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*Research Article*

# Robust Anti-Windup Control Considering Multiple Design Objectives

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A new saturation control technique is proposed to design multiobjective and robust anti-windup controllers for linear systems with input saturations. Based on the characterization of saturation nonlinearities and modeling uncertainties via integral quadratic constraints (IQCs), this method considers a mixed $H_2/H_\infty$ performance indexes while maintaining dynamic constraints on the controller. The analysis and synthesis conditions are presented in terms of scaled linear matrix inequalities (LMIs). The proposed control algorithm can improve the performance of the input-constrained system while also guaranteeing robustness with respect to the modeling uncertainties. Finally, a numerical example is given to illustrate the effectiveness of the developed techniques.

## 1. Introduction

Nonlinear control was one of the most active areas of control research. A number of different approaches have recently emerged to discuss this challenging problems, such as the fuzzy control [1–5] and robust sliding mode control [6–8]. Saturation nonlinearities are very common in feedback control systems [9], nearly all physical systems are subjected to some type of control input saturation. If input constraints are not taken into account, harmful effects on system performance and stability may appear. Numerous methods have been proposed to handle such nonlinearities, among which the anti-windup strategy is related to pratical use closely. The basic idea underlining anti-windup designs is to introduce control modifications in order to recover, as much as possible, the performance induced by a previous design carried out on the basis of the unsaturated system. First results on anti-windup consisted on ad hoc methods intended to work with standard PID controllers, which are commonly used in present commercial controllers. Nonetheless, major improvements have been achieved in the last decade as it can be researched rigorousy in theory.

A general framework that unifies a large class of existing anti-windup control schemes in terms of two matrix parameters was proposed in [10]. In [11], a rigorous definition of anti-windup compensation was provided in terms of $L_2$ stability and performance. The rigorous stability analysis based on passivity concept was developed in [12]. The synthesis condition of static anti-windup controllers was formulated as an LMI problem in [13]. References [14, 15] further derived the dynamic anti-windup controller synthesis condition with linear matrix inequality (LMI) constraints. In addition, based on the linear fractional transformation (LFT)/linear parameter-varying (LPV) framework, extended anti-windup schemes were introduced in [16, 17]. In these contributions, the saturations are modeled as sector-bounded nonlinearities and the anti-windup control design is recast as a convex optimization problem by absolute stability theory provided that no uncertainty affects the plant.

The problems associated with robustness to plant uncertainty and the problems associated with actuator saturation have often been considered in isolation. There has been little literature which attempts to handle them simultaneously in the anti-windup framework. As noted in [18], nominal linear robustness is only a necessary, but not sufficient condition for the robustness of the overall anti-windup compensated system. Furthermore, [18] introduced an approach to synthesizing anti-windup compensators for input constrained systems subject to additive dynamic uncertainty. Reference [19] considered anti-windup design problem for a closed-loop LFT model whose structured perturbation block contains parametric uncertainties.

In this paper, we propose a unified synthesis method for the construction of multiobjective and robust anti-windup controller for linear systems with actuator saturations, time-varying parametric and dynamic uncertainties. Through an equivalent representation, actuator saturations are treated as sector-bounded nonlinear uncertainty and are included in a block-diagonal operator $\Delta$ together with the other uncertainties. Inspired by the research work in [20], the problems associated with robustness are handled within the integral quadratic constraints (IQCs) framework characterizing the properties and structure of $\Delta$. The performance objectives are specified in terms of $H_\infty$ norm, $H_2$ norm, and additional regional constraints on the closed-loop poles. Interestingly, the regional closed-loop poles placement also ensures the pole-placement constraints on the anti-windup controller in that the closed-loop poles exactly consist of the poles of nominal system and those of anti-windup controller. As observed in [21], this helps to prohibit the slow dynamics which remain visible on the plant outputs even when the saturations are no longer active. The overall analysis conditions are cast as an optimization over LMIs using $S$-procedure technique and a common quadratic Lyapunov function. The controller synthesis procedure requires solving scaled LMIs with a $D/K$-like iteration and provides a full-order dynamic anti-windup controller.

*Notation.* Let $\Lambda^{n\times n}$ denote $n$-dimensional diagonal matrix. For compact presentation, given a square matrix $X$ we denote $\operatorname{He} X := X + X^T$. A block-diagonal structure with sub-blocks $X_1, X_2, \ldots, X_p$ in its diagonal will be denoted by diag $(X_1, X_2, \ldots, X_p)$. $L_{2e}^n$ denotes $n$-dimensional functional space whose members only need to be square integrable on finite intervals. $\epsilon$ is a sufficiently small value. Other notations are standard.

**Table 1:** IQC characterization for specified $\Delta_i$.

| Type | $\Phi \in [0, K_\Phi]$ | $\text{diag}(\delta_1, \ldots, \delta_{n_i})$ | $\delta_i(t) I_{n_i}$ | $\|\Delta_i(s)\|_\infty < 1$ |
|---|---|---|---|---|
| | $Q = -2V \in \Lambda^{n_i \times n_i}$ | $Q \in \Lambda^{n_i \times n_i}$ | $Q \in \Re^{n_i \times n_i}$ | $Q = qI_{n_i},\ q \in \Re$ |
| Scalings | $S = VK_\Phi$ | $S = 0$ | $S + S^T = 0$ | $S = 0$ |
| | $R = \epsilon I$ | $R = -Q$ | $R = -Q$ | $R = -Q$ |

## 2. Problem Statement

The anti-windup control problem is sketched in Figure 1(a). The block $P(s)$ denotes the stable nominal system and typically includes a model of the plant with uncertainties, nominal controller together with weighing functions specified by the user. Note that $\Phi = z - \Psi(z)$, where $\Psi$ denotes the standard saturation operator. For clearness, the anti-windup control diagram in Figure 1(a) is equivalently reformulated as LFT structure in Figure 1(b) with $\widetilde{P}(s)$ described by

$$\begin{aligned}
\dot{x}_p &= A_p x_p + B_r w_r + B_p w_p + B_u u, \\
z_r &= C_r x_p + D_{rr} w_r + D_{rp} w_p + D_{ru} u, \\
z_\infty &= C_\infty x_p + D_{\infty r} w_r + D_{\infty p} w_p + D_{\infty u} u, \\
z_2 &= C_2 x_p + D_{2r} w_r + D_{2p} w_p + D_{2u} u, \\
w &= D_{wr} w_r, \\
w_r &= \Delta z_r.
\end{aligned} \tag{2.1}$$

Here, $x_p \in \Re^n$ are the states. The input/output channels associated with the robustness are $w_r, z_r \in \Re^{n_r}$. The input/output channels associated with the performance criterion are $w_p \in \Re^{n_p}$, $z_\infty \in \Re^{n_\infty}$, and $z_2 \in \Re^{n_2}$. $u \in \Re^{n_u}$ are the compensated controls, and $w \in \Re^{n_w}$ are the saturation error feedback. For well posedness, we will assume that $D_{2p} = 0$.

$\Delta$ is a causal operator from $L_{2e}^r[0, \infty]$ to $L_{2e}^r[0, \infty]$ with its inputs and outputs satisfying the following time-domain integral quadratic constraint

$$\int_0^t \begin{bmatrix} w_r(t) \\ z_r(t) \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} w_r(t) \\ z_r(t) \end{bmatrix} dt \geq 0, \quad \forall t \geq 0. \tag{2.2}$$

Let $Q$, $S$, $R$ be constant scaling matrices such that $Q < 0$, $R > 0$. We assume that $\Delta$ is block diagonal: $\Delta = \text{diag}(\Delta_1, \ldots, \Delta_r)$, where $\Delta_i$ denotes a "troublemaking" component. The IQC characterizations for the typical cases considered here are listed in Table 1. Reference [20] provides a fairly complete overview of IQCs. For application, all of the individual IQC are collected in block-diagonal matrices $Q = \text{diag}\ (Q_1, \ldots, Q_r)$, $R = \text{diag}\ (R_1, \ldots, R_r)$, and $S = \text{diag}\ (S_1, \ldots, S_r)$ to characterize the associated composition of $\Delta$.

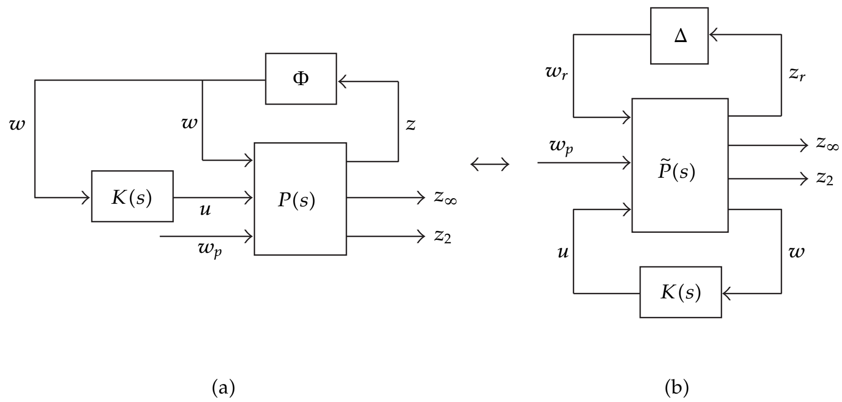


**Figure 1:** (a) Anti-windup control structure; (b) equivalent LFT formulation.

Considering system (2.1), we assume that a full-order dynamic anti-windup compensator is of the form

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k w, \\ u &= C_k x_k + D_k w, \end{aligned} \tag{2.3}$$

where $x_k \in \Re^n$ is the controller state, and $A_k$, $B_k$, $C_k$, $D_k$ are constant matrices of appropriate dimensions. Then, the final closed-loop system admits the realization

$$\begin{aligned} \dot{x}_c &= \mathcal{A}_c x_c + \mathcal{B}_r w_r + \mathcal{B}_p w_p, \\ z_r &= \mathcal{C}_r x_c + \mathfrak{D}_{rr} w_r + \mathfrak{D}_{rp} w_p, \\ z_\infty &= \mathcal{C}_\infty x_c + \mathfrak{D}_{\infty r} w_r + \mathfrak{D}_{\infty p} w_p, \\ z_2 &= \mathcal{C}_2 x_c + \mathfrak{D}_{2r} w_r, \end{aligned} \tag{2.4}$$

where $x_c = [x_p^T \;\; x_k^T]^T$ and

$$\begin{bmatrix} \mathcal{A}_c & \mathcal{B}_i \\ \mathcal{C}_j & \mathfrak{D}_{ji} \end{bmatrix} = \left[\begin{array}{cc|c} A_p & B_u C_k & B_i + B_u D_k D_{\omega i} \\ 0 & A_k & B_k D_{\omega i} \\ \hline C_j & D_{ju} C_k & D_{ji} + D_{ju} D_k D_{\omega i} \end{array}\right] \tag{2.5}$$

with $i = r, p$ and $j = r, \infty, 2$.

Denoting by $T_\infty(s)$ and $T_2(s)$ the closed-loop transfer functions from $w_p$ to $z_\infty$ and $z_2$ respectively, we consider the following multiobjective synthesis problem: design an dynamic anti-windup controller (2.3) such that as follows.

(1) The closed-loop system (2.4) is robustly stable with respect to the perturbation block $\Delta$.

(2) Minimize $\|T_2(s)\|_2$ subject to $\|T_\infty(s)\|_\infty < \gamma$.

(3) The closed-loop poles can be placed in the prescribed complex plane which is described by LMI region.

## 3. LMI Formulation of System Analysis

In this section, we will provide robust stability and performance analysis conditions for the closed-loop system (2.4) in the LMI framework. The specifications and objectives under consideration include $H_\infty$ performance, $H_2$ performance. Additional regional constraints on the closed-loop poles can also be imposed.

**Theorem 3.1** (robust $H_\infty$ performance). *Given the closed-loop system* (2.4) *with perturbation block* $\Delta$ *satisfying the integral quadratic constraint* (2.2) *and a scalar* $\gamma$*, if there exist a positive-definite matrix* $P_\infty$ *and scaling matrices* $Q, S, R$ *such that*

$$\text{He}\begin{bmatrix} P_\infty \mathcal{A}_c & P_\infty \mathcal{B}_r + \mathcal{C}_r^T S & P_\infty \mathcal{B}_p & 0 & 0 \\ 0 & \frac{1}{2}Q + S^T \mathfrak{D}_{rr} & S^T \mathfrak{D}_{rp} & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\gamma I & 0 & 0 \\ R\mathcal{C}_r & R\mathfrak{D}_{rr} & R\mathfrak{D}_{rp} & -\frac{1}{2}R & 0 \\ \mathcal{C}_\infty & \mathfrak{D}_{\infty r} & \mathfrak{D}_{\infty p} & 0 & -\frac{1}{2}\gamma I \end{bmatrix} < 0 \tag{3.1}$$

*then the closed-loop system is robustly stable against the perturbation block* $\Delta$*, and one has* $\|T_\infty(s)\|_\infty < \gamma$ *with zero-state initial conditions.*

*Proof.* Consider a Lyapunov function $V(x_c) = x_c^T P_\infty x_c$ for the closed-loop system (2.4). A sufficient condition for the robust $H_\infty$ performance specification can be established from the inequality

$$\dot{V} + \begin{bmatrix} w_r \\ z_r \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} w_r \\ z_r \end{bmatrix} + \frac{1}{\gamma} z_\infty^T z_\infty - \gamma w_p^T w_p < 0. \tag{3.2}$$

First, consider the robust stability with the performance channel removed, the inequality (3.2) is rewritten as

$$\frac{d}{dt}\left( V + \int_0^t \begin{bmatrix} w_r \\ z_r \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} w_r \\ z_r \end{bmatrix} dt \right) < 0. \tag{3.3}$$

Note that the second term is always nonnegative. According to standard arguments from Lyapunov theory, the closed-loop system is stable. Here, the function $V$ decreases to zero, but not necessarily monotonically. Next, consider robust performance, integrating (3.2) from 0 to $\infty$ with initial condition $x_c(0) = 0$ yields $\|z_\infty\|_2 < \gamma \|w_p\|_2$. As a result, robust $H_\infty$ performance can be guaranteed. Inequality (3.2) is equivalent to the LMI condition (3.1) by Schur complement. □

**Theorem 3.2** (robust $H_2$ performance). *Given the closed-loop system* (2.4) *with perturbation block* $\Delta$ *satisfying the integral quadratic constraint* (2.2) *and a scalar* $\nu$, *if there exist a positive-definite matrix* $P_2$ *and scaling matrices* $Q, S, R$ *such that*

$$\text{He}\begin{bmatrix} P_2\mathcal{A}_c & P_2\mathcal{B}_r + \mathcal{C}_r^T S & 0 & 0 \\ 0 & \dfrac{1}{2}Q + S^T\mathfrak{D}_{rr} & 0 & 0 \\ R\mathcal{C}_r & R\mathfrak{D}_{rr} & -\dfrac{1}{2}R & 0 \\ \mathcal{C}_2 & \mathfrak{D}_{2r} & 0 & -\dfrac{1}{2}I \end{bmatrix} < 0,$$
$$\begin{bmatrix} P_2 & P_2\mathcal{B}_p \\ \mathcal{B}_p^T P_2 & W \end{bmatrix} > 0, \tag{3.4}$$
$$\text{Tr}(W) < \nu^2$$

*then the closed-loop system is robustly stable against the perturbation block* $\Delta$, *and one has* $\|T_2(S)\|_2 < \nu$.

*Proof.* Let $\{e_1, \ldots, e_{n_p}\}$ be a basis of the input space $\Re^{n_p}$. Let $x_{c0\cdot i} = \mathcal{B}_p e_i$, $i = 1, \ldots, n_p$ be the initial conditions of the closed-loop system (2.4). Let $z_{2\cdot i}$ denote the output response subject to initial condition $x_{c0\cdot i}$ and $w_p = 0$. Then the $H_2$ norm $\|T_2(s)\|_2$ can be equivalently defined as [22]

$$\|T_2(s)\|_2^2 := \sum_{i=1}^{n_p} \|z_{2\cdot i}\|_2^2. \tag{3.5}$$

With these results, a Lyapunov function $V(x_c) = x_c^T P_2 x_c$ can be constructed to satisfy the following inequality

$$\dot{V} + \begin{bmatrix} w_r \\ z_r \end{bmatrix}^T \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \begin{bmatrix} w_r \\ z_r \end{bmatrix} + z_2^T z_2 < 0. \tag{3.6}$$

The robust stability proof is the same as the one in Theorem 3.1. As for robust performance, integrating (3.6) from 0 to $\infty$ with $x_c(\infty) = 0$ guaranteed by stability, we can obtain $\|z_2\|_2^2 < V(x_c(0))$. As a result, the output energy is bounded by

$$\sum_{i=1}^{n_p} \|z_{2\cdot i}\|_2^2 < \sum_{i=1}^{n_p} e_i^T \mathcal{B}_p^T P_2 \mathcal{B}_p e_i = \text{Tr}\left(\mathcal{B}_p^T P_2 \mathcal{B}_p\right). \tag{3.7}$$

With an auxiliary parameter $W$ such that $\mathcal{B}_p^T P_2 \mathcal{B}_p < W$, the LMI conditions (9~11) can be obtained by Schur complement. □

Pole assignment in convex regions of the left-half plane can be expressed as LMI constraints on the Lyapunov matrix. An LMI region is any region $D$ of the complex plane that can be defined as

$$D = \left\{ z \in C : L + Mz + M^T \overline{z} < 0 \right\} \tag{3.8}$$

with $L = L^T = \{\lambda_{ij}\}_{1 \le i, j \le m}$ and $M = \{\mu_{ij}\}_{1 \le i, j \le m}$ being constant real matrices. Reference [23] gives a thorough discussion for various types of the convex region.

**Theorem 3.3** (see [23] (pole placement)). *The closed-loop state matrix $\mathcal{A}_c$ has all its eigenvalues in the LMI region $D$* (3.8) *if and only if there exists a positive definite matrix $P_{\text{pol}}$ such that*

$$\left[ \lambda_{ij} P_{\text{pol}} + \mu_{ij} \mathcal{A}_c^T P_{\text{pol}} + \mu_{ji} P_{\text{pol}} \mathcal{A}_c \right]_{1 \le i, j \le m} < 0. \tag{3.9}$$

Note that the closed-loop poles of system (2.4) exactly consist of the poles of system (2.1) and those of controller (2.3); LMI region $D$ should include the poles of system (2.1) to ensure the feasibility of the problem. Furthermore, the dynamics of the controller (2.3) can be constrained by the LMI region $D$.

## 4. LMI Approach to Multiobjective Synthesis

Based on the analysis results stated in the above section, in this section we aim to present a constructive procedure to design an anti-windup controller of the form (2.3), satisfying the multiobjective synthesis purposes proposed in Section 2. This procedure relies on a simple change of controller variables to map all LMIs of Section 3 into a set of affine constraints on the new controller variables and the closed-loop Lyapunov matrix.

For tractability in the LMI framework, we must seek a common Lyapunov matrix

$$P := P_\infty = P_2 = P_{\text{pol}} \tag{4.1}$$

that satisfies Theorems 3.1, 3.2, and 3.3. This restriction has been extensively used in multiobjective control problem such as [23, 24]. Partition $P$ and $P^{-1}$ as

$$P = \begin{bmatrix} Y & N \\ N^T & * \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} X & M \\ M^T & * \end{bmatrix}, \tag{4.2}$$

where $X, Y \in \Re^{n \times n}$ are symmetric. Factorizing $P$ as

$$PX_1 = X_2, \qquad X_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix} \tag{4.3}$$

we define the change of controller variables as follows:

$$\begin{aligned}
\mathcal{A}_k &:= YA_pX + YB_uC_kM^T + NA_kM^T,\\
\mathcal{B}_k &:= YB_uD_k + NB_k,\\
\mathcal{C}_k &:= C_kM^T,\\
\mathfrak{D}_k &:= D_k.
\end{aligned} \tag{4.4}$$

For full-order design, one can always assume that $M, N$ are $n \times n$ square and invertible matrices. Hence the controller variables $A_k, B_k, C_k, D_k$ can be determined by $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathfrak{D}_k, X, Y$ uniquely. Then through suitable congruence transformation, the analysis results of Section 3 are readily turned into inequality constraints on the variables $X, Y, \mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathfrak{D}_k$ as well as auxiliary variable $W$ and scaling matrices $Q, S, R$, and we arrive at Theorem 4.1.

**Theorem 4.1** (multiobjective synthesis for robust anti-windup controller). *Given the generalized plant* (2.1) *with perturbation block $\Delta$ satisfies the integral quadratic constraint* (2.2) *and the LMI region $D$* (3.7). *There exists a controller* (2.3) *which robustly stabilizes plant* (2.1) *and enforces a tight upper bound $\sqrt{\operatorname{Tr}(W)}$ on $\|T_2(s)\|_2$ subject to $\|T_\infty(s)\|_\infty < \gamma$ and closed-loop poles constraints specified by $D$, if there exist matrices $X, Y, \mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathfrak{D}_k$ as well as auxiliary variable $W$ and scaling matrices $Q, S, R$ such that the inequalities hold as shown in (20~22) at the top of the next page, together with*

$$\operatorname{He}\begin{bmatrix}
A_pX + B_u\mathcal{C}_k & A_p + \mathcal{A}_k^T & B_r + B_u\mathfrak{D}_kD_{wr} + XC_r^TS + \mathcal{C}_k^TD_{ru}^TS & B_p & 0 & 0\\
0 & YA_p & YB_r + \mathcal{B}_kD_{wr} + C_r^TS & YB_p & 0 & 0\\
0 & 0 & \frac{1}{2}Q + S^TD_{rr} + S^TD_{ru}\mathfrak{D}_kD_{wr} & S^TD_{rp} & 0 & 0\\
0 & 0 & 0 & -\frac{1}{2}\gamma I & 0 & 0\\
RC_rX + RD_{ru}\mathcal{C}_k & RC_r & RD_{rr} + RD_{ru}\mathfrak{D}_kD_{wr} & RD_{rp} & -\frac{1}{2}R & 0\\
C_\infty X + D_{\infty u}\mathcal{C}_k & C_\infty & D_{\infty r} + D_{\infty u}\mathfrak{D}_kD_{wr} & D_{\infty p} & 0 & -\frac{1}{2}\gamma I
\end{bmatrix} < 0,$$

$$\operatorname{He}\begin{bmatrix}
A_pX + B_u\mathcal{C}_k & A_p + \mathcal{A}_k^T & B_r + B_u\mathfrak{D}_kD_{wr} + XC_r^TS + \mathcal{C}_k^TD_{ru}^TS & 0 & 0\\
0 & YA_p & YB_r + \mathcal{B}_kD_{wr} + C_r^TS & 0 & 0\\
0 & 0 & \frac{1}{2}Q + S^TD_{rr} + S^TD_{ru}\mathfrak{D}_kD_{wr} & 0 & 0\\
RC_rX + RD_{ru}\mathcal{C}_k & RC_r & RD_{rr} + RD_{ru}\mathfrak{D}_kD_{wr} & -\frac{1}{2}R & 0\\
C_2X + D_{2u}\mathcal{C}_k & C_2 & D_{2r} + D_{2u}\mathfrak{D}_kD_{wr} & 0 & -\frac{1}{2}I
\end{bmatrix} < 0,$$

$$\left[\lambda_{ij}\begin{pmatrix} X & I\\ I & Y\end{pmatrix} + \mu_{ij}\begin{pmatrix} A_pX + B_u\mathcal{C}_k & A_p\\ \mathcal{A}_k & YA_p\end{pmatrix}^T + \mu_{ji}\begin{pmatrix} A_pX + B_u\mathcal{C}_k & A_p\\ \mathcal{A}_k & YA_p\end{pmatrix}\right]_{1\le i,\, j\le m} < 0,$$

$$\begin{bmatrix} X & I & B_p \\ I & Y & YB_p \\ B_p^T & B_p^T Y & W \end{bmatrix} > 0,$$

$$\text{Minimizing } \operatorname{Tr}(W). \tag{4.5}$$

Due to the fact that the matrix variables $X, Y, \mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathfrak{D}_k$ and scaling matrices $Q, S, R$ enter the inequalities (21~22) in nonlinear fashion, synthesis conditions are no longer convex optimization problem. In order to overcome this difficulty, one will resort to the following iterative scheme based on LMI.

*Step 1.* Initialize scaling matrices $Q, S, R$.

*Step 2.* With fixed $Q, S, R$, perform control synthesis according to Theorem 4.1. Compute two invertible matrices $M, N \in \Re^{n\times n}$ such that

$$MN^T = I - XY. \tag{4.6}$$

Equation (4.4) can be solved for $D_k, C_k, B_k, A_k$ in this order.

*Step 3.* Apply Theorems 3.1, 3.2, 3.3, and (4.1) to the closed-loop system (2.4) to solve scaling matrices $Q, S, R$ minimizing $\operatorname{Tr}(W)$.

*Step 4.* Iterate over Step 2 to Step 3 until $\operatorname{Tr}(W)$ cannot be decreased significantly.

It is important to mention that the previously described iterative scheme, although not guaranteeing a global solution theoretically, has proven very efficient in practice.

## 5. Application Example

As an application, a missile benchmark problem [25] will be used to demonstrate the effectiveness of the results discussed. The model is linearized at $\alpha = 10\,\text{deg}$ (angle of attack) and $Ma = 3$ (Mach number), and admits the realization

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\vartheta} \end{bmatrix} = \begin{bmatrix} Z_\alpha & 1 & 0 \\ M_\alpha & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \vartheta \end{bmatrix} + \begin{bmatrix} Z_\delta \\ M_\delta \\ 0 \end{bmatrix} \delta, \tag{5.1}$$

where $q$, $\vartheta$, and $\delta$ denote pitch rate, pitch angle, and elevator deflection, respectively. The measurement outputs are the flight path angle $r = \vartheta - \alpha$ and the pitch rate $q$. The parametric uncertainties originate from the aerodynamic force $Z$ and moment $M$ with uncertainty level of ±20%. The actuator dynamics are given by $G_{\text{act}}(s) = 150^2/(s^2 + 210s + 150^2)$ with saturation limit $\delta \in [-15, 15]$ deg.

Ignoring the saturation, a PID controller can be designed as $\delta_c = [1.5\int(r - r_c)dt + 2r + 0.3q]$. $\delta_c$ and $r_c$ denote the commanded signal to the actuator and the commanded flight path angle, respectively. According to the analysis results in Section 3, the PID controller can

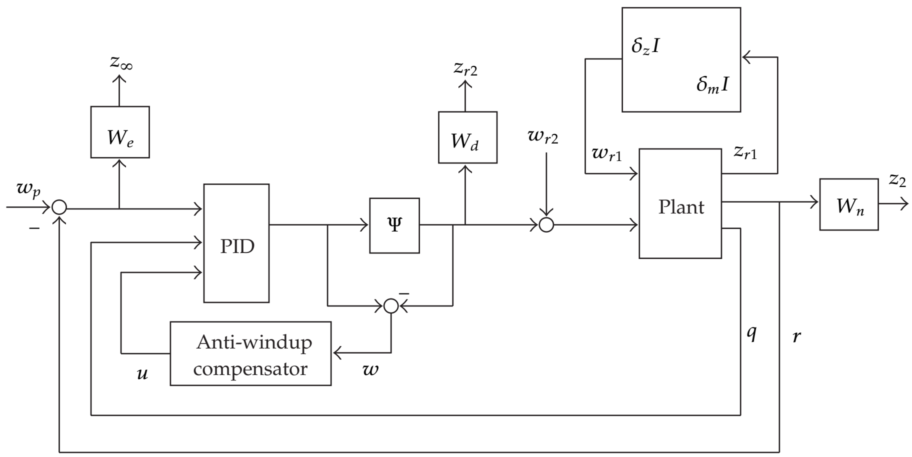


**Figure 2:** Interconnection structure for anti-windup design.

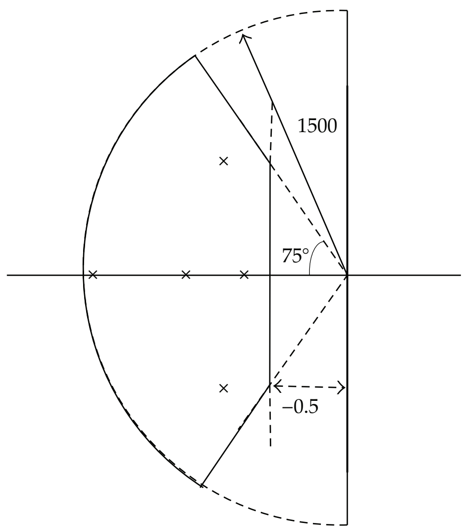


**Figure 3:** LMI region with poles $[-1346, -61, -1, -9.8 \pm 10.2i]$.

guarantee global stability for the saturated plant with $K_\Phi = 1$. Although the PID controller provides adequate stability and nominal performance, the tracking trajectory of the nominal system under saturation deteriorates and exhibits great overshoot (see Figure 4). This clearly necessitates the anti-windup compensation scheme.

In the anti-windup design, firstly parametric uncertainties in $Z$ and $M$ are extracted from the plant in a linear fractional way and rescaled to $[-1, 1]$. Secondly, to avoid excitation of unmodeled high-frequency dynamics, a multiplicative input uncertainty $\Delta_d(s)$ weighted by $W_d(s) = 1.5[(s+2)/(s+80)]$ is placed at the actuator. Finally, we end up with the control interconnection as shown in Figure 2. Constant weights $W_e = 1$ and $W_n = 0.001$ are used to reflect the tracking performance and measurements with noise.

We combine the sector-bounded nonlinearity $\Phi = I - \Psi$ with the modeling uncertainties as a block-diagonal uncertainty structure given by $\Delta = \text{diag}\,(\Phi, \delta_z I, \delta_m I, \Delta_d)$.

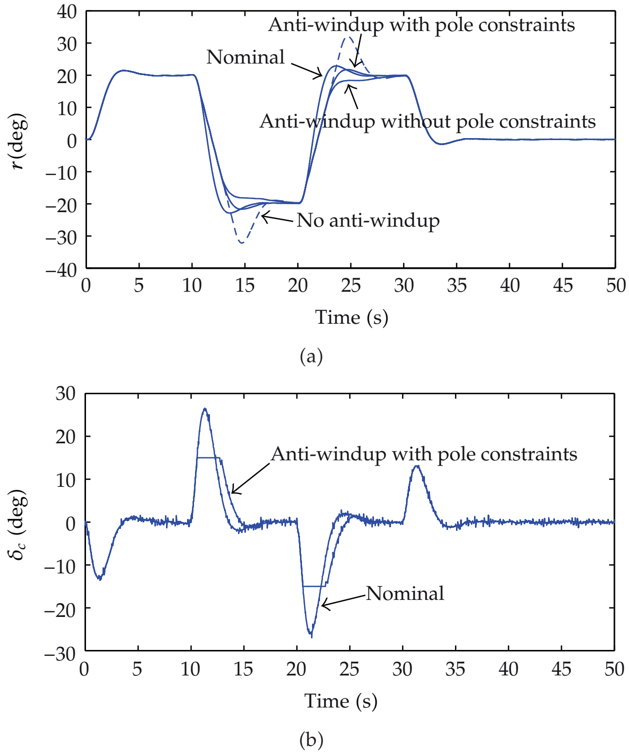


**Figure 4:** Time-domain responses to a double pulse reference.

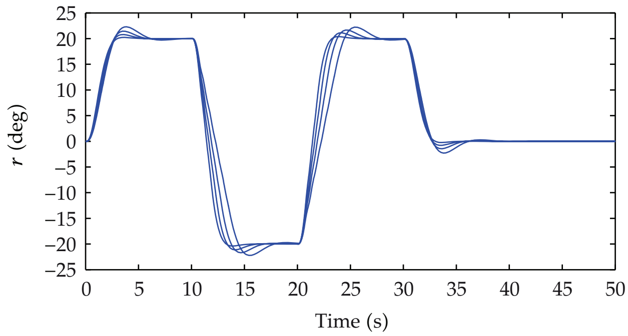


**Figure 5:** Time-domain responses for all combinations of perturbed aerodynamics.

Then, the anti-windup control diagram in Figure 2 is equivalently reformulated as LFT structure in Figure 1(b) for design. For low-order compensator, the actuator dynamics are ignored in design. This is justified by the fact that the bandwidth of the system is far below that of the actuator. The LMI region $D$ specified in Figure 3 is used to constrain the dynamics of the compensator. We choose to minimize $\|T_2(s)\|_2$ subject to $\|T_\infty(s)\|_\infty < \gamma$. $K_\Phi = 0.8$ is used to allocate the partial design freedom for coping with robustness and performance at the cost of global stability. As a result, we achieve $\gamma = 38.6$ and $\|T_2(s)\|_2 = 4.2$. The control deflection should satisfy the condition $|\delta| \le (1/(1 - K_\Phi))15$ deg. The distribution of the poles of the compensator is shown in Figure 3.

For numerical simulations, the measurement noise is chosen as band-limited white noise of power $10^{-6}$ passed through a zero order holder with sampling time $10^{-3}$ s. The resulting anti-windup response almost coincides with the linear response (see Figure 4). We can see that the designed anti-windup controlled guarantees the stability and recovers the nominal performance when the actuator is saturated deeply. For comparison, the anti-windup response of the nondynamically constrained compensator become worse because of the existence of a slow compensator mode −0.002. Figure 5 shows the time-domain robust performance behaves. As expected from previous results, Figure 5 illustrated that the anti-windup performance of the obtained controller is robust with respect to the error in model parameters.

## 6. Conclusion

This paper presents a unified synthesis method for the construction of multiobjective and robust anti-windup compensator for linear systems with actuator saturations, time-varying parametric and dynamic uncertainties. Motivated by the capability of integral quadratic constraints in characterizing saturation nonlinearities and modeling uncertainties, the concerned anti-windup and robustness problems are addressed in the framework of IQCs. The performance objectives are specified in terms of a mixed $H_2/H_\infty$ norm and additional constraints on the poles of the controller. The controller synthesis procedure requires solving scaled LMIs with a $D/K$-like iteration and provides dynamically constrained anti-windup compensators. Finally, simulation example demonstrates the effectiveness of the results.

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