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Research Article

Solution of $(3+1)$-Dimensional Nonlinear Cubic Schrodinger Equation by Differential Transform Method

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Four-dimensional differential transform method has been introduced and fundamental theorems have been defined for the first time. Moreover, as an application of four-dimensional differential transform, exact solutions of nonlinear system of partial differential equations have been investigated. The results of the present method are compared very well with analytical solution of the system. Differential transform method can easily be applied to linear or nonlinear problems and reduces the size of computational work. With this method, exact solutions may be obtained without any need of cumbersome work, and it is a useful tool for analytical and numerical solutions.

1. Introduction

In this paper, we study system of nonlinear partial differential equations (PDEs). In order to solve systems of differential equations, the commonly used methods are the method of characteristics and the Riemann invariants among other methods. The existing techniques have difficulties in related to the size of computational work, especially when the system has several PDEs. More recently, Adomian decomposition method, were used to handle systems of PDEs by [1], and here we present an alternative method for the system. The concept of differential transform was first introduced by Zhou [2], who solved linear and nonlinear initial value problems in electric circuit analysis. The transformation method, called three-dimensional differential transform, is different from the high-order Taylor series method, which consists of computing the coefficients of the Taylor series of the solution using the initial data and the PDE. But the Taylor series method requires more computational work for large orders. The present method is well addressed in [3–7]. The differential transform technique is an iterative procedure for obtaining Taylor series solutions of differential equations and systems of PDEs. This method reduces the size of computational domain and is applicable to many problems easily. In the present work, in order to extend applications of the differential transform method to different problems, four-dimensional differential transform have been defined and new theorems have been prooved. In this paper, Differential Transformation Method (DTM) is employed to obtain the solution of the $(3+1)$-dimensional nonlinear cubic Schrödinger equation

$$i\psi_t + \psi_{xx} + \psi_{yy} + \psi_{zz} + \psi\left(|\psi|^2 - S(x,y,z,t)\right) = 0. \tag{1.1}$$

We put

$$\psi = u + iv. \tag{1.2}$$

From (1.1) we obtain that

$$\begin{aligned} M(u,v) &= -v_t + u_{xx} + u_{yy} + u_{zz} + u\left(u^2 + v^2 - S(x,y,z,t)\right) = 0, \\ N(u,v) &= u_t + v_{xx} + v_{yy} + v_{zz} + v\left(u^2 + v^2 - S(x,y,z,t)\right) = 0. \end{aligned} \tag{1.3}$$

2. Basic Idea of the Differential Equation

2.1. One-Dimensional Differential Transform Method

We introduce in this section the basic definition of the one-dimensional differential transformation.

Definition 2.1. If $u(t)$ is analytic in the domain T, then it will be differentiated continuously with respect to time t:

$$\frac{d^k u(t)}{dt^k} = \phi(t,k), \qquad \forall t \in T, \tag{2.1}$$

for $t = t_i$, where $\phi(t,k) = \phi(t_i,k)$, and k and k belong to the set of nonnegative integer denoted as the K domain.

Therefore, (1.2) can be written as

$$U_i(k) = \phi(t_i,k) = \left[\frac{d^k u(t)}{dt^k}\right]_{t=t_i}, \qquad \forall k \in K, \tag{2.2}$$

where $U_i(k)$ is called spectrum of $u(t)$ at $t = t_i$, in the K domain.

Definition 2.2. If $u(t)$ can be expressed by Taylor's series, then $u(t)$ can be represented as

$$u(t) = \sum_{k=0}^{\infty} \frac{(t = t_i)}{k!} U(k). \tag{2.3}$$

Equation (2.3) is known as the inverts transformation of $U(k)$. If $U(k)$ is defined as

$$U(k) = M(k)\left[\frac{d^k q(t)u(t)}{dt^k}\right]_{t=t_i}, \tag{2.4}$$

where $k = 0, 1, 2, \ldots, \infty$, then the function $u(t)$ can be described as

$$u(t) = \frac{1}{q(t)}\sum_{k=0}^{\infty}\frac{(t-t_i)}{k!}\frac{U(k)}{M(k)}, \tag{2.5}$$

where $M(k) \neq 0$, $q(t) \neq 0$. The function $M(k)$ is called the weighting factor and $q(t)$ is regarded as kernel corresponding to $u(t)$. If $M(k) = 1$ and $q(t) = 1$, then (1.3) and (2.2) are equivalent. In this way, (2.1) can be treated as a special case of (2.3). In this paper, the transformation with $M(k) = 1/k!$ and $q(t) = 1$ is applied.

Then (2.4) becomes

$$U(k) = \left[\frac{d^k u(t)}{dt^k}\right]_{t=t_i}, \quad \text{where } k = 0, 1, 2, \ldots, \infty. \tag{2.6}$$

Using the differential transform, a differential equation in the domain of interest can be transformed to be an algebraic equation in the K domain and $u(t)$ can be obtained by finite-term Taylor series plus a remainder, as

$$u(t) = \frac{1}{q(t)}\sum_{k=0}^{\infty}\frac{(t-t_i)}{k!}\frac{U(k)}{M(k)} + R_{n+1}(t) = \sum_{k=0}^{\infty}(t-t_i)^k\, U(k) + R_{n+1}(t). \tag{2.7}$$

In order to speed up the convergent rate and the accuracy of calculation, the entire domain of t needs to be split into subdomains.

2.2. Two-Dimensional Differential Transform Method

Definition 2.3. Given an w function which has two components such as x, t to w-dimensional differential transform of $w(x,t)$ is defined

$$W(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}W(x,t)}{\partial x^k \partial y^h}\right]_{(0,0)}, \tag{2.8}$$

where $W(x,y)$ the original is function and $W(k,h)$ is the transformed function. Again, the transformation can be called T-function, and the lower case and upper case letters represent the original and transformed functions, respectively (Table 1).

Definition 2.4. The differential inverse transform of $W(k,h)$ is defined as

$$W(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)x^k t^h. \tag{2.9}$$

Table 1: The fundamental theorems.

Original function	Transformed function
(1) $W(x,t) = u(x,t) \pm v(x,t)$,	$W(k,h) = U(k,h) \pm V(k,h)$.
(2) $W(x,t) = \lambda u(x,t)$,	$W(k,h) = \lambda U(k,h)$, λ is constant
(3) $W(x,t) = \partial u(x,t)/\partial x$,	$W(k,h) = (k+1)U(k+1,h)$
(4) $W(x,t) = \partial u(x,t)/\partial t$	$W(k,h) = (h+1)U(k,h+1)$
(5) $W(x,t) = \partial^{r+s}u(x,t)/\partial x^r \partial t^s$	$W(k,h) = (k+1)(k+2)\cdots(k+r)(h+1)$ $\times(h+2)\cdots(h+s)U(k+r,h+s)$
(6) $W(x,t) = u(x,t)v(x,t)$	$W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{k} U(r,h-s)V(k-r,s)$
(7) $W(x,t) = (\partial u(x,t)/\partial x)(\partial v(x,t)/\partial x)$,	$W(k,h) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}(r+1)(k-r+1)U(r+1,h-s)$ $\times U(k-r+1,s)$
(8) $W(x,t) = u(x,t)(\partial u(x,t)/\partial x)$,	$W(k,h) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}(k-r+1)U(r,h-s)U(k-r+1,s)$

and from (2.8) and (2.9) can be concluded as follows:

$$W(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}W(x,t)}{\partial x^k \partial t^h}\right]_{(0,0)} x^k t^h. \tag{2.10}$$

2.3. *Three-Dimensional Differential Transform*

Definition 2.5. Given an w function which has three components such as x, y, t. Three-dimensional differential transform of $w(x,y,t)$ is defined as

$$W(k,h,m) = \frac{1}{k!h!m!}\left[\frac{\partial^{h+k+m}w(x,y,t)}{\partial x^k \partial y^h \partial t^m}\right]_{(0,0,0)}, \tag{2.11}$$

where $w(x,y,t)$ is the original function and $W(k,h,m)$ is the transformed function. Again, the transformation can be called T-function, and the lower case and upper case letters represent the original and transformed functions, respectively (Table 2).

Definition 2.6. The differential inverse transform of $W(k,h,m)$ is defined as

$$w(x,y,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}W(k,h,m)x^k y^h t^m \tag{2.12}$$

and from (2.11) and (2.12) can be concluded as

$$w(x,y,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{k!h!m!}\left[\frac{\partial^{h+k+m}w(x,y,t)}{\partial x^k \partial y^h \partial t^m}\right]_{(0,0,0)} x^k y^h t^m. \tag{2.13}$$

Table 2: Fundamental theorems for three-dimensional case.

Original function	Transformed function
(1) $w(x,y,t) = u(x,y,t) \pm v(x,y,t)$	$W(h,k,m) = U(k,h,m) \pm V(k,h,m)$
(2) $w(x,y,t) = cu(x,y,t)$	$W(h,k,m) = cU(k,h,m)$
(3) $w(x,y,t) = \partial u(x,y,t)/\partial x$,	$W(h,k,m) = (k+1)U(k+1,h,m)$
(4) $w(x,y,t) = \partial u(x,y,t)/\partial y$	$W(h,k,m) = (h+1)U(k,h+1,m)$
(5) $w(x,y,t) = \partial u(x,y,t)/\partial t$	$W(h,k,m) = (m+1)U(k,h,m+1)$
(6) $w(x,y,t) = \partial^{r+s+p}u(x,y,t)/\partial x^r \partial y^s \partial t^p$	$W(h,k,m) = (k+1)(k+2)\cdots(k+r) \times (h+2)\cdots (h+s)(m+1)(m+2)\cdots(m+p)$
(7) $w(x,y,t) = u(x,y,t)v(x,y,t)$	$W(h,k,m) = \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m} U(r,h-s,m-p)V(k-r,s,p)$
(8) $w(x,y,t) = (\partial u(x,y,t)/\partial x)(\partial v(x,y,t)/\partial y)$	$W(h,k,m) = \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(k-r+1)(h-s+1) \times U(k-r+1,s,p) \times V(r,h-s+1,m-p)$

Table 3: Fundamental theorems for four-dimensional case.

Original function	Transformed function
(1) $w(x,y,z,t) = u(x,y,z,t) \pm v(x,y,z,t)$	$W(h,k,m,n) = U(k,h,m,n) \pm V(k,h,m,n)$
(2) $w(x,y,z,t) = cu(x,y,z,t)$	$W(h,k,m,n) = cU(k,h,m,n)$
(3) $w(x,y,z,t) = \partial^{r+s+p+q}u(x,y,z,t)/\partial x^r \partial y^s \partial z^p \partial t^q$	$W(h,k,m,n) = (k+1)(k+2)\cdots(k+r)(h+1)(h+2)\cdots (h+s)(m+1)(m+2)\times\cdots(m+p)(n+1)(n+2)\cdots (n+q)U(k+r,h+s,m+p,n+q)$
(4) $w(x,y,z,t) = \partial u(x,y,z,t)/\partial t$	$W(h,k,m,n) = (m+1)U(k,h,m,n+1)$
(5) $w(x,y,z,t) = u(x,y,z,t)v(x,y,z,t)$	$W(h,k,m,n) = \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}\sum_{q=0}^{n} U(r,h-s,m-p,n-q)V(k-r,s,p,q)$

2.4. Four-Dimensional Differential Transform

Definition 2.7. Given an w function which has four components such as x, y, z, t. Four-dimensional differential transform of $w(x,y,z,t)$ is defined as

$$W(k,h,m,n) = \frac{1}{k!h!m!n!}\left[\frac{\partial^{h+k+m+n}w(x,y,z,t)}{\partial x^k \partial y^h \partial z^m \partial t^n}\right]_{(0,0,0,0)}, \tag{2.14}$$

where $w(x,y,z,t)$ is the original function and $W(k,h,m,n)$ is the transformed function. Again, the transformation can be called T-function, and the lower case and upper case letters represent the original and transformed functions, respectively (Table 3).

Definition 2.8. The differential inverse transform of $W(k,h,m,n)$ is defined as

$$w(x,y,z,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} W(k,h,m,n)x^k y^h z^m t^n \tag{2.15}$$

and from (2.14) and (2.15) can be concluded as

$$w(x,y,z,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{k!h!m!n!}\left[\frac{\partial^{h+k+m}w(x,y,t)}{\partial x^k \partial y^h \partial t^m}\right]_{(0,0,0,0)} x^k y^h z^m t^n. \tag{2.16}$$

Theorem 2.9. *If $w(x,y,z,t)=u(x,y,z,t)v(x,y,z,t)\omega(x,y,z,t)$, then*

$$W(h,k,m,n)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{i=0}^{h-s}\sum_{p=0}^{m}\sum_{j=0}^{m-p}\sum_{q=0}^{n}\sum_{a=0}^{n-q}U[r,s,p,n-q-a]V[t,i,j,q] \times \mathbb{W}[k-r-t,h-s-i,m-p-j,a]. \tag{2.17}$$

Proof. From the definition of transform,

$$\begin{aligned}
W(0,0,0,0) &= U[0,0,0,0]V[0,0,0,0]\omega[0,0,0,0],\\
W(1,0,0,0) &= U[1,0,0,0]V[0,0,0,0]\omega[0,0,0,0]+U[0,0,0,0]V[1,0,0,0]\omega[0,0,0,0]\\
&\quad+U[0,0,0,0]V[0,0,0,0]\omega[1,0,0,0],\\
W(2,0,0,0) &= U[2,0,0,0]V[0,0,0,0]\omega[0,0,0,0]+U[1,0,0,0]V[1,0,0,0]\omega[0,0,0,0]\\
&\quad+U[0,0,0,0]V[2,0,0,0]\omega[0,0,0,0]+U[1,0,0,0]V[0,0,0,0]\omega[1,0,0,0]\\
&\quad+U[0,0,0,0]V[1,0,0,0]\omega[1,0,0,0]+U[0,0,0,0]V[0,0,0,0]\omega[2,0,0,0],\\
W(0,1,0,0) &= U[0,1,0,0]V[0,0,0,0]\omega[0,0,0,0]+U[0,0,0,0]V[0,1,0,0]\omega[0,0,0,0]\\
&\quad+U[0,0,0,0]V[0,0,0,0]\omega[0,1,0,0],\\
W(0,2,0,0) &= U[0,2,0,0]V[0,0,0,0]\omega[0,0,0,0]+U[0,1,0,0]V[0,1,0,0]\omega[0,0,0,0]\\
&\quad+U[0,0,0,0]V[0,2,0,0]\omega[0,0,0,0]+\cdots,\\
W(0,0,1,0) &= U[0,0,1,0]V[0,0,0,0]\omega[0,0,0,0]+U[0,0,0,0]V[0,0,1,0]\omega[0,0,0,0]\\
&\quad+U[0,0,0,0]V[0,0,0,0]\omega[0,0,1,0].
\end{aligned} \tag{2.18}$$

In general, we have

$$W(h,k,m,n)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{i=0}^{h-s}\sum_{p=0}^{m}\sum_{j=0}^{m-p}\sum_{q=0}^{n}\sum_{a=0}^{n-q}U[r,s,p,n-q-a]V[t,i,j,q] \times \mathbb{W}[k-r-t,h-s-i,m-p-j,a]. \tag{2.19}$$

□

3. Analysis of Method

To investigate the solution of (1.3), we first construct a system by differential transformation method as follows:

$$\begin{aligned}
(n+1)V[k,h,m,n+1] &= (k+1)(k+2)U[k+2,h,m,n]+(h+1)(h+2)U[k,h+2,m,n]\\
&\quad+(m+1)(m+2)U[k,h,m+2,n]\\
&\quad+\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{i=0}^{h-s}\sum_{p=0}^{m}\sum_{j=0}^{m-p}\sum_{q=0}^{n}\sum_{a=0}^{n-a}U[r,s,p,n-q-a]U[t,i,j,q]\\
&\quad\times U[k-r-t,h-s-i,m-p-j,a]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{i=0}^{h-s}\sum_{p=0}^{m}\sum_{j=0}^{m-p}\sum_{q=0}^{n}\sum_{a=0}^{n-a}U[r,s,p,n-q-a]V[t,i,j,q]\\
&\times V[k-r-t,h-s-i,m-p-j,a]\\
&-\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}\sum_{q=0}^{p}U[r,h-s,m-p,n-q]S[k-r,s,p,q],\\
(n+1)U[k,h,m,n+1] = &-(k+1)(k+2)V[k+2,h,m,n] -(h+1)(h+2)V[k,h+2,m,n]\\
&-(m+1)(m+2)V[k,h,m+2,n]\\
&-\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{i=0}^{h-s}\sum_{p=0}^{m}\sum_{j=0}^{m-p}\sum_{q=0}^{n}\sum_{a=0}^{n-a}U[r,s,p,n-q-a]U[t,i,j,q]\\
&\times V[k-r-t,h-s-i,m-p-j,a]\\
&-\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{i=0}^{h-s}\sum_{p=0}^{m}\sum_{j=0}^{m-p}\sum_{q=0}^{n}\sum_{a=0}^{n-a}V[r,s,p,n-q-a]V[t,i,j,q]\\
&\times V[k-r-t,h-s-i,m-p-j,a]\\
&+\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}\sum_{q=0}^{p}V[r,h-s,m-p,n-q]S[k-r,s,p,q].
\end{aligned}
\tag{3.1}
$$

Subject to Taylor's expand of initial condition

$$
\begin{aligned}
u(x,y,z,0) = G(x,y,z) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}U(k,h,m)x^k y^h z^m\\
&= F(0,0,0) + \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}\frac{F^{(k+h+m)}(0,0,0)}{k!h!m!}x^k y^h z^m,\\
v(x,y,z,0) = G(x,y,z) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}V(k,h,m)x^k y^h z^m\\
&= G(0,0,0) + \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}\frac{G^{(k+h+m)}(0,0,0)}{k!h!m!}x^k y^h z^m.
\end{aligned}
\tag{3.2}
$$

Substituting (3.2) into (3.1) and using operation of Table 3, we get

$$
\begin{aligned}
&V[k,h,m,n+1]\\
&\quad= \frac{1}{(n+1)}\Bigg[(k+1)(k+2)U[k+2,h,m,n] + (h+1)(h+2)U[k,h+2,m,n]\\
&\qquad + (m+1)(m+2)U[k,h,m+2,n]\\
&\qquad + \sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{i=0}^{h-s}\sum_{p=0}^{m}\sum_{j=0}^{m-p}\sum_{q=0}^{n}\sum_{a=0}^{n-a}U[r,s,p,n-q-a]U[t,i,j,q]
\end{aligned}
$$

$$\times U[k-r-t,h-s-i,m-p-j,a]$$
$$+\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{i=0}^{h-s}\sum_{p=0}^{m}\sum_{j=0}^{m-p}\sum_{q=0}^{n}\sum_{a=0}^{n-a}U[r,s,p,n-q-a]V[t,i,j,q]$$
$$\times V[k-r-t,h-s-i,m-p-j,a]$$
$$\left.-\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}\sum_{q=0}^{p}U[r,h-s,m-p,n-q]S[k-r,s,p,q]\right],$$

$$\begin{aligned}
U[k,h,m,n+1] &= \frac{1}{(n+1)}\Bigg[-(k+1)(k+2)V[k+2,h,m,n]-(h+1)(h+2)V[k,h+2,m,n]\\
&\quad-(m+1)(m+2)V[k,h,m+2,n]\\
&\quad-\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{i=0}^{h-s}\sum_{p=0}^{m}\sum_{j=0}^{m-p}\sum_{q=0}^{n}\sum_{a=0}^{n-a}U[r,s,p,n-q-a]U[t,i,j,q]\\
&\quad\times V[k-r-t,h-s-i,m-p-j,a]\\
&\quad-\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{i=0}^{h-s}\sum_{p=0}^{m}\sum_{j=0}^{m-p}\sum_{q=0}^{n}\sum_{a=0}^{n-a}V[r,s,p,n-q-a]V[t,i,j,q]\\
&\quad\times V[k-r-t,h-s-i,m-p-j,a]\\
&\quad+\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}\sum_{q=0}^{p}V[r,h-s,m-p,n-q]S[k-r,s,p,q]\Bigg].
\end{aligned}\tag{3.3}$$

In order to obtain the unknowns of $U(k,h,m,n)$ and $V(k,h,m,n)$, $k,h,m,n=0,1,2,\ldots$, we must construct and solve the above equation and substitute in (2.16).

4. Application

In this section, the differential transformation technique is applied to solve (1.3). This method only needs the initial condition of covering PDEs. Firstly, we consider the solution of (1.3) with the initial conditions:

$$\begin{aligned}
u(x,y,z,0) &= 1+\mathrm{Tanh}(x+y+z),\\
v(x,y,z,0) &= 1-\mathrm{Tanh}(x+y+z),\\
S(x,y,z,0) &= -4+8\mathrm{Tanh}(x+y+z)^2.
\end{aligned}\tag{4.1}$$

Suppose that $x_0 = y_0 = z_0 = t_0 = 0$, in Definition 2.7, then we have

$$\begin{gathered}
U[0,0,0,0] = 1, \qquad U[1,0,0,0] = 1, \qquad U[2,0,0,0] = 0, \\
U[3,0,0,0] = -\frac{1}{3}, \qquad U[4,0,0,0] = 0, \qquad U[5,0,0,0] = \frac{2}{15}, \\
U[0,1,0,0] = 1, \qquad U[0,0,2,0] = 0, \qquad U[0,0,3,0] = -\frac{1}{3}, \\
U[2,3,0,0] = \frac{4}{3}, \qquad U[0,0,4,0] = 0, \qquad U[0,0,5,0] = \frac{2}{15}, \qquad U[2,1,0,0] = -1, \\
U[4,1,0,0] = \frac{2}{3}, \qquad U[1,2,0,0] = -1, \qquad U[0,2,0,0] = 0, \\
U[0,3,0,0] = -\frac{1}{3}, \qquad U[0,4,0,0] = 0, \qquad U[0,5,0,0] = \frac{2}{15}, \\
U[0,0,1,0] = 1, \qquad U[4,3,0,0] = -\frac{17}{9}, \qquad U[1,4,0,0] = \frac{2}{3}, \\
U[3,4,0,0] = -\frac{17}{9}, \qquad U[2,0,1,0] = -1, \qquad U[4,0,1,0] = \frac{2}{3}, \\
U[3,0,2,0] = \frac{4}{3}, \qquad U[4,0,3,0] = -\frac{17}{9}, \qquad U[1,0,4,0] = \frac{2}{3}, \\
U[3,0,4,0] = -\frac{17}{9}, \qquad U[0,2,1,0] = -1, \ldots.
\end{gathered} \tag{4.2}$$

Using (3.3) and (4.2) and by recursive method, we get

$$\begin{gathered}
U[0,0,0,1] = -6, \qquad U[0,0,1,1] = 0, \qquad U[0,1,0,1] = 0, \qquad U[1,0,0,1] = 0, \\
U[0,1,1,1] = 12, \qquad U[1,0,1,1] = 12, \\
U[1,1,0,1] = 12, \qquad U[1,1,1,1] = 0, \qquad U[0,0,2,1] = 6, \qquad U[0,2,0,1] = 6, \\
U[2,0,0,1] = 6, \qquad U[0,0,0,2] = 0, \qquad U[2,1,0,1] = 0, \\
U[0,1,2,1], \qquad U[0,2,1,1], \qquad U[2,0,1,1], \qquad U[1,0,2,1], \ldots, \\
V[0,0,0,1] = 6, \qquad V[0,0,1,1] = 0, \qquad V[0,1,0,1] = 0, \\
V[1,0,0,1] = 0, \qquad V[0,1,1,1] = -12, \qquad V[1,0,1,1] = -12, \\
V[1,1,0,1] = -12, \qquad V[1,1,1,1] = 0, \qquad V[0,0,2,1] = -6, \qquad V[0,2,0,1] = -6, \\
V[2,0,0,1] = -6, \qquad V[0,0,0,2] = 0, \\
V[2,1,0,1] = 0, \qquad V[0,1,2,1] = 0, \qquad V[0,2,1,1] = 0, \qquad V[1,0,2,1] = 0, \\
V[1,2,0,1] = 0, \qquad V[1,2,1,1] = 48, \qquad V[2,1,1,1] = -16, \\
V[2,2,0,1] = 24, \qquad V[2,0,2,1] = 24, \qquad V[1,1,2,1] = -16, \\
V[0,2,2,1] = 24, \qquad V[2,2,1,1] = -64, \qquad V[1,2,2,1] = -64, \ldots.
\end{gathered} \tag{4.3}$$

Substituting all $U(k,h,m,n)$ and $V(k,h,m,n)$ into (1.2) yields

$$\begin{aligned}
u(x,y,z,t) =&1-6t+x+20t^2x+6tx^2-56t^2x^2-\frac{x^3}{3}-4tx^4+y+20t^2y+12txy-x^2y+\frac{2x^4y}{3}\\
&+6ty^2-56t^2y^2-xy^2-24tx^2y^2+\frac{4x^3y^2}{3}-\frac{y^3}{3}+\frac{4x^2y^3}{3}-\frac{17x^4y^3}{9}-4ty^4+\frac{2xy^4}{3}\\
&-\frac{17x^3y^4}{9}+z+20t^2z+12txz-x^2z+\frac{2x^4z}{3}+12tyz-48t^2yz-2xyz+16tx^2yz\\
&+\frac{8}{3}x^3yz-y^2z-48txy^2z+4x^2y^2z-64tx^2y^2z-\frac{17}{3}x^4y^2z-16ty^3z+\frac{8}{3}xy^3z\\
&+\frac{2y^4z}{3}-\frac{17}{3}x^2y^4z+6tz^2-56t^2z^2-xz^2-24tx^2z^2+\frac{4x^3z^2}{3}-yz^2+16txyz^2\\
&+4x^2yz^2-128tx^2yz^2-\frac{17}{3}x^4yz^2-24ty^2z^2+4xy^2z^2-64txy^2z^2+204tx^2y^2z^2\\
&-\frac{34}{3}x^3y^2z^2+\frac{4y^3z^2}{3}-\frac{34}{3}x^2y^3z^2-\frac{17}{3}xy^4z^2-\frac{z^3}{3}+\frac{4x^2z^3}{3}-\frac{17x^4z^3}{9}-16tyz^3+\frac{8}{3}xyz^3\\
&+\frac{4y^2z^3}{3}-\frac{34}{3}x^2y^2z^3-\frac{17y^4z^3}{9}-4tz^4+\frac{2xz^4}{3}-\frac{17x^3z^4}{9}+\frac{2yz^4}{3}-\frac{17}{3}x^2yz^4\\
&-\frac{17}{3}xy^2z^4-\frac{17y^3z^4}{9}+\cdots,
\end{aligned}\tag{4.4}$$

$$\begin{aligned}
v(x,y,z,t) =&1+6t-x-20t^2x-6tx^2-56t^2x^2+\frac{x^3}{3}+4tx^4-y-20t^2y-12txy+x^2y-\frac{2x^4y}{3}-6ty^2\\
&-56t^2y^2+xy^2+24tx^2y^2-\frac{4x^3y^2}{3}+\frac{y^3}{3}-\frac{4x^2y^3}{3}+\frac{17x^4y^3}{9}+4ty^4-\frac{2xy^4}{3}+\frac{17x^3y^4}{9}\\
&-z-20t^2z-12txz+x^2z-\frac{2x^4z}{3}-12tyz-48t^2yz+2xyz-16tx^2yz-\frac{8}{3}x^3yz\\
&+y^2z+48txy^2z-4x^2y^2z-64tx^2y^2z+\frac{17}{3}x^4y^2z+16ty^3z-\frac{8}{3}xy^3z-\frac{2y^4z}{3}\\
&+\frac{17}{3}x^2y^4z-6tz^2-56t^2z^2+xz^2+24tx^2z^2-\frac{4x^3z^2}{3}+yz^2-16txyz^2-4x^2yz^2\\
&-128tx^2yz^2+\frac{17}{3}x^4yz^2+24ty^2z^2-4xy^2z^2-64txy^2z^2-204tx^2y^2z^2+\frac{34}{3}x^3y^2z^2\\
&-\frac{4y^3z^2}{3}+\frac{34}{3}x^2y^3z^2+\frac{17}{3}xy^4z^2+\frac{z^3}{3}-\frac{4x^2z^3}{3}+\frac{17x^4z^3}{9}+16tyz^3-\frac{8}{3}xyz^3-\frac{4y^2z^3}{3}\\
&+\frac{34}{3}x^2y^2z^3+\frac{17y^4z^3}{9}+4tz^4-\frac{2xz^4}{3}+\frac{17x^3z^4}{9}-\frac{2yz^4}{3}+\frac{17}{3}x^2yz^4\\
&+\frac{17}{3}xy^2z^4+\frac{17y^3z^4}{9}+\cdots,
\end{aligned}\tag{4.5}$$

Table 4: Exact and approximate solutions and absolute errors of $u(x, y, z, t)$, $(y = z = 0.01)$.

x	t	Exact solutions	Approximate solutions with DTM	Absolute errors
0.01	0.01	0.970008996761179	0.970102829745246	0.00009383
0.02	0.02	0.920170230888868	0.920475254085735	0.00030502
0.03	0.03	0.870727416393941	0.871248374408655	0.00052095
0.04	0.04	0.821919131882669	0.822542425564273	0.00062329
0.05	0.05	0.773971647721329	0.774464061599229	0.00049241
0.06	0.06	0.727094919436867	0.727106307756535	0.00001138
0.07	0.07	0.681479223097229	0.680548512475579	0.00093071
0.08	0.08	0.637292532421948	0.634856299392122	0.00243623
0.09	0.09	0.594678691310537	0.590081519338295	0.00459717
0.10	0.10	0.553756389751220	0.546262202342606	0.00749418

Table 5: Exact and approximate solutions and absolute errors of $v(x, y, z, t)$, $(y = z = 0.01)$.

x	t	Exact solutions	Approximate solutions with DTM	Absolute errors
0.01	0.01	1.0299910032388202	1.0298928497427533	0.000098153
0.02	0.02	1.0798297691111314	1.0794940223302638	0.000335747
0.03	0.03	1.1292725836060584	1.1286320940713441	0.00064049
0.04	0.04	1.1780808681173303	1.1771196278117255	0.00096124
0.05	0.05	1.226028352278671	1.2247558872007704	0.00127247
0.06	0.06	1.2729050805631328	1.2713268846914636	0.00157820
0.07	0.07	1.318520776902771	1.316605429540419	0.00191535
0.08	0.08	1.362707467578051	1.3603511758078772	0.00235629
0.09	0.09	1.4053213086894631	1.4023106703577044	0.00301064
0.10	0.10	1.4462436102487797	1.4422174008573931	0.00402621

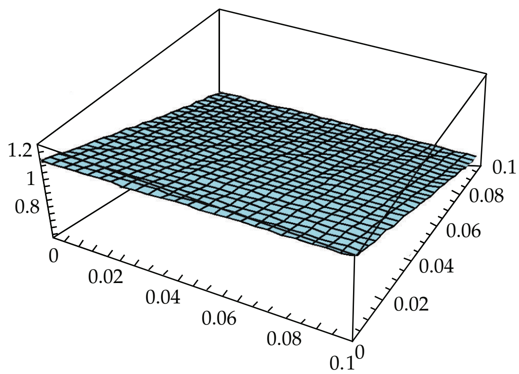

Figure 1: The analytic solution of $u(x, y, z, t)$.

and the analytical solution of the problem by Tanh method is

$$u(x, y, z, t) = 1 + \text{Tanh}(x + y + z - 6t),$$
$$v(x, y, z, t) = 1 - \text{Tanh}(x + y + z - 6t). \tag{4.6}$$

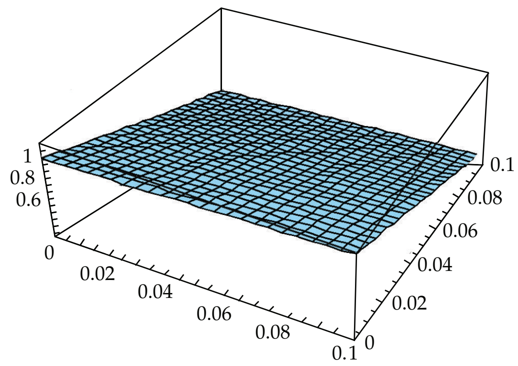

Figure 2: The approximate solution with DTM of $u(x, y, z, t)$.

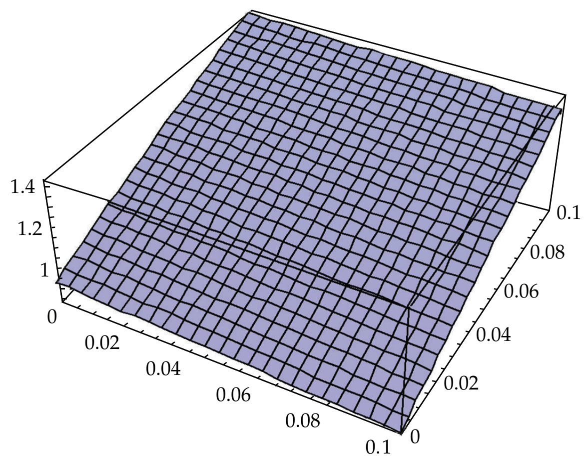

Figure 3: The analytic solution of $v(x, y, z, t)$.

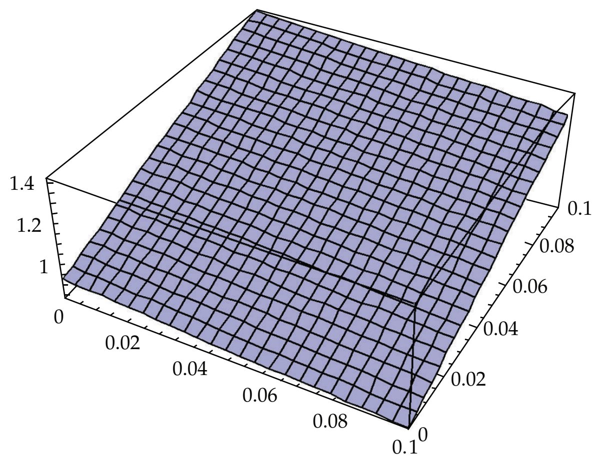

Figure 4: The approximate solution with DTM of $v(x, y, z, t)$.

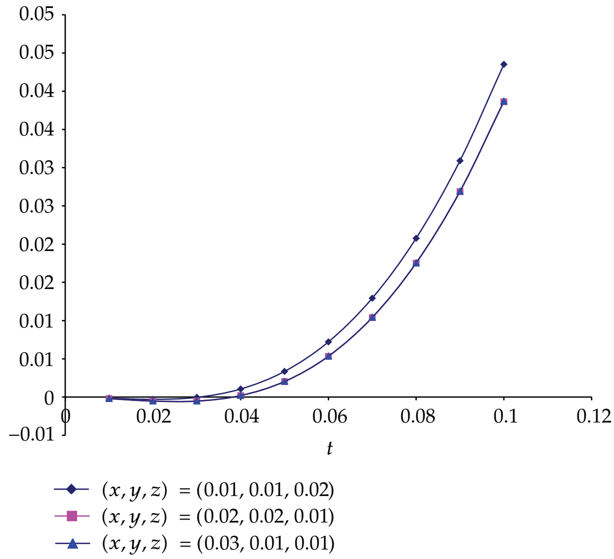

Figure 5: The error of u_{exact}-u_{DTM}.

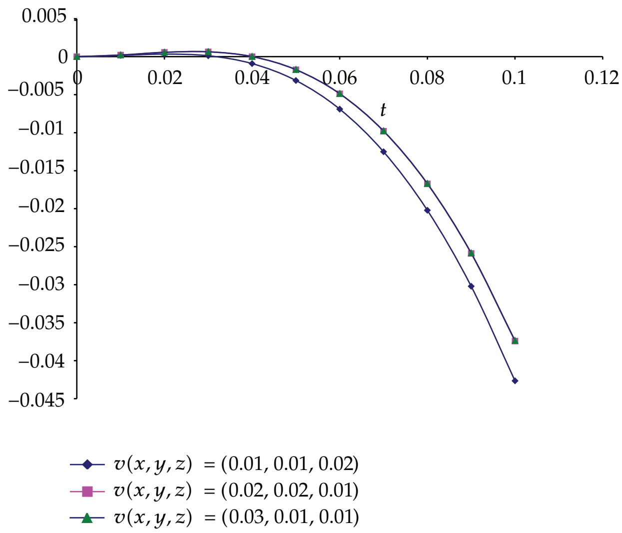

Figure 6: The error of v_{exact}-v_{DTM}.

When (4.4), (4.5), and (4.6) are compared, it can be seen that these two results are quite compatible for small values of t. For example, for $x = 0.1, y = 0.01$, $z = 0.01$, and $t = 0.001$, analytic solution of $u(x, y, z, t)$ is 1.1135087 and transform solution is 1.1135137, and analytic solution of $v(x, y, z, t)$ is 0.886491 and transform solution is 0.886485. For $x = 0.1$, $y = 0.01$, $z = 0.01$, and $t = 0.0001$, analytic solution of $u(x, y, z, t)$ is 1.1188358 and transform solution is 1.1188345, and analytic solution of $v(x, y, z, t)$ is 0.8811642 and transform solution is 0.8811654 (Tables 4 and 5, Figures 1 to 6).

5. Conclusion

Four-dimensional differential transform has been applied to nonlinear system of PDEs. Analytic solution and transform solution are compared; the results are quite compatible for small values of t. The present method reduces the computational difficulties of the other methods, and all the calculations can be made by simple manipulations. On the other hand, the results are quite reliable. Therefore, this method can be applied to many complicated linear and nonlinear PDEs and system of PDEs and does not require linearization, discretization, or perturbation.

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