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Research Article

Highly Efficient Calculation Schemes of Finite-Element Filter Approach for the Eigenvalue Problem of Electric Field

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This paper discusses finite-element highly efficient calculation schemes for solving eigenvalue problem of electric field. Multigrid discretization is extended to the filter approach for eigenvalue problem of electric field. With this scheme one solves an eigenvalue problem on a coarse grid just at the first step, and then always solves a linear algebraic system on finer and finer grids. Theoretical analysis and numerical results show that the scheme has high efficiency. Besides, we use interpolation postprocessing technique to improve the accuracy of solutions, and numerical results show that the scheme is an efficient and significant method for eigenvalue problem of electric field.

1. Introduction

In recent years, eigenvalue problems of electric field has attracted increasing attention in the fields of physics and mathematics, and its numerical methods (the filter approach, the parameterized approach, and the mixed approach) are also developed further (see [1–7]). Although the filter approach is an effective and important method for solving eigenvalue problems of electric field, its computation costs and accuracy of numerical solutions still need to be improved.

In fact, it is really a challenging job to reduce the computation costs without decreasing the accuracy of finite-element solutions. As we know, two-grid discretization and multigrid discretization are reliable and important methods satisfying the above requirements. Two-grid discretization was first introduced by Xu for nonsymmetric and nonlinear elliptic problems, and so forth (see [8–10]). Later on, it was successfully applied to Stokes equations, semilinear eigenvalue problems and linear eigenvalue problems, and so forth (see [11–16]). Recently, Yang and Bi [16] established two-grid finite-element discretization and multigrid discretization Schemes based on shifted-inverse power method. Referecnces [6, 7] applied the two Schemes to the mixed approach for eigenvalue problem of electric field, and [17] applied them to conforming finite element for the Steklov eigenvalue problem. Based on the work mentioned above, this paper discusses two-grid discretization and multigrid discretization Schemes of the filter approach for eigenvalue problem of electric field and analyzes error estimates. They are extensions of Scheme 2 and Scheme 3 in [16], respectively.

From 1989 to 1991, Lin and Yang firstly pointed out and proved that the function, obtained by using nodes of lower-order element as interpolation nodes to make a higher order interpolation of lower order finite-element solutions, can have global gradient superconvergence. The technique used to obtain global superconvergence was called finite-element interpolation postprocessing or finite-element interpolation correction (see reviews paper [18] and the references cited therein). For over 20 years, finite element interpolation postprocessing technique has been developed greatly and was applied to a variety of partial differential equations (see [19–25]). It is applied to this paper too. We give Theorem 4.1, and our numerical results show that interpolation postprocessing is an efficient and significant method for solving eigenvalue problems of electric field.

The rest of this paper is organized as follows. In the next section, some preliminaries which are needed are provided. In Section 3, two kinds of finite-element discretization schemes for eigenvalue problem of electric field are given and the error estimates are established. In Section 4, we introduce interpolation postprocessing technique. Finally, numerical experiments are presented.

2. Preliminaries

Let $\Omega \subset R^n$ $(n = 2, 3)$ be a bounded polyhedron domain with boundary $\partial\Omega$. We denote by $\mathbf{n}$ the unit outward normal vector to $\partial\Omega$, by $\mathbf{u}$ the electric field, and by ω the time frequency. Let $c = 3.0 \times 10^8$ m/s be the light velocity in vacuum, **curl** curl operator, and div divergence operator.

Consider the following eigenvalue problem of electric field:

$$\begin{aligned} c^2 \mathbf{curl}\mathbf{curl}\, \mathbf{u} &= \omega^2 \mathbf{u}, \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0, \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= 0, \quad \text{on} \in \partial\Omega. \end{aligned} \tag{2.1}$$

Let $\lambda = \omega^2/c^2$ named eigenvalue.

Define function spaces as follows:

$$\begin{aligned} H(\mathbf{curl}, \Omega) &= \left\{ \mathbf{v} \in L_2(\Omega)^n : \mathbf{curl}\, \mathbf{v} \in L_2(\Omega)^n \right\}, \\ H_0(\mathbf{curl}, \Omega) &= \{ \mathbf{v} \in H(\mathbf{curl}, \Omega) : \mathbf{v} \times \mathbf{n}|_{\partial\Omega} = 0 \}. \end{aligned} \tag{2.2}$$

When Ω is a convex polyhedron, we define the following function space:

$$X = \left\{ \mathbf{v} \in H_0(\mathbf{curl}, \Omega) : \operatorname{div} \mathbf{v} \in L^2(\Omega) \right\}. \tag{2.3}$$

Denote

$$\begin{gathered} b(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v})_0 = \int_\Omega \mathbf{u} \cdot \mathbf{v} dx, \qquad \|\mathbf{u}\|_0 = (\mathbf{u}, \mathbf{u})_0^{1/2}, \\ a(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v})_X = (\mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{v})_0 + (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v})_0, \qquad \|\mathbf{u}\|_X = (\mathbf{u}, \mathbf{u})_X^{1/2}. \end{gathered} \tag{2.4}$$

Let $\sigma_\Delta^D \in (3/2, 2)$ be the following smallest singular exponent in the Laplace problem with homogenous Dirichlet boundary condition:

$$\begin{gathered} \left\{ \phi \in H^1(\Omega) : \Delta\phi \in L_2(\Omega), \psi\big|_{\partial\Omega} = 0 \right\} \subset \cap_{s<\sigma_\Delta^D} H^s(\Omega), \\ \left\{ \phi \in H^1(\Omega) : \Delta\phi \in L_2(\Omega), \psi\big|_{\partial\Omega} = 0 \right\} \not\subset H^{\sigma_\Delta^D}(\Omega). \end{gathered} \tag{2.5}$$

Set $\gamma_{\min} = 2 - \sigma_\Delta^D$ and $\gamma \in (\gamma_{\min}, 1)$.

When Ω is a nonconvex polyhedron, let E denote a set of edges of reentrant dihedral angles on $\partial\Omega$, and let $d = d(x)$ denote the distance to the set $E : d(x) = \operatorname{dist}(x, \cup_{e \in E} \overline{e})$. We introduce a weight function ω_γ which is a nonnegative smooth function corresponding to x. It can be represented by d^γ in reentrant edge and angular domain. We shall write $\omega_\gamma \simeq d^\gamma$. Define the weighted functional spaces:

$$\begin{gathered} L_\gamma^2(\Omega) = \left\{ w \in L_{\text{loc}}^2(\Omega) : \omega_\gamma w \in L_2(\Omega) \right\}, \\ X_\gamma = \left\{ \mathbf{v} \in L_2(\Omega)^n : \mathbf{curl}\, \mathbf{v} \in L_2(\Omega)^n, \mathbf{v} \times \mathbf{n}|_{\partial\Omega} = 0, \operatorname{div} \mathbf{v} \in L_\gamma^2(\Omega) \right\}. \end{gathered} \tag{2.6}$$

Denote

$$\begin{gathered} (w, v)_{L_\gamma^2} = \int_\Omega \omega_\gamma^2 w v dx, \qquad \|w\|_{0,\gamma} = (w, w)_{L_\gamma^2}^{1/2}, \\ a(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v})_{X_\gamma} = (\mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{v})_0 + (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v})_{L_\gamma^2}, \qquad \|\mathbf{u}\|_{X_\gamma} = (\mathbf{u}, \mathbf{u})_{X_\gamma}^{1/2}. \end{gathered} \tag{2.7}$$

Note that $X_\gamma = X$ when Ω is a convex polyhedron, namely, in the case of $\gamma = 0$. Consider the variational formulation: Find $(\lambda, \mathbf{u}) \in R^+ \times X_\gamma$ with $\|\mathbf{u}\|_{X_\gamma} = 1$, such that

$$a(\mathbf{u}, \mathbf{v}) = \lambda b(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in X_\gamma. \tag{2.8}$$

Let π_h be a regular simplex partition, and let X_h be a space of piecewise polynomial of degree less than or equal to k defined on π_h:

$$X_h = \left\{ \mathbf{v} \in C^0\left(\overline{\Omega}\right)^n : \mathbf{v} \times \mathbf{n}|_{\partial\Omega} = 0, \mathbf{v}|_\kappa \in P_k(\kappa)^n, \forall \kappa \in \pi_h \right\}. \tag{2.9}$$

Then, $X_h \subseteq X_\gamma$.

The discrete variational form of (2.8): Find $(\lambda_h, \mathbf{u}_h) \in R^+ \times X_h$ with $\|\mathbf{u}_h\|_{X_\gamma} = 1$, such that

$$a(\mathbf{u}_h, \mathbf{v}) = \lambda_h b(\mathbf{u}_h, \mathbf{v}), \quad \forall \mathbf{v} \in X_h. \tag{2.10}$$

The eigenpairs of (2.1) must be that of (2.8). But the converse of this statement may not be true, namely, (2.8) has spurious pairs. Hence, (2.10) has spurious pairs.

It is easy to prove that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are symmetric bilinear forms. Next we shall prove that $a(\cdot,\cdot)$ is continuous and V-elliptic.

From the definition of $a(\cdot,\cdot)$, we have

$$\begin{aligned} |a(\mathbf{w}, \mathbf{v})| &= \left| \int_\Omega \mathbf{curl}\, \mathbf{w} \cdot \mathbf{curl}\, \mathbf{v} dx + \int_\Omega \omega_\gamma^2 \operatorname{div} \mathbf{w} \operatorname{div} \mathbf{v}\, dx \right| \\ &\le \|\mathbf{curl}\, \mathbf{w}\|_0 \|\mathbf{curl}\, \mathbf{v}\|_0 + \|\operatorname{div} \mathbf{w}\|_{0,\gamma} \|\operatorname{div} \mathbf{v}\|_{0,\gamma} \\ &\le \sqrt{\left(\|\mathbf{curl}\, \mathbf{w}\|_0^2 + \|\operatorname{div} \mathbf{w}\|_{0,\gamma}^2\right)\left(\|\mathbf{curl}\, \mathbf{v}\|_0^2 + \|\operatorname{div} \mathbf{v}\|_{0,\gamma}^2\right)} \\ &\le \|\mathbf{w}\|_{X_\gamma} \|\mathbf{v}\|_{X_\gamma}. \end{aligned} \tag{2.11}$$

Therefore, continuity of $a(\cdot,\cdot)$ is valid. And

$$a(\mathbf{w}, \mathbf{w}) = \|\mathbf{w}\|_{X_\gamma}^2, \tag{2.12}$$

which indicates that $a(\cdot,\cdot)$ is V-elliptic.

Define operator $T : X_\gamma \to X_\gamma$ satisfying

$$a(T\mathbf{f}, \mathbf{v}) = b(\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in X_\gamma. \tag{2.13}$$

Define operator $T_h : X_h \to X_h$ satisfying

$$a(T_h\mathbf{f}, \mathbf{v}) = b(\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in X_h. \tag{2.14}$$

It is easy to prove that $T : X_\gamma \to X_\gamma$, $T_h : X_h \to X_h$ is self-adjoint completely continuous operator, respectively. Actually, for all $\mathbf{f}, \mathbf{g} \in X_\gamma$, we have

$$a(\mathbf{f}, T\mathbf{g}) = a(T\mathbf{g}, \mathbf{f}) = b(\mathbf{g}, \mathbf{f}) = b(\mathbf{f}, \mathbf{g}) = a(T\mathbf{f}, \mathbf{g}), \tag{2.15}$$

which shows that $T : X_\gamma \to X_\gamma$ is self-adjoint in the sense of inner product $a(\cdot,\cdot)$. Similarly, we can prove that $T_h : X_h \to X_h$ is self-adjoint in the sense of inner product $a(\cdot,\cdot)$.

From [2, 4], we get $X_\gamma \hookrightarrow L_2(\Omega)^3$ (compactly imbedded). Hence, we derive that operator $T : X_\gamma \to X_\gamma$ is completely continuous. Obviously, $T_h : X_h \to X_h$ is a finite-rank operator.

By [3, 26], we know that (2.8) has the following equivalent operator form:

$$T\mathbf{u} = \mu\mathbf{u}. \tag{2.16}$$

Denote $\mu_k = 1/\lambda_k$, $\mu_{k,h} = 1/\lambda_{k,h}$.

Then, the eigenvalues of (2.8) are sorted as

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \nearrow +\infty. \tag{2.17}$$

We can construct a complete orthogonal system of X_γ by using the eigenfunctions corresponding to $\{\lambda_k\}$:

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \ldots. \tag{2.18}$$

Equation (2.10) has the following equivalent operator form:

$$\lambda_h T_h \mathbf{u}_h = \mathbf{u}_h. \tag{2.19}$$

Then the eigenvalues of (2.10) are sorted as

$$0 < \lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{k,h} \le \cdots \lambda_{N_h,h}, \tag{2.20}$$

and the corresponding eigenfunctions are

$$\mathbf{u}_{1,h}, \mathbf{u}_{2,h}, \ldots, \mathbf{u}_{k,h}, \ldots, \mathbf{u}_{N_h,h}, \tag{2.21}$$

where $N_h = \dim X_h$.

In this paper, μ_k and $\mu_{k,h}$, λ_k and $\lambda_{k,h}$ are all called eigenvalues.

Suppose that the algebraic multiplicity of μ_k is equal to q. $\mu_k = \mu_{k+1} = \cdots = \mu_{k+q-1}$. Let $M(\mu_k)$ be the space spanned by all eigenfunctions corresponding to μ_k of T, and let $M_h(\mu_k)$ be the space spanned by all eigenfunctions corresponding to all eigenvalues of T_h that converge to μ_k. Let $\widehat{M}(\mu_k) = \{v : v \in M(\mu_k), \|v\|_{X_\gamma} = 1\}$, $\widehat{M}_h(\mu_k) = \{v : v \in M_h(\mu_k), \|v\|_{X_\gamma} = 1\}$. We also write $M(\lambda_k) = M(\mu_k)$, $M_h(\lambda_k) = M_h(\mu_k)$, $\widehat{M}(\lambda_k) = \widehat{M}(\mu_k)$, and $\widehat{M}_h(\lambda_k) = \widehat{M}_h(\mu_k)$.

The Filter Approach

Let $(\lambda_h, \mathbf{u}_h)$ be an eigenpair of (2.10), we know that some of these eigenvalues are "real," but some are spurious (namely, not divergence free). We should filter out the spurious pairs to obtain "real" eigenpairs. Hence, ones designed a filter ratio:

$$\frac{\|\operatorname{div} \mathbf{u}_h\|_{0,\gamma}}{\|\operatorname{curl} \mathbf{u}_h\|_0}. \tag{2.22}$$

The corresponding value of filter ratio is small for "real" pairs since the divergence part of the eigenvector is small, whereas it is large for spurious ones since the curl part small. Noting that when a multiple eigenvalue is dealt with, an additional step must be carried out (see [3, 5]).

Next we introduce error estimates for the filter approach.

Define $\|(T-T_h)|_{M(\lambda_k)}\|_{X_\gamma} = \max_{\mathbf{u}\in M(\lambda_k)}(\|(T-T_h)\mathbf{u}\|_{X_\gamma}/\|\mathbf{u}\|_{X_\gamma})$.

Denote

$$\delta_h(\lambda_k) = \sup_{\mathbf{u}\in\widehat{M}(\lambda_k)} \inf_{\mathbf{v}\in X_h} \|\mathbf{u}-\mathbf{v}\|_{X_\gamma}. \tag{2.23}$$

Let $P_h : X_\gamma \to X_h$ be orthogonal projection, namely,

$$a(\mathbf{u}-P_h\mathbf{u},\mathbf{v}) = 0, \quad \forall \mathbf{v} \in X_h. \tag{2.24}$$

Then, $T_h = P_h T$.

Using the spectral theory (see [26]), [3] discussed error estimates for the filter approach and gave the following lemmas.

Lemma 2.1. $\|T_h - T\|_{X_\gamma} \to 0$ $(h \to 0)$.

Lemma 2.2. *Let* $(\lambda_{k,h}, \mathbf{u}_{k,h})$ *be the kth eigenpair of* (2.10) *with* $\|\mathbf{u}_{k,h}\|_{X_\gamma} = 1$. *Let* λ_k *be the kth eigenvalue of* (2.8). *Then, there exists* $\mathbf{u}_k \in \widehat{M}(\lambda_k)$ *such that*

$$|\lambda_{k,h} - \lambda_k| \le C_1 \delta_h^2(\lambda_k), \tag{2.25}$$

$$\|\mathbf{u}_{k,h} - \mathbf{u}_k\|_{X_\gamma} \le C_2 \delta_h(\lambda_k). \tag{2.26}$$

For any $\mathbf{u}_k \in \widehat{M}(\lambda_k)$, *there exists* $\mathbf{u}_h \in M_h(\lambda_k)$ *such that*

$$\|\mathbf{u}_h - \mathbf{u}_k\|_{X_\gamma} \le C_3 \delta_h(\lambda_k), \tag{2.27}$$

where C_1, C_2, *and* C_3 *are constants independent of mesh diameter.*

In this paper, we will use the following lemma.

Lemma 2.3. *Let* $(\lambda, \mathbf{u})$ *be an eigenpair of* (2.8), *then for any* $\mathbf{w} \in X_\gamma$, $\|\mathbf{w}\|_0 \neq 0$, *the Rayleigh quotient* $a(\mathbf{w},\mathbf{w})/\|\mathbf{w}\|_0^2$ *satisfies*

$$\frac{a(\mathbf{w},\mathbf{w})}{\|\mathbf{w}\|_0^2} - \lambda = \frac{\|\mathbf{w}-\mathbf{u}\|_{X_\gamma}^2}{\|\mathbf{w}\|_0^2} - \lambda \frac{\|\mathbf{w}-\mathbf{u}\|_0^2}{\|\mathbf{w}\|_0^2}. \tag{2.28}$$

Proof. The proof is completed by using the same proof steps as that of Lemma 9.1 in [26]. □

3. Two-Grid Discretization Scheme and Multigrid Discretization Scheme

Consider (2.19) on X_h (inner product $a(\cdot,\cdot)$ and norm $\|\cdot\|_{X_Y}$). We will discuss the high efficiency of two-grid discretization scheme and multigrid discretization scheme next.

Lemma 3.1. *For all nonzero* $\mathbf{u},\mathbf{v}\in X_Y$,

$$\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|_{X_Y}}-\frac{\mathbf{v}}{\|\mathbf{v}\|_{X_Y}}\right\|_{X_Y}\le 2\frac{\|\mathbf{u}-\mathbf{v}\|_{X_Y}}{\|\mathbf{u}\|_{X_Y}},\qquad \left\|\frac{\mathbf{u}}{\|\mathbf{u}\|_{X_Y}}-\frac{\mathbf{v}}{\|\mathbf{v}\|_{X_Y}}\right\|_{X_Y}\le 2\frac{\|\mathbf{u}-\mathbf{v}\|_{X_Y}}{\|\mathbf{v}\|_{X_Y}}. \tag{3.1}$$

Proof. See [16]. □

Lemma 3.2. *Let* $(\mu_0,\mathbf{u}_0)$ *be an approximation for* $(\mu_k,\mathbf{u}_k)$, *where* μ_0 *is not an eigenvalue of* T_h, *and* $\mathbf{u}_0\in X_h$ *with* $\|\mathbf{u}_0\|_{X_Y}=1$. *Suppose that* $\max_{k\le j\le k+q-1}|(\mu_{j,h}-\mu_{k,h})/(\mu_0-\mu_{j,h})|\le 1/2$, $\operatorname{dist}(\mathbf{u}_0,M_h(\mu_k))\le 1/2$, $|\mu_0-\mu_{j,h}|\ge\rho/2$, $(j\neq k,k+1,\ldots,k+q-1)$, *and* $\mathbf{u}\in X_h$, $\mathbf{u}_k^h\in X_h$ *satisfy*

$$(\mu_0-T_h)\mathbf{u}=\mathbf{u}_0,\qquad \mathbf{u}_k^h=\frac{\mathbf{u}}{\|\mathbf{u}\|_{X_Y}}. \tag{3.2}$$

Then

$$\operatorname{dist}\left(\mathbf{u}_k^h,\widehat{M}_h(\mu_k)\right)\le\frac{16}{\rho}\left|\mu_0-\mu_{k,h}\right|\operatorname{dist}(\mathbf{u}_0,M_h(\mu_k)), \tag{3.3}$$

where $\rho=\min_{\mu_j\neq\mu_k}|\mu_j-\mu_k|$ *is the separation constant of the eigenvalue* μ_k.

Proof. See [16]. □

3.1. Two-Grid Discretization Scheme

Reference [16] established the two-grid discretization scheme based on shifted-inverse power method. Next, we will apply the scheme to eigenvalue problem of electric field.

Let π_H and π_h be regular meshes (see [3]) with diameters H and h, respectively. Let $\delta_h(\lambda_k)=\delta_H(\lambda_k)^{t_2}$, $t_2\in[1+\delta,3-\delta]$, and δ be a properly small positive number.

Scheme 1. Two-grid Discretization.

Step 1. Solve (2.8) on a coarse grid π_H: Find $(\lambda_H,\mathbf{u}_H)\in R^+\times X_H$, such that $\|\mathbf{u}_H\|_{X_Y}=1$, and

$$a(\mathbf{u}_H,\mathbf{v})=\lambda_H b(\mathbf{u}_H,\mathbf{v}),\quad \forall\mathbf{v}\in X_H. \tag{3.4}$$

And obtain the "real" eigenpair $(\lambda_{k,H},\mathbf{u}_{k,H})$ by filtering process.

Step 2. Solve a linear system on a fine grid π_h: Find $\mathbf{u} \in X_h$, such that

$$a(\mathbf{u}, \mathbf{v}) - \lambda_{k,H} b(\mathbf{u}, \mathbf{v}) = b(\mathbf{u}_{k,H}, \mathbf{v}), \quad \forall \mathbf{v} \in X_h. \tag{3.5}$$

And set $\mathbf{u}_k^h = \mathbf{u} / \|\mathbf{u}\|_{X_Y}$.

Step 3. Compute the Rayleigh quotient

$$\lambda_k^h = \frac{a\left(\mathbf{u}_k^h, \mathbf{u}_k^h\right)}{b\left(\mathbf{u}_k^h, \mathbf{u}_k^h\right)}. \tag{3.6}$$

We use $(\lambda_k^h, \mathbf{u}_k^h)$ as the approximate eigenpair of (2.1).

Theorem 3.3. *Suppose that H is properly small. Let $(\lambda_k^h, \mathbf{u}_k^h)$ be the approximate eigenpair obtained by Scheme 1. Then there exists eigenpair $(\lambda_k, \mathbf{u}_k)$ of* (2.1)*, such that*

$$\left\|\mathbf{u}_k^h - \mathbf{u}_k\right\|_{X_Y} \le \frac{48}{\rho} C_1 C_2 C_4 C_5 \delta_H^3(\lambda_k) + 3C_2 q \delta_h(\lambda_k), \tag{3.7}$$

$$\left|\lambda_k^h - \lambda_k\right| \le 2\lambda_k (1 + C_6 \lambda_k) \left\|\mathbf{u}_k^h - \mathbf{u}_k\right\|_{X_Y}^2, \tag{3.8}$$

where C_4, C_5, and C_6 are positive constants independent of mesh diameters, and these constants are decided by (3.11), (3.13), *and* (3.30) *in the following proof.*

Proof. We use Lemma 3.2 to complete the proof. Select $\mu_0 = 1/\lambda_{k,H}$ and $\mathbf{u}_0 = \lambda_{k,H} T_h \mathbf{u}_{k,H} / \|\lambda_{k,H} T_h \mathbf{u}_{k,H}\|_{X_Y}$. Obviously,

$$\|\mathbf{u}_0\|_{X_Y} = 1. \tag{3.9}$$

Noting that $\lambda_k = \lambda_{k+1} = \cdots \lambda_{k+q-1}$, for $j = k, k+1, \ldots, k+q-1$, we have

$$\begin{aligned} \left|\mu_{j,h} - \mu_{k,h}\right| &= \left|\frac{\lambda_{k,h} - \lambda_{j,h}}{\lambda_{k,h}\lambda_{j,h}}\right| = \left|\frac{\lambda_{k,h} - \lambda_k + \lambda_j - \lambda_{j,h}}{\lambda_{k,h}\lambda_{j,h}}\right| \\ &\le \frac{\left|\lambda_{k,h} - \lambda_k\right| + \left|\lambda_j - \lambda_{j,h}\right|}{\lambda_{k,h}\lambda_{j,h}}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \left|\mu_0 - \mu_{j,h}\right| &= \left|\frac{\lambda_{j,h} - \lambda_{k,H}}{\lambda_{j,h}\lambda_{k,H}}\right| \\ &= \left|\frac{\lambda_{j,h} - \lambda_j + \lambda_k - \lambda_{k,H}}{\lambda_{j,h}\lambda_{k,H}}\right| \\ &\le C_4 |\lambda_{k,H} - \lambda_k|. \end{aligned} \tag{3.11}$$

Combining the above two inequalities with (2.25) and noting that $\delta_h(\lambda_k)$ is a small quantity of higher order than $\delta_H(\lambda_k)$, we obtain

$$\max_{k\le j\le k+q-1}\left|\frac{\mu_{j,h}-\mu_{k,h}}{\mu_0-\mu_{j,h}}\right|\le\frac{1}{2}. \tag{3.12}$$

From Lemma 2.1, we know that $\|T_h-T\|_{X_\gamma}\to 0$ $(h\to 0)$, then there exists a constant C_5 independent of h, such that

$$\|T_h\mathbf{f}\|_{X_\gamma}\le C_5\|\mathbf{f}\|_{X_\gamma},\quad \forall\mathbf{f}\in X_h. \tag{3.13}$$

Obviously, there exists $\mathbf{u}^k\in\widehat{M}(\lambda_k)$, such that

$$\left\|\mathbf{u}_{k,H}-\mathbf{u}^k\right\|_{X_\gamma}=\operatorname{dist}\left(\mathbf{u}_{k,H},\widehat{M}(\lambda_k)\right). \tag{3.14}$$

Then, we derive

$$\begin{aligned}\left\|\lambda_{k,H}T_h\mathbf{u}_{k,H}-\mathbf{u}^k\right\|_{X_\gamma}&=\left\|\lambda_{k,H}T_h\mathbf{u}_{k,H}-\lambda_kT\mathbf{u}^k\right\|_{X_\gamma}\\&\le\|(\lambda_{k,H}-\lambda_k)T_h\mathbf{u}_{k,H}\|_{X_\gamma}+\left\|\lambda_kT_h(\mathbf{u}_{k,H}-\mathbf{u}^k)\right\|_{X_\gamma}+\left\|\lambda_k(T_h-T)\mathbf{u}^k\right\|_{X_\gamma}\\&\le C_5|\lambda_{k,H}-\lambda_k|+C_5\lambda_k\left\|\mathbf{u}_{k,H}-\mathbf{u}^k\right\|_{X_\gamma}+\left\|(P_h-I)\mathbf{u}^k\right\|_{X_\gamma}\\&\le C_5|\lambda_{k,H}-\lambda_k|+C_5\lambda_k\left\|\mathbf{u}_{k,H}-\mathbf{u}^k\right\|_{X_\gamma}+\delta_h(\lambda_k).\end{aligned} \tag{3.15}$$

Hence, by Lemma 3.1, (3.15) and (2.28), we have

$$\begin{aligned}\operatorname{dist}\left(\mathbf{u}_0,\widehat{M}(\lambda_k)\right)&\le\left\|\mathbf{u}_0-\mathbf{u}^k\right\|_{X_\gamma}\\&\le 2\left\|\lambda_{k,H}T_h\mathbf{u}_{k,H}-\mathbf{u}^k\right\|_{X_\gamma}\\&\le 2C_5|\lambda_{k,H}-\lambda_k|+2C_5\lambda_k\left\|\mathbf{u}_{k,H}-\mathbf{u}^k\right\|_{X_\gamma}+2\delta_h(\lambda_k)\\&\le 3C_5\lambda_k\left\|\mathbf{u}_{k,H}-\mathbf{u}^k\right\|_{X_\gamma}+2\delta_h(\lambda_k)\\&=3C_5\lambda_k\operatorname{dist}\left(\mathbf{u}_{k,H},\widehat{M}(\lambda_k)\right)+2\delta_h(\lambda_k).\end{aligned} \tag{3.16}$$

Combining the triangle inequality, (2.27) and (3.16), we deduce

$$\begin{aligned}\operatorname{dist}(\mathbf{u}_0, M_h(\lambda_k)) &\le \operatorname{dist}\Big(\mathbf{u}_0, \widehat{M}(\lambda_k)\Big) + C_3\delta_h(\lambda_k)\\ &\le 3C_5\lambda_k \operatorname{dist}\Big(\mathbf{u}_{k,H}, \widehat{M}(\lambda_k)\Big) + (C_3+2)\delta_h(\lambda_k).\end{aligned} \tag{3.17}$$

Since H is small enough and $\delta_h(\lambda_k) = \delta_H(\lambda_k)^{t_2}$, from (2.26) and (3.17), we know

$$\operatorname{dist}(\mathbf{u}_0, M_h(\lambda_k)) \le \frac{1}{2}. \tag{3.18}$$

For $j \ne k,\ k+1, \dots, k+q-1$, since H is small enough, ρ is the separation constant, we have

$$\left|\mu_0 - \mu_{j,h}\right| \ge \frac{\rho}{2}. \tag{3.19}$$

From the Step 2 in Scheme 1 and (2.14), we get

$$a(\mathbf{u},\mathbf{v}) - \lambda_{k,H} a(T_h\mathbf{u},\mathbf{v}) = a(T_h\mathbf{u}_{k,H},\mathbf{v}), \tag{3.20}$$

namely,

$$\mathbf{u} - \lambda_{k,H} T_h \mathbf{u} = T_h \mathbf{u}_{k,H}. \tag{3.21}$$

Thus $(1/\lambda_{k,H})\mathbf{u} - T_h\mathbf{u} = (1/\lambda_{k,H})T_h\mathbf{u}_{k,H}$ and $\mathbf{u}_k^h = \mathbf{u}/\|\mathbf{u}\|_{X_Y}$. Note that $(1/\lambda_{k,H})T_h\mathbf{u}_{k,H} = \|(1/\lambda_{k,H})T_h\mathbf{u}_{k,H}\|_{X_Y}\mathbf{u}_0$ differs from $\mathbf{u}_0$ by only a constant; then Step 2 is equivalent to

$$\left(\frac{1}{\lambda_{k,H}} - T_h\right)\mathbf{u} = \mathbf{u}_0, \qquad \mathbf{u}_k^h = \frac{\mathbf{u}}{\|\mathbf{u}\|_{X_Y}}. \tag{3.22}$$

From the arguments of (3.9), (3.12), (3.18), (3.19), and (3.22), we see that the conditions of Lemma 3.2 hold. Hence, substituting (3.11) and (3.17) into (3.3), we obtain

$$\operatorname{dist}\Big(\mathbf{u}_k^h, \widehat{M}_h(\lambda_k)\Big) \le \frac{16}{\rho} C_4 |\lambda_{k,H} - \lambda_k| \Big(3C_5\lambda_k \operatorname{dist}\Big(\mathbf{u}_{k,H}, \widehat{M}(\lambda_k)\Big) + (C_3+2)\delta_h(\lambda_k)\Big). \tag{3.23}$$

Let eigenvectors $\{\mathbf{u}_{j,h}\}_k^{k+q-1}$ be an orthogonal basis of $M_h(\lambda_k)$ (in the sense of inner product $a(\cdot,\cdot)$), then

$$\operatorname{dist}\Big(\mathbf{u}_k^h, M_h(\lambda_k)\Big) = \left\| \mathbf{u}_k^h - \sum_{j=k}^{k+q-1} a(\mathbf{u}_k^h, \mathbf{u}_{j,h})\mathbf{u}_{j,h} \right\|_{X_Y}. \tag{3.24}$$

Set

$$\mathbf{u}^* = \sum_{j=k}^{k+q-1} a\left(\mathbf{u}_k^h, \mathbf{u}_{j,h}\right)\mathbf{u}_{j,h}. \tag{3.25}$$

From (3.23), we directly get

$$\begin{aligned}\left\|\mathbf{u}_k^h - \mathbf{u}^*\right\|_{X_\gamma} &= \operatorname{dist}\left(\mathbf{u}_k^h, M_h(\lambda_k)\right) \\ &\leq \frac{16}{\rho} C_4 |\lambda_{k,H} - \lambda_k| \left(3C_5\lambda_k \operatorname{dist}\left(\mathbf{u}_{k,H}, \widehat{M}(\lambda_k)\right) + (C_3+2)\delta_h(\lambda_k)\right).\end{aligned} \tag{3.26}$$

From Lemma 2.2, we know that there exist $\{\mathbf{u}_j^0\}_k^{k+q-1} \subset M(\lambda_k)$ making $\mathbf{u}_{j,h} - \mathbf{u}_j^0$ satisfy (2.26). Let $\mathbf{u}_k = \sum_{j=k}^{k+q-1} a(\mathbf{u}_k^h, \mathbf{u}_{j,h})\mathbf{u}_j^0$, then

$$\begin{aligned}\|\mathbf{u}^* - \mathbf{u}_k\|_{X_\gamma} &= \left\|\sum_{j=k}^{k+q-1} a\left(\mathbf{u}_k^h, \mathbf{u}_{j,h}\right)\left(\mathbf{u}_{j,h} - \mathbf{u}_j^0\right)\right\|_{X_\gamma} \\ &\leq \left(\sum_{j=k}^{k+q-1} \left\|\mathbf{u}_{j,h} - \mathbf{u}_j^0\right\|_{X_\gamma}^2\right)^{1/2} \\ &\leq C_2 \sum_{j=k}^{k+q-1} \delta_h(\lambda_j) \\ &\leq C_2 q \delta_h(\lambda_k).\end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27), we obtain

$$\begin{aligned}\left\|\mathbf{u}_k^h - \mathbf{u}_k\right\|_{X_\gamma} &\leq \left\|\mathbf{u}_k^h - \mathbf{u}^*\right\|_{X_\gamma} + \|\mathbf{u}^* - \mathbf{u}_k\|_{X_\gamma} \\ &\leq \frac{48}{\rho} C_4 C_5 \lambda_k |\lambda_{k,H} - \lambda_k| \operatorname{dist}\left(\mathbf{u}_{k,H}, \widehat{M}(\lambda_k)\right) + 3C_2 q \delta_h(\lambda_k).\end{aligned} \tag{3.28}$$

Besides, by (2.26), we easily know

$$\operatorname{dist}\left(\mathbf{u}_{k,H}, \widehat{M}(\lambda_k)\right) \leq C_2 \delta_H(\lambda_k), \tag{3.29}$$

which together with (3.28) and (2.25) leads to (3.7).

From the continuous embedding of X_γ into $L_2(\Omega)^n$, we get that there exists a constant C_6 independent of meshes, such that

$$\|\mathbf{u}\|_0^2 \leq C_6 \|\mathbf{u}\|_{X_\gamma}^2. \tag{3.30}$$

Equation (3.7) indicates that $\mathbf{u}_k^h$ converges to $\mathbf{u}_k$ in the sense of norm $\|\cdot\|_{X_\gamma}$, then $\mathbf{u}_k^h$ converges to $\mathbf{u}_k$ in the sense to norm $\|\cdot\|_0$; thus, $1/\|\mathbf{u}_k^h\|_0^2 \to 1/\|\mathbf{u}_k\|_0^2 = \lambda_k$. Therefore, when h is small enough, we have

$$\begin{aligned}\left|\lambda_k^h - \lambda_k\right| &\le \frac{\left\|\mathbf{u}_k^h - \mathbf{u}_k\right\|_{X_\gamma}^2}{\left\|\mathbf{u}_k^h\right\|_0^2} + \lambda_k \frac{\left\|\mathbf{u}_k^h - \mathbf{u}_k\right\|_0^2}{\left\|\mathbf{u}_k^h\right\|_0^2} \\ &\le 2\lambda_k \left\|\mathbf{u}_k^h - \mathbf{u}_k\right\|_{X_\gamma}^2 + 2\lambda_k^2 \left\|\mathbf{u}_k^h - \mathbf{u}_k\right\|_0^2 \\ &\le 2\lambda_k (1 + C_6\lambda_k) \left\|\mathbf{u}_k^h - \mathbf{u}_k\right\|_{X_\gamma}^2 .\end{aligned} \tag{3.31}$$

The proof of Theorem 3.3 is completed. □

Let σ_Δ^N be the smallest singular exponent in the Laplace problem with homogenous Neumann boundary condition, then $\sigma_\Delta^N \in (3/2, 2)$. Denote $\tau = \min(\gamma - \gamma_{\min}, \sigma_\Delta^N - 1)$.

Corollary 3.4. *Suppose that H is properly small. Let $(\lambda_k^h, \mathbf{u}_k^h)$ be an approximate eigenpair obtained by Scheme 1. Then there exists an eigenpair $(\lambda_k, \mathbf{u}_k)$ of* (2.1)*, such that when Ω is a convex domain,*

$$\left\|\mathbf{u}_k^h - \mathbf{u}_k\right\|_{X_\gamma} \le 6qC_2C'h, \tag{3.32}$$

$$\left|\lambda_k^h - \lambda_k\right| \le 2\lambda_k (1 + C_6\lambda_k) \left\|\mathbf{u}_k^h - \mathbf{u}_k\right\|_{X_\gamma}^2, \tag{3.33}$$

when Ω is a nonconvex domain,

$$\left\|\mathbf{u}_k^h - \mathbf{u}_k\right\|_{X_\gamma} \le 6qC_2C''h^\mu, \quad \forall \mu \in (0, \tau), \tag{3.34}$$

$$\left|\lambda_k^h - \lambda_k\right| \le 2\lambda_k (1 + C_6\lambda_k) \left\|\mathbf{u}_k^h - \mathbf{u}_k\right\|_{X_\gamma}^2, \tag{3.35}$$

where C' and C'' are stated in the proof as follows.

Proof. From [1, 4], we know that when Ω is a convex domain, there exists a constant C' independent of h, such that

$$\delta_h(\lambda_k) \le C'h. \tag{3.36}$$

Substituting the above inequality into (3.7), and noting that $\delta_h(\lambda_k)$ is an infinitesimal of lower order comparing with $\delta_H(\lambda_k)^3$, we know that (3.32) is valid.

And when Ω is a nonconvex domain, there exists a constant C'' independent of h, such that

$$\delta_h(\lambda_k) \le C'' h^{\mu}, \tag{3.37}$$

where $\mu \in (0, \tau)$.

Substituting the above inequality into (3.7), we know that (3.34) is valid. □

3.2. Multigrid Discretization Scheme

Next, we will discuss finite-element multigrid discretization scheme based on Rayleigh quotient iteration method. Assume that partition satisfies the following condition.

Condition (A). $\{\pi_{h_i}\}_1^l$ is a family of regular meshes (see [3]) with diameters h_i, $\delta_{h_{i+1}}(\lambda_k) = \delta_{h_i}(\lambda_k)^{t_{i+1}}$, $t_{i+1} \in [1+\delta, 3-\delta]$, $i = 1, 2, \ldots$, and δ is a properly small positive number.

Let $\{X_{h_i}\}_1^l$ be the finite-element spaces defined on $\{\pi_{h_i}\}_1^l$. Further, let $\pi_H = \pi_{h_1}$, $X_H = X_{h_1}$.

Scheme 2. Multigrid Discretization.

Step 1. Solve (2.8) on a coarse grid π_H: Find $(\lambda_H, \mathbf{u}_H) \in R^+ \times X_H$, with $\|\mathbf{u}_H\|_{X_Y} = 1$, such that

$$a(\mathbf{u}_H, \mathbf{v}) = \lambda_H b(\mathbf{u}_H, \mathbf{v}), \quad \forall \mathbf{v} \in X_H. \tag{3.38}$$

And obtain the "real" eigenpair $(\lambda_{k,H}, \mathbf{u}_{k,H})$ by filtering process.

Step 2. $\mathbf{u}_k^{h_1} = \mathbf{u}_{k,H}$, $\lambda_k^{h_1} = \lambda_{k,H}$, $i \Leftarrow 2$.

Step 3. Solve a linear system on a fine grid π_{h_i}: Find $\mathbf{u} \in X_{hi}$, such that

$$a(\mathbf{u}, \mathbf{v}) - \lambda_k^{h_{i-1}} b(\mathbf{u}, \mathbf{v}) = b\left(\mathbf{u}_k^{h_{i-1}}, \mathbf{v}\right), \quad \forall \mathbf{v} \in X_{h_i}. \tag{3.39}$$

Set $\mathbf{u}_k^{h_i} = \mathbf{u}/\|\mathbf{u}\|_{X_Y}$.

Step 4. Compute the Rayleigh quotient

$$\lambda_k^{h_i} = \frac{a\left(\mathbf{u}_k^{h_i}, \mathbf{u}_k^{h_i}\right)}{b\left(\mathbf{u}_k^{h_i}, \mathbf{u}_k^{h_i}\right)}, \tag{3.40}$$

Step 5. If $i = l$, then output $(\lambda_k^{h_l}, \mathbf{u}_k^{h_l})$, stop. Else, $i \Leftarrow i + 1$, and return to Step 3.

We use $(\lambda_k^{h_l}, \mathbf{u}_k^{h_l})$ obtained by Scheme 2 as the approximate eigenpair of (2.1). Next, we will discuss the efficiency of Scheme 2.

Theorem 3.5. *Suppose that H is properly small and Condition (A) holds. Let $(\lambda_k^{h_l}, \mathbf{u}_k^{h_l})$ be an approximate eigenpair obtained by Scheme 2. Then there exists an eigenpair $(\lambda_k, \mathbf{u}_k)$ of* (2.1)*, such that*

$$\left\|\mathbf{u}_k^{h_l} - \mathbf{u}_k\right\|_{X_Y} \leq \frac{48}{\rho} C_4 C_5 \lambda_k \left|\lambda_k^{h_{l-1}} - \lambda_k\right| \operatorname{dist}\left(\mathbf{u}_k^{h_{l-1}}, \widehat{M}(\lambda_k)\right) + 3C_2 q \delta_{h_l}(\lambda_k), \tag{3.41}$$

$$\left|\lambda_k^{h_l} - \lambda_k\right| \leq 2\lambda_k (1 + C_6 \lambda_k) \left\|\mathbf{u}_k^{h_l} - \mathbf{u}_k\right\|_{X_Y}, \quad l \geq 2. \tag{3.42}$$

Proof. We use induction to complete the proof of (3.41).

For $l = 2$, Scheme 2 is actually Scheme 1. Hence, (3.41) is easily obtained from (3.28).

Suppose that (3.41) holds for $l = 3, 4, \ldots, l-1$. Next, we shall prove that (3.41) holds for l.

Select $\mu_0 = 1/\lambda_k^{h_{l-1}}$, $\mu_{k,h_l} = 1/\lambda_{k,h_l}$, and $\mathbf{u}_0 = \lambda_k^{h_{l-1}} T_{h_l} \mathbf{u}_k^{h_{l-1}} / \|\lambda_k^{h_{l-1}} T_{h_l} \mathbf{u}_k^{h_{l-1}}\|_{X_Y}$. Using the proof method of Theorem 3.3, we deduce

$$\begin{gathered} \left|\mu_0 - \mu_{k,h_l}\right| \leq C_4 \left|\lambda_k^{h_{l-1}} - \lambda_k\right|, \\ \max_{k \leq j \leq k+q-1} \left|\frac{\mu_{j,h_l} - \mu_{k,h_l}}{\mu_0 - \mu_{j,h_l}}\right| \leq \frac{1}{2}. \end{gathered} \tag{3.43}$$

Using the triangle inequality and (2.27), we get

$$\operatorname{dist}(\mathbf{u}_0, M_{h_l}(\lambda_k)) \leq \operatorname{dist}\left(\mathbf{u}_0, \widehat{M}(\lambda_k)\right) + C_3 \delta_{h_l}(\lambda_k), \tag{3.44}$$

and together with the induction assumption, yields

$$\operatorname{dist}(\mathbf{u}_0, M_{h_l}(\lambda_k)) \leq \frac{1}{2}. \tag{3.45}$$

From Step 3 of Scheme 2, we know that $\mathbf{u}_k^{h_l}$ satisfies

$$\left(\frac{1}{\lambda_k^{h_{l-1}}} - T_h\right)\mathbf{u} = \mathbf{u}_0, \qquad \mathbf{u}_k^{h_l} = \frac{\mathbf{u}}{\|\mathbf{u}\|_{X_Y}}. \tag{3.46}$$

From the above arguments, we know that the conditions of Lemma 3.2 hold.

Define $\mathbf{u}^*$ and $\mathbf{u}_k$ as those in Theorem 3.3 (using $\mathbf{u}_k^{h_l}$ instead of $\mathbf{u}_k^h$, $\mathbf{u}_{j,h_l}$ instead of $\mathbf{u}_{j,h}$), then

$$\begin{gathered} \mathbf{u}^* = \sum_{j=k}^{k+q-1} a\left(\mathbf{u}_k^{h_l}, \mathbf{u}_{j,h_l}\right) \mathbf{u}_{j,h_l}, \\ \mathbf{u}_k = \sum_{j=k}^{k+q-1} a\left(\mathbf{u}_k^{h_l}, \mathbf{u}_{j,h_l}\right) \mathbf{u}_j^0, \end{gathered} \tag{3.47}$$

where $\mathbf{u}_{j,h_l} - \mathbf{u}_j^0$ satisfies (2.26). We can derive by Lemma 3.2 and the proof of (3.11) that

$$\begin{aligned}\left\|\mathbf{u}_k^{h_l} - \mathbf{u}^*\right\|_{X_Y} &= \operatorname{dist}\left(\mathbf{u}_k^{h_l}, M_{h_l}(\lambda_k)\right) \\ &\leq \frac{16}{\rho}\left|\mu_0 - \mu_{k,h_l}\right| \operatorname{dist}(\mathbf{u}_0, M_{h_l}(\lambda_k)) \\ &\leq \frac{16}{\rho} C_4 \left|\lambda_k^{h_{l-1}} - \lambda_k\right| \operatorname{dist}(\mathbf{u}_0, M_{h_l}(\lambda_k)).\end{aligned} \tag{3.48}$$

Substituting (3.44) into the above inequality, we deduce

$$\left\|\mathbf{u}_k^{h_l} - \mathbf{u}^*\right\|_{X_Y} \leq \frac{16}{\rho} C_4 \left|\lambda_k^{h_{l-1}} - \lambda_k\right| \left(\operatorname{dist}\left(\mathbf{u}_0, \widehat{M}(\lambda_k)\right) + C_3 \delta_{h_l}(\lambda_k)\right). \tag{3.49}$$

Like the proof method of (3.27), we get

$$\|\mathbf{u}^* - \mathbf{u}_k\|_{X_Y} \leq C_2 q \delta_{h_l}(\lambda_k). \tag{3.50}$$

From the above two inequalities, we obtain

$$\left\|\mathbf{u}_k^{h_l} - \mathbf{u}_k\right\|_{X_Y} \leq \frac{16}{\rho} C_4 \left|\lambda_k^{h_{l-1}} - \lambda_k\right| \operatorname{dist}\left(\mathbf{u}_0, \widehat{M}(\lambda_k)\right) + 2C_2 q \delta_{h_l}(\lambda_k). \tag{3.51}$$

There exists a constant C_5 independent of h_l such that

$$\|T_{h_l}\mathbf{f}\|_{X_Y} \leq C_5 \|\mathbf{f}\|_{X_Y}, \quad \forall \mathbf{f} \in X_{h_l}. \tag{3.52}$$

Like the proof method of (3.16), we can derive

$$\operatorname{dist}\left(\mathbf{u}_0, \widehat{M}(\lambda_k)\right) \leq 3C_5 \lambda_k \operatorname{dist}\left(\mathbf{u}_k^{h_{l-1}}, \widehat{M}(\lambda_k)\right) + 2\delta_{h_l}(\lambda_k). \tag{3.53}$$

Combining (3.51) and (3.53), we know that (3.41) is valid. Like the proof method of (3.8), we get (3.42), namely, Theorem 3.5 is valid. □

Corollary 3.6. *Suppose that Condition (A) holds and h_1 (namely, H) is properly small. Let $(\lambda_k^{h_l}, \mathbf{u}_k^{h_l})$ be an approximate eigenpair obtained by Scheme 2. Then there exists an eigenpair $(\lambda_k, \mathbf{u}_k)$ of (2.1), such that the following error estimates hold: when Ω is a convex domain,*

$$\begin{aligned}\left\|\mathbf{u}_k^{h_l} - \mathbf{u}_k\right\|_{X_Y} &\leq 6qC_2C' h_l, \\ \left|\lambda_k^{h_l} - \lambda_k\right| &\leq 2\lambda_k(1 + C_6\lambda_k)\left\|\mathbf{u}_k^{h_l} - \mathbf{u}_k\right\|_{X_Y}^2;\end{aligned} \tag{3.54}$$

when Ω is a nonconvex domain,

$$\left\|\mathbf{u}_k^{h_l} - \mathbf{u}_k\right\|_{X_\Upsilon} \leq 6qC_2C'' h_l^{\mu}, \quad \forall \mu \in (0, \tau),$$
$$\left|\lambda_k^{h_l} - \lambda_k\right| \leq 2\lambda_k (1 + C_6\lambda_k) \left\|\mathbf{u}_k^{h_l} - \mathbf{u}_k\right\|_{X_\Upsilon}^2, \tag{3.55}$$

where the C' and C'' are the ones in Corollary 3.4.

4. Interpolation Postprocessing Technique

In this section, we apply interpolation postprocessing technique to the filter approach for eigenvalue problem of electric field.

Let π_{2h} be a regular simplex mesh of Ω. When $n = 2$, the mesh π_h is obtained by dividing each element of the mesh π_{2h} into four congruent triangular elements; when $n = 3$, the mesh π_h is obtained by connecting the midpoints on each edge of the tetrahedral element, which divides each element of tetrahedralization π_{2h} into eight tetrahedral elements.

Let $I_h : C^0(\overline{\Omega})^n \to X_h$ with $k = 1$ be a piecewise linear node interpolation operator on π_h. Let $I_{2h}^{(2)} : C^0(\overline{\Omega})^n \to X_{2h}$ with $k = 2$ be a piecewise quadratic node interpolation operator on π_{2h} by using the corners of the mesh π_h as interpolation nodes.

Scheme 3. Interpolation Postprocessing Technique.

Step 1. Use linear finite-element filter approach to solve the problem (2.1) on the mesh π_h, and obtain the "*real*" eigenpair $(\lambda_{k,h}, \mathbf{u}_{k,h})$.

Step 2. On π_{2h}, use the value of the function $\mathbf{u}_{k,h}$ on the corners of the mesh π_h as interpolation conditions to construct a piecewise quadratic interpolation $I_{2h}^{(2)}\mathbf{u}_{k,h}$.

Step 3. Compute the Rayleigh quotient:

$$\lambda_{k,h}^r = \frac{a\left(I_{2h}^{(2)}\mathbf{u}_{k,h}, I_{2h}^{(2)}\mathbf{u}_{k,h}\right)}{b\left(I_{2h}^{(2)}\mathbf{u}_{k,h}, I_{2h}^{(2)}\mathbf{u}_{k,h}\right)}. \tag{4.1}$$

Here, $(\lambda_{k,h}^r, I_{2h}^{(2)}\mathbf{u}_{k,h})$ is the eigenpair corrected.

We develop the work in [18] to get the following theorem.

Theorem 4.1. *Let $(\lambda_{k,h}^r, I_{2h}^{(2)}\mathbf{u}_{k,h})$ be an approximate eigenpair obtained by Scheme 3. Assume that $M(\lambda_k) \subset H^{2+\alpha}(\Omega)$ and there exists an $\mathbf{u}_k \in M(\lambda_k)$ such that $\|I_h\mathbf{u}_k - \mathbf{u}_{k,h}\|_{X_\Upsilon} \leq Ch^{1+\alpha}$, for some $\alpha > 0$. Then*

$$\left\|I_{2h}^{(2)}\mathbf{u}_{k,h} - \mathbf{u}_k\right\|_{X_\Upsilon} \leq Ch^{1+\alpha}, \tag{4.2}$$

$$\left|\lambda_{k,h}^r - \lambda_k\right| \leq Ch^{2+2\alpha}. \tag{4.3}$$

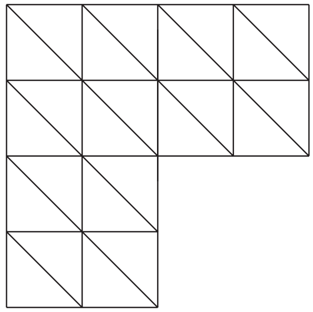

Figure 1: The L-shaped domain Ω.

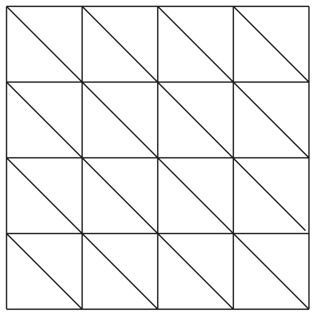

Figure 2: The square domain Ω.

Table 1: The results on square by Scheme 2 for eigenvalue problem of electric field ($\gamma = 0$): Set $h_1 = H = \sqrt{2}/8, h_2 = h_1/4, h_i = h_{i-1}/2, i = 3, 4, \ldots$.

l	$\lambda_1^{h_l}$	$\lambda_3^{h_l}$	$\lambda_4^{h_l}$
2	1.000000128059	2.000001795837	4.000008182019
3	1.000000008035	2.000000112617	4.000000514003
4	1.000000000503	2.000000007048	4.000000032195
5	1.000000000033	2.000000000442	4.000000002014

Table 2: The results on L-shape domain by Scheme 2 for eigenvalue problem of electric field ($\gamma = 0.5$): Set $h_1 = H = \sqrt{2}/8, h_2 = h_1/4, h_i = h_{i-1}/2, i = 3, 4, \ldots$.

l	$\lambda_1^{h_l}$	$\lambda_2^{h_l}$	$\lambda_3^{h_l}$	$\lambda_5^{h_l}$
2	1.821670160961	3.541016446362	9.869624525814	11.392623917025
3	1.747568405543	3.536558996225	9.869605667282	11.390593949705
4	1.690869606785	3.534929019047	9.869604480461	11.389874997804
5	1.646185577455	3.534344665373	9.869604406059	11.389617861812

Table 3: The results on L-shape domain by Scheme 2 for eigenvalue problem of electric field ($\gamma = 0.95$): Set $h_1 = H = \sqrt{2}/8, h_2 = h_1/4, h_i = h_{i-1}/2, i = 3, 4, \ldots$.

l	$\lambda_1^{h_l}$	$\lambda_2^{h_l}$	$\lambda_3^{h_l}$	$\lambda_5^{h_l}$
2	1.442172956105	3.534742382807	9.869624451909	11.390286991090
3	1.424394098212	3.534137592831	9.869605664957	11.389497235196
4	1.437944305790	3.534047685310	9.869604480377	11.389493626284
5	1.473860172568	3.534033892611	9.869604406055	11.389480149393

Proof. From boundedness of interpolation $I_{2h}^{(2)}$, we have

$$\begin{aligned} \left\| I_{2h}^{(2)}\mathbf{u}_k - I_{2h}^{(2)}\mathbf{u}_{k,h} \right\|_{X_\gamma} &= \left\| I_{2h}^{(2)} I_h \mathbf{u}_k - I_{2h}^{(2)}\mathbf{u}_{k,h} \right\|_{X_\gamma} \\ &\le C\|I_h\mathbf{u}_k - \mathbf{u}_{k,h}\|_{X_\gamma} \le Ch^{1+\alpha}, \end{aligned} \tag{4.4}$$

by triangle inequality and interpolation error estimate, we get

$$\begin{aligned} \left\| I_{2h}^{(2)}\mathbf{u}_{k,h} - \mathbf{u}_k \right\|_{X_\gamma} &\le \left\| I_{2h}^{(2)}\mathbf{u}_{k,h} - I_{2h}^{(2)}\mathbf{u}_k \right\|_{X_\gamma} + \left\| I_{2h}^{(2)}\mathbf{u}_k - \mathbf{u}_k \right\|_{X_\gamma} \\ &\le Ch^{1+\alpha}, \end{aligned} \tag{4.5}$$

namely, (4.2) is valid. Combining (2.28) and (4.2), we know that (4.3) is valid. □

Remark 4.2. Generally, to 2nd-order elliptic eigenvalue problems, condition $\|I_h\mathbf{u}_k - \mathbf{u}_{k,h}\|_{H^1(\Omega)} \le Ch^{1+\alpha}$ is valid (see [18–25]). But to eigenvalue problems of electric field, it is very difficult to prove that $\|I_h\mathbf{u}_k - \mathbf{u}_{k,h}\|_{X_\gamma} \le Ch^{1+\alpha}$. In Section 5, we will verify this theorem by the numerical experiments.

5. Numerical Experiments

In this section, we consider numerical solutions of problem (2.1) on the L-shaped domain $[-1,0]\times[-1,0]\cup[-1,1]\times[0,1]$ and on the square domain $[0,\pi]\times[0,\pi]$. The smallest five exact eigenvalues are $\lambda_1 \approx 1.47562182408, \lambda_2 \approx 3.53403136678, \lambda_3 = \lambda_4 \approx 9.86960440109, \lambda_5 \approx 11.3894793979$, and $\lambda_1 = \lambda_2 = 1, \lambda_3 = 2, \lambda_4 = \lambda_5 = 4$, respectively.

We adopt a uniform isosceles right triangulation for Ω (the edge in each element is along three fixed directions, see Figures 1 and 2) to produce the meshes π_{h_l} with mesh diameter h_l.

Here the weight is $\omega = d^\gamma$ ($d = \sqrt{x_1^2 + x_2^2}$). In the numerical experiments, when Ω is the L-shaped domain, let $\gamma = 0.5$ or 0.95; when Ω is the square domain, we choose $\gamma = 0$. And we use the numerical integral formula with accuracy of order 2 in our experiments.

From the following tables, we know that these three schemes are reliable for solving Maxwell eigenvalue problems. In addition, the accuracy of solutions is improved highly by these schemes.

Example 5.1. Solve problem (2.1) on the L-shaped domain $[-1,0]\times[-1,0]\cup[-1,1]\times[0,1]$ by using Scheme 1 with quadratic finite element. The eigenvalues obtained by Scheme 1 can be seen in [27].

Example 5.2. Solve problem (2.1) on the square domain $[0,\pi]\times[0,\pi]$ and the L-shaped domain $[-1,0]\times[-1,0]\cup[-1,1]\times[0,1]$ by using Scheme 2 with quadratic finite element.

We compute the first five approximate eigenvalues by using Scheme 2. The numerical results are listed in Tables 1–3; here λ_k^h, $(k = 1, 2, \ldots, 5)$ denote the first five "real" eigenvalues obtained by Scheme 2.

Table 4: The results on square by Scheme 3 for eigenvalue problem of electric field ($\gamma = 0$).

k	h	$\lambda_{k,h}$	$\lambda^r_{k,h}$	$\lambda^{(2)}_{k,2h}$
1	$\sqrt{2}/32$	1.000803263068	1.000002233535	1.000002031741
1	$\sqrt{2}/64$	1.000200802028	1.000000140453	1.000000128059
1	$\sqrt{2}/128$	1.000050199677	1.000000008794	1.000000008035
1	$\sqrt{2}/256$	1.000012549868	1.000000000550	1.000000000503
3	$\sqrt{2}/32$	2.004819363681	2.000031493140	2.000028466776
3	$\sqrt{2}/64$	2.001204798563	2.000001982275	2.000001795837
3	$\sqrt{2}/128$	2.000301197210	2.000000124127	2.000000112617
3	$\sqrt{2}/256$	2.000075299154	2.000000007762	2.000000007048
4	$\sqrt{2}/32$	4.012861235873	4.000136435109	4.000129172679
4	$\sqrt{2}/64$	4.003213391903	4.000008625833	4.000008182018
4	$\sqrt{2}/128$	4.000803229708	4.000000540798	4.000000514003
4	$\sqrt{2}/256$	4.000200800066	4.000000033828	4.000000032195

Table 5: The results on L-shape domain by Scheme 3 for eigenvalue problem of electric field ($\gamma = 0.5$).

k	h	$\lambda_{k,h}$	$\lambda^r_{k,h}$	$\lambda^{(2)}_{k,2h}$
1	$\sqrt{2}/32$	2.011577759579	1.943699371956	1.922638950089
1	$\sqrt{2}/64$	1.898872106646	1.846871485478	1.821288431396
1	$\sqrt{2}/128$	1.820217989064	1.778546113807	1.747565968425
1	$\sqrt{2}/256$	1.761155941126	1.726985873864	1.690868193292
2	$\sqrt{2}/32$	3.572934947967	3.554733448789	3.552868211051
2	$\sqrt{2}/64$	3.550750124217	3.542510419003	3.541016085614
2	$\sqrt{2}/128$	3.541430954385	3.537608542327	3.536558996172
2	$\sqrt{2}/256$	3.537364244380	3.535586886107	3.534929019046
3	$\sqrt{2}/32$	9.901333191694	9.869943232779	9.869921180772
3	$\sqrt{2}/64$	9.877532690720	9.869626032863	9.869624525812
3	$\sqrt{2}/128$	9.871586259667	9.869605763991	9.869605667282
3	$\sqrt{2}/256$	9.870099853293	9.869604486559	9.869604480461
5	$\sqrt{2}/32$	11.460149814384	11.399622345282	11.398934132563
5	$\sqrt{2}/64$	11.410106013972	11.393258438593	11.392624308452
5	$\sqrt{2}/128$	11.396038772982	11.391051676895	11.390593949719
5	$\sqrt{2}/256$	11.391771543060	11.390164853843	11.389874997804

From Tables 1–3, we see that Scheme 2 is highly efficient for solving eigenvalue problem of electric field.

Example 5.3. Solve problem (2.1) on the square domain $[0,\pi] \times [0,\pi]$ and the L-shaped domain $[-1,0] \times [-1,0] \cup [-1,1] \times [0,1]$ by using Scheme 3.

In Tables 4–6, $\lambda_{k,h}$ ($k = 1,2,\ldots,5$) denote the first five "real" eigenvalues obtained by linear element filter approach directly, $\lambda^r_{k,h}$ ($k = 1,2,\ldots,5$) denote the first five "real" eigenvalues obtained by Scheme 3, $\lambda^{(2)}_{k,2h}$ ($k = 1,2,\ldots,5$) denote the eigenvalues obtained by quadratic element filter approach directly.

To the square domain, eigenfunctions are smooth enough. And from Table 4, we see that $\lambda^r_{k,h}$ obtained by interpolation postprocessing technique achieve the accuracy order of quadratic finite element. To the L-shape domain, eigenfunctions are not smooth enough, generally. For example, the first eigenfunction has a strong singularity to L-shape domain

Table 6: The results on L-shape domain by Scheme 3 for eigenvalue problem of electric field ($\gamma = 0.95$).

k	h	$\lambda_{k,h}$	$\lambda^r_{k,h}$	$\lambda^{(2)}_{k,2h}$
1	$\sqrt{2}/32$	1.508480704616	1.465444529094	1.450478468682
1	$\sqrt{2}/64$	1.466277799473	1.448244634750	1.442139772480
1	$\sqrt{2}/128$	1.448544881396	1.440581165707	1.424272599464
1	$\sqrt{2}/256$	1.439836736426	1.434823427838	1.490040481056
2	$\sqrt{2}/32$	3.540250087930	3.528411456749	3.543755446693
2	$\sqrt{2}/64$	3.514437651295	3.458119260291	3.534742392880
2	$\sqrt{2}/128$	3.538194948635	3.533617471424	3.534137592922
2	$\sqrt{2}/256$	3.535161265275	3.534215865622	3.534047685310
3	$\sqrt{2}/32$	9.901333191694	9.869943232779	9.869918238162
3	$\sqrt{2}/64$	9.877532690720	9.869626032863	9.869624447221
3	$\sqrt{2}/128$	9.871586259667	9.869605763991	9.869605663393
3	$\sqrt{2}/256$	9.870099853293	9.869604486559	9.869604480222
5	$\sqrt{2}/32$	11.460149814384	11.399622345282	11.390599176482
5	$\sqrt{2}/64$	11.410106013972	11.393258438593	11.390317414966
5	$\sqrt{2}/128$	11.396038772982	11.391051676895	11.389497235201
5	$\sqrt{2}/256$	11.391771543060	11.390164853843	11.389493626284

(see [28]). Tables 5 and 6 show that the accuracy of $\lambda^r_{3,h}$ is improved obviously, and the improvement of $\lambda^r_{1,h}$ is not obvious.

Remark 5.4. Wang established two-grid discretization scheme of finite-element parameterized approach for eigenvalue problem of electric field (see [29]). And she also proved error estimates of the Scheme. It will still be meaningful to extend the multigrid discretization scheme and the interpolation postprocessing technique discussed in our paper to parameterized approach.

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