371	0.871733059	0.8714249118	0.8713748860
	Order	2	6	2	2	6	3
	A.A.R.E.	$2.54568e(-06)$	0.002714577	0.002320107	0.002044737	0.019244326	$2.05e(-05)$

Besides, HAM [14] has an relatively poor value of A.A.R.E., despite the fact that it is a sixth-order approximation. Furthermore, HPM [12] shows divergence from the numerical solution. In addition, (3.14) do not require an adjustment parameter, nevertheless, VIM solution required a specific adjustment parameter calculated for each value of Troesch's parameter n. Therefore, the proposed solution is easier to use than VIM solution.

In Table 4, is presented a comparison of initial slope $y'(0)$ and the results reported in [4, 5, 15, 28, 29] for the range of $1 \leq n \leq 20$; resulting that the proposed solution is the only one reported in literature with high accuracy in the complete aforementioned range. Moreover, in

Table 3: Comparison between (3.14), numerical solution [15], and other reported approximate solutions, using $n = 10$.

x	Numerical [15]	This work (3.14)	HAM [14]	VIM [15]	HPM [12]
0.100	$4.211183679705e(-05)$	$4.211189927237e(-05)$	$4.19038e(-05)$	$4.211189936715e(-05)$	17.617500000
0.200	$1.299639238293e(-04)$	$1.299641158237e(-04)$	$1.32094e(-04)$	$1.299641161162e(-04)$	33.693333333
0.300	$3.589778855481e(-04)$	$3.589784013896e(-04)$	$3.60051e(-04)$	$3.589784021976e(-04)$	46.785833333
0.400	$9.779014227050e(-04)$	$9.779027718029e(-04)$	$9.24818e(-04)$	$9.779027740037e(-04)$	55.653333333
0.500	$2.659017178062e(-03)$	$2.659020490351e(-03)$	$2.57642e(-03)$	$2.659020496335e(-03)$	59.354166667
0.600	$7.228924695208e(-03)$	$7.228931212877e(-03)$	$7.78968e(-03)$	$7.228931229141e(-03)$	57.346666667
0.700	$1.966406025665e(-02)$	$1.966406309701e(-02)$	$2.22598e(-02)$	$1.966406314118e(-02)$	49.589166667
0.800	$5.373032958567e(-02)$	$5.373032935060e(-02)$	$5.59715e(-02)$	$5.373032947016e(-02)$	36.640000000
0.900	$1.521140787863e(-01)$	$1.521140764047e(-01)$	$1.28433e(-01)$	$1.521140767248e(-01)$	19.757500000
0.999	$8.889931171768e(-01)$	$8.889931181558e(-01)$	—	$8.889931185202e(-01)$	—
	Order	2	6	2	2
	A.A.R.E.	$8.09027e(-07)$	0.057403529	$8.1021e(-07)$	99824.7587

order to compare the derivative of (3.14) for $n > 20$, we use the approximate $y'(0)$ reported in [29]

$$y'(0) = 10^{-12} \exp(29.71 - n), \qquad y'(0) < 10^{-4}, \tag{4.1}$$

resulting in a remarkable accuracy for the range $n \in [22, 1e(+12)]$.

In the same fashion, in Table 5, we compare $y'(1)$ for proposed solution and other reported numerical solutions [4, 5, 28, 30]. The results shows that (3.14) has a good accuracy for $y'(1)$ at least in the range $n \in [1, 20]$.

Figure 2 results from Tables 1, 2, and 3 and Maple routines of numerical solution of differential equations. We can observe the overlap of numerical solution and (3.14), for different values of $n \in [0.5, 10]$. Therefore, (3.14) is a high accurate approximate solution, valid for a wide range of values of Troesch's parameter.

Further research should be performed in order to verify the accuracy of (3.14) for large values of n. Nevertheless, as we know from above discussion $y'(0)$ and $y'(1)$ are valid and accurate in the ranges $1 \le n \le 1e(+12)$ and $1 \le n \le 20$, respectively, therefore, we can expect a high accuracy of (3.14) in the same ranges.

5. Concluding Remarks

In this work, we obtained an approximate solution for Troesch's problem. Besides, we presented a comparison between the numerical solution, the proposed solution, and other approximations reported in the literature. The numerical and graphical results show that the proposed solution is the most accurate one, for a wide range of values of Troesch's parameter (n). Moreover, Troesch's problem approximation exhibit a remarkable accuracy for $y'(0)$ at least in the range $1 \le n \le 1e(+12)$. In the same fashion, $y'(1)$ exhibits a good accuracy at least in the range $1 \le n \le 20$. Accordingly, we can expect a good accuracy of proposed solution for the range $0 \le n \le 1e(+12)$ and maybe for even more larger values of n. Additionally, the

Table 4: Initial slope $y'(0)$ of Troesch's problem.

	$y'(0)$			
n	This work	Other methods	[5]	VIM [15]
1	8.45197542e(−01)	8.452026845e(−01) [4]	8.452026884e(−01)	8.452114213e(−01)
2	5.186322404e(−01)	5.186212200e(−01) [4]	—	—
3	2.55607567e(−01)	2.556042136e(−01) [4]	2.556042156e(−01)	2.557127118e(−01)
4	1.118803125e(−01)	1.118801662e(−01) [4]	—	—
5	4.575046433e(−02)	4.575046116e(−02) [4]	4.575046140e(−02)	4.575261507e(−02)
6	1.795094954e(−02)	1.795094997e(−02) [4]	—	—
7	6.867509691e(−03)	6.867509416e(−03) [4]	—	—
8	2.587169418e(−03)	2.587169460e(−03) [4]	2.587169399e(−03)	
9	9.655845408e(−04)	9.655844857e(−04) [4]	—	—
10	3.583377845e(−04)	3.583377707e(−04) [4]	3.583377675e(−04)	3.583377854e(−04)
12	4.891062176e(−05)	4.8910e(−05) [28]	4.891060963e(−05)	—
15	2.444513025e(−06)	2.44450e(−06) [28]	2.444506324e(−06)	—
20	1.648773182e(−08)	1.65e(−08) [29]	1.648666618e(−08)	—
22	2.231499933e(−09)	2.230542258e(−09) [29]	—	—
25	1.111027228e(−10)	1.110521599e(−10) [29]	—	—
28	5.531510890e(−12)	5.528961478e(−12) [29]	—	—
30	7.486093793e(−13)	7.482635676e(−13) [29]	—	—
50	1.542999878e(−21)	1.542286163e(−21) [29]	—	—
1e(+02)	2.976060781e(−43)	2.974684202e(−43) [29]	—	—
2e(+02)	1.107117221e(−86)	1.106605124e(−86) [29]	—	—
5e(+02)	5.699661125e(−217)	5.697024744e(−217) [29]	—	—
1e(+03)	4.060767116e(−434)	4.058888808e(−434) [29]	—	—
1e(+04)	9.083870920e(−4343)	9.079669173e(−4343) [29]	—	—
1e(+05)	2.850359652e(−43429)	2.849041217e(−43429) [29]	—	—
1e(+06)	2.637465182e(−434294)	2.636245221e(−434294) [29]	—	—
1e(+12)	4.479838276e(−434294481903)	4.477766121e(−434294481903) [29]	—	—

Table 5: Slope $y'(1)$ of Troesch's problem.

	$y'(1)$		
n	This work	Other methods	[5]
1	$1.341828780e(+00)$	$1.341837966e(+00)$ [4]	$1.341837867e(+00)$
2	$2.406790318e(+00)$	$2.406939711e(+00)$ [4]	—
3	$4.266151411e(+00)$	$4.266223175e(+00)$ [4]	$4.266222862e(+00)$
4	$7.254574096e(+00)$	$7.254582910e(+00)$ [4]	—
5	$1.210049478e(+01)$	$1.210049568e(+01)$ [4]	$1.210049546e(+01)$
6	$2.003575791e(+01)$	$2.003575367e(+01)$ [4]	—
7	$3.308525498e(+01)$	$3.308526669e(+01)$ [4]	—
8	$5.457983465e(+01)$	$5.461322834e(+01)$ [4]	$5.457983447e(+01)$
9	$9.000602248e(+01)$	$9.000618074e(+01)$ [4]	—
10	$1.484064126e(+02)$	$1.484065422e(+02)$ [4]	$1.484064212e(+02)$
12	$4.034263503e(+02)$	$4.034263147e(+02)$ [28]	$4.034263147e(+02)$
15	$1.808042673e(+03)$	$1.808041861e(+03)$ [28]	$1.808041861e(+03)$
20	$2.202629966e(+04)$	$2.20264657e(+04)$ [30]	$2.202646577e(+04)$

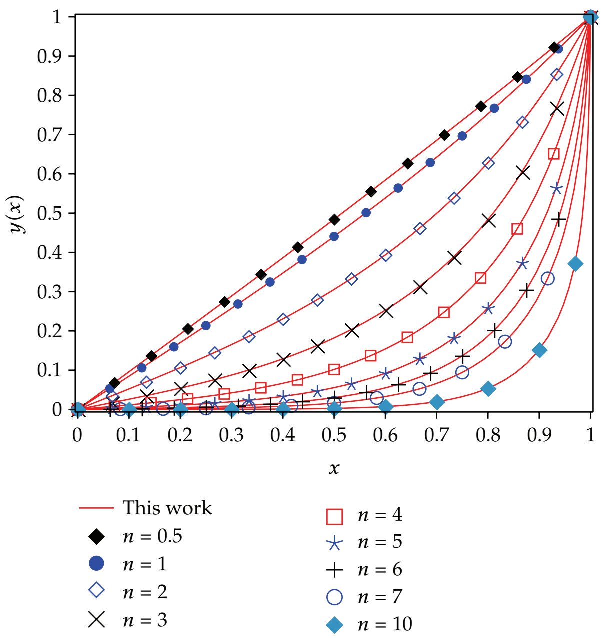

Figure 2: Numerical solution (different symbols) of (1.1) and proposed solution (3.14) (solid line) for different values of n.

proposed approximated solution does not require any adjustment parameter as reported for solutions obtained by using LDTM and VIM methods, which makes our proposed solution easier to use than those approximations. Finally, further research is necessary in order to verify the accuracy of our proposed approximation for large values of n.

Acknowledgments

The authors gratefully acknowledge the financial support of the National Council for Science and Technology of Mexico (CONACYT) through Grant CB-2010-01 no. 157024. The authors would like to thank Rogelio-Alejandro Callejas-Molina and Roberto Ruiz-Gomez for their contribution to this project.

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