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Research Article

Bifurcation of Traveling Wave Solutions for a Two-Component Generalized θ-Equation

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We study the bifurcation of traveling wave solutions for a two-component generalized θ-equation. We show all the explicit bifurcation parametric conditions and all possible phase portraits of the system. Especially, the explicit conditions, under which there exist kink (or antikink) solutions, are given. Additionally, not only solitons and kink (antikink) solutions, but also peakons and periodic cusp waves with explicit expressions, are obtained.

1. Introduction

In 2008, Liu [1] introduced a class of nonlocal dispersive models, that is, θ-equations, as follows:

$$u_t - u_{xxt} + uu_x = (1-\theta)u_x u_{xx} + \theta u u_{xxx}, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

where $u(x,t)$ denotes the velocity field at time t in the spatial x direction.

Recently, Ni [2] further investigated the cauchy problem for the following two-component generalized θ-equations:

$$\begin{gathered} u_t - u_{xxt} + uu_x - (1-\theta)u_x u_{xx} - \theta u u_{xxx} + \sigma\rho\rho_x = 0, \quad x \in \mathbb{R},\ t > 0, \\ \rho_t + \theta\rho_x u + (1-2\theta)\rho u_x = 0, \quad x \in \mathbb{R},\ t > 0, \end{gathered} \tag{1.2}$$

where σ takes 1 or -1. This system includes two components $u(x,t)$ and $\rho(x,t)$. The first one describes the horizontal velocity of the fluid, while the other one describes the horizontal deviation of the surface from equilibrium, both are measured in dimensionless units.

In this paper, we study the bifurcation of traveling wave solutions for the following system:

$$\begin{aligned} &u_t - u_{xxt} + uu_x - \frac{3}{5}u_x u_{xx} - \frac{2}{5}uu_{xxx} + \rho\rho_x = 0, \quad x \in \mathbb{R},\ t > 0, \\ &\rho_t + \frac{2}{5}\rho_x u + \frac{1}{5}\rho u_x = 0, \quad x \in \mathbb{R},\ t > 0, \end{aligned} \tag{1.3}$$

which is a special form of system (1.2) through taking $\theta = 2/5$ and $\sigma = 1$, by employing the bifurcation method and qualitative theory of dynamical systems [3–7]. We give all the explicit bifurcation parametric conditions for various solutions and all possible phase portraints of the system, from which not only solitons and kink (antikink) solutions, but also peakons and periodic cusp waves are obtained.

2. Bifurcation of Phase Portraits

For given constant c, multiplying both sides of the second equation of system (1.3) by $\rho(x,t)$ and substituting $u(x,t) = \varphi(\xi)$, $\rho = \psi(\xi)$ with $\xi = x - ct$ into system (1.3), it follows that

$$\begin{aligned} &-c\varphi' + c\varphi''' + \varphi\varphi' - \frac{3}{5}\varphi'\varphi'' - \frac{2}{5}\varphi\varphi''' + \psi\psi' = 0, \\ &-c\psi\psi' + \frac{2}{5}\psi\psi'\varphi + \frac{1}{5}\psi^2\varphi' = 0. \end{aligned} \tag{2.1}$$

Integrating system (2.1) once leads to

$$\begin{aligned} &-c\varphi + c\varphi'' + \frac{1}{2}\varphi^2 - \frac{1}{10}(\varphi')^2 - \frac{2}{5}\varphi\varphi'' + \frac{1}{2}\psi^2 = g, \\ &-\frac{c}{2}\psi^2 + \frac{1}{5}\psi^2\varphi = G, \end{aligned} \tag{2.2}$$

where both g and G are integral constants, respectively.

From the second equation of system (2.2), we obtain

$$\psi^2 = \frac{5G}{\varphi - (5/2)c}. \tag{2.3}$$

Substituting (2.3) into the first equation of system (2.2), it leads to

$$\left(\varphi - \frac{5}{2}c\right)^2 \varphi'' = -\frac{1}{4}\left(\varphi - \frac{5}{2}c\right)(\varphi')^2 + \frac{5}{4}\left[\left(\varphi - \frac{5}{2}c\right)\left(\varphi^2 - 2c\varphi - 2g\right) + 5G\right]. \tag{2.4}$$

By setting $\varphi = \phi + (3/2)c$, (2.4) becomes

$$\begin{aligned}(\phi - c)^2\phi'' &= -\frac{1}{4}(\phi - c)(\phi')^2 + \frac{5}{4}\left[(\phi - c)\left(\left(\phi - \frac{3}{2}c\right)^2 - 2c\left(\phi - \frac{3}{2}c\right) - 2g\right) + 5G\right] \\ &= -\frac{1}{4}(\phi - c)(\phi')^2 + \frac{5}{4}\left[\phi^3 - \left(\frac{7}{4}c^2 + 2g\right)\phi + \frac{3}{4}c^3 + 5G + 2cg\right].\end{aligned} \tag{2.5}$$

Letting $y = \varphi'$, we obtain a planar system

$$\begin{aligned}\frac{d\phi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= \frac{-(1/4)(\phi - c)y^2 + (5/4)\left[\phi^3 - ((7/4)c^2 + 2g)\phi + (3/4)c^3 + 5G + 2cg\right]}{(\phi - c)^2},\end{aligned} \tag{2.6}$$

with first integral

$$H(\phi, y) = \frac{1}{2}\sqrt{\phi - c}y^2 - \frac{4(\phi - c)^3 + 20c(\phi - c)^2 + (25c^2 - 40g)(\phi - c) - 100G}{8\sqrt{\phi - c}}, \quad \text{for } \phi > c, \tag{2.7}$$

or

$$H(\phi, y) = \frac{1}{2}\sqrt{c - \phi}y^2 - \frac{4(c - \phi)^3 - 20c(c - \phi)^2 + (25c^2 - 40g)(c - \phi) + 100G}{8\sqrt{c - \phi}}, \quad \text{for } \phi < c. \tag{2.8}$$

Note that when $G = 0$, systems (2.6), (2.7), and (2.8) become, respectively,

$$\begin{aligned}\frac{d\phi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= \frac{-(1/4)y^2 + (5/4)(\phi^2 + c\phi - (7/4)c^2 - 2g)}{\phi - c},\end{aligned} \tag{2.9}$$

$$H(\phi, y) = \frac{1}{2}\sqrt{\phi - c}y^2 - \frac{1}{8}\left(4(\phi - c)^{5/2} + 20c(\phi - c)^{3/2} + \left(25c^2 - 40g\right)(\phi - c)^{1/2}\right), \quad \text{for } \phi > c, \tag{2.10}$$

$$H(\phi, y) = \frac{1}{2}\sqrt{c - \phi}y^2 - \frac{1}{8}\left(4(c - \phi)^{5/2} - 20c(c - \phi)^{3/2} + \left(25c^2 - 40g\right)(c - \phi)^{1/2}\right), \quad \text{for } \phi < c. \tag{2.11}$$

Transformed by $\mathrm{d}\xi = (\phi - c)^2 \mathrm{d}\tau$, system (2.6) becomes a Hamiltonion system

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = (\phi - c)^2 y,$$
$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = -\frac{1}{4}(\phi - c)y^2 + \frac{5}{4}\left[\phi^3 - \left(\frac{7}{4}c^2 + 2g\right)\phi + \frac{3}{4}c^3 + 5G + 2cg\right]. \tag{2.12}$$

Since the first integral of system (2.6) is the same as that of the Hamiltonian system (2.12), system (2.6) should have the same topological phase portraits as system (2.12) except the straight line $l : \phi = c$. Therefore, we should be able to obtain the topological phase portraits of system (2.6) from those of system (2.12).

Let

$$f(\phi) = \phi^3 - \left(\frac{7}{4}c^2 + 2g\right)\phi + \frac{3}{4}c^3 + 5G + 2cg. \tag{2.13}$$

It is easy to obtain the two extreme points of $f(\phi)$ as follows:

$$\phi_{\pm}^* = \pm\sqrt{\frac{7c^2 + 8g}{12}}, \quad \text{for } g > -\frac{7}{8}c^2, \tag{2.14}$$

from which we can obtain a critical curve for g as follows:

$$g_0(c) = -\frac{7}{8}c^2. \tag{2.15}$$

We obtain two bifurcation curves:

$$G_1 = -\frac{1}{180}\left[72cg + 27c^3 + \left(8g + 7c^2\right)\sqrt{21c^2 + 24g}\right],$$
$$G_2 = -\frac{1}{180}\left[72cg + 27c^3 - \left(8g + 7c^2\right)\sqrt{21c^2 + 24g}\right], \tag{2.16}$$

from $f(\phi_-^*) = 0$ and $f(\phi_+^*) = 0$, respectively. Note that when $g < g_0(c)$, obviously $G_1 < G_2$. For convenience, we assume that $g \propto c^2$ in this paper, then we have $G_1 \propto c^3$ and $G_2 \propto c^3$.

Further, from $G_1 = 0$ or $G_2 = 0$, we can obtain another two critical curves for g, that is,

$$g_1(c) = -\frac{1}{2}c^2, \tag{2.17}$$

$$g_2(c) = \frac{5}{8}c^2. \tag{2.18}$$

Note that (2.18) can also be obtained by letting $\phi_+^* = c$, $c > 0$ or $\phi_-^* = c$, $c < 0$.

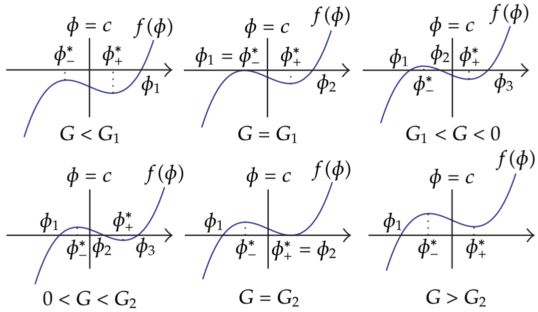


Figure 1: The graphics of $f(\phi)$ when $g > g_2(c)$.

Let $(\phi^*, 0)$ be one of the singular points of system (2.12), then the characteristic values of the linearized system of system (2.12) at the singular point $(\phi^*, 0)$ are

$$\lambda_{\pm} = \pm\frac{1}{2}\sqrt{5(\phi^* - c)^2 f'(\phi^*)}. \tag{2.19}$$

From the qualitative theory of dynamical systems, we can determine the property of singular point $(\phi^*, 0)$ by the sign of $f'(\phi^*)$ and whether ϕ^* equals to c or not. However, we also know that $H(c, y) = \infty$ from (2.7) and (2.8). Therefore, $\phi = c$ is an isolated orbit, dividing (ϕ, y)-plane into two parts.

Based on the above analysis, we give the property of the singular points for system (2.12) and their relationship with ϕ_-^*, ϕ_+^* and c in the following lemma.

Lemma 2.1. *For $g > g_2(c)$, one has $G_1 < 0 < G_2$ and the singular points of system* (2.12) *can be described as follows.*

(a) *If $G < G_1$, then there is only one singular point denoted as $S_1(\phi_1, 0)$ $(\phi_-^* < c < \phi_+^* < \phi_1)$. S_1 is a saddle point.*

(b) *If $G = G_1$, then there are two singular points denoted as $S_1(\phi_1, 0)$ and $S_2(\phi_2, 0)$ $(\phi_1 = \phi_-^* < c < \phi_+^* < \phi_2)$, respectively. S_1 is a degenerate saddle point and S_2 is a saddle point.*

(c) *If $G_1 < G < 0$, then there are three singular points denoted as $S_1(\phi_1, 0)$, $S_2(\phi_2, 0)$, and $S_3(\phi_3, 0)$ $(\phi_1 < \phi_-^* < \phi_2 < c < \phi_+^* < \phi_3)$, respectively. S_1 and S_3 are saddle points and S_2 is a center.*

(d) *If $0 < G < G_2$, then there are three singular points denoted as $S_1(\phi_1, 0)$, $S_2(\phi_2, 0)$, and $S_3(\phi_3, 0)$ $(\phi_1 < \phi_-^* < c < \phi_2 < \phi_+^* < \phi_3)$, respectively. S_1 and S_3 are saddle points and S_2 is a center.*

(e) *If $G = G_2$, then there are two singular points denoted as $S_1(\phi_1, 0)$ and $S_2(\phi_2, 0)$ $(\phi_1 < \phi_-^* < c < \phi_+^* = \phi_2)$, respectively. S_1 is a saddle point and S_2 is a degenerate saddle point.*

(f) *If $G > G_2$, then there is only one singular point denoted as $S_1(\phi_1, 0)$ $(\phi_1 < \phi_-^* < c < \phi_+^*)$. S_1 is a saddle point.*

Proof. Lemma 2.1 follows easily from the graphics of the function $f(\phi)$ which can be obtained directly and shown in Figure 1. □

For the other cases, the similar analysis can be taken to make the conclusions. We just omit these processes for the ease of simplicity. However, it is worth mentioning that, when $g_0(c) < g < g_2(c)$ and $G_1 < G < G_2 \quad (G \neq 0)$, there exist two saddle points and one center lie on the same side of singular line $\phi = c$. Hence, there may exist heteroclinic orbits for system (2.6). We will show the existence of heteroclinic orbits for system (2.6) in the following analysis.

If $G_1 < G < G_2$, we set three solutions of $f(\phi) = 0$ be ϕ_s, ϕ_m, and ϕ_b $(\phi_s < \phi_m < \phi_b)$, respectively. Through simple calculation, we can express ϕ_s and ϕ_b as the function of ϕ_m, that is,

$$\phi_s = \frac{-\phi_m - \sqrt{8g + 7c^2 - 3\phi_m^2}}{2}, \qquad \phi_b = \frac{-\phi_m + \sqrt{8g + 7c^2 - 3\phi_m^2}}{2}. \tag{2.20}$$

It follows from $\phi_s < \phi_m < \phi_b$ that ϕ_m must satisfy condition

$$\phi_m^2 < \frac{8g + 7c^2}{12}. \tag{2.21}$$

From $H(\phi_s, 0) = H(\phi_b, 0)$, we obtain the expression of G as the function of ϕ_m,

$$G = \frac{1}{100}\Big[9c^3 + 24cg - \left(8g + 15c^2\right)\phi_m - 8c\phi_m^2 + 4\phi_m^3 + \left(2c^2 - 16g - 4c\phi_m\right)\sqrt{4\phi_m^2 - 8g + 4c\phi_m - 3c^3}\Big]. \tag{2.22}$$

Substituting (2.22) into $f(\phi_m) = 0$, we obtain the expression of ϕ_m from $f(\phi_m) = 0$ as follows:

$$\phi_{m1} = \frac{1}{6}\left(5c - 2\sqrt{c^2 - 6g}\right), \tag{2.23}$$

$$\phi_{m2} = \frac{1}{6}\left(5c + 2\sqrt{c^2 - 6g}\right), \tag{2.24}$$

$$\phi_{m3} = -\sqrt{\frac{7c^2 + 8g}{3}}, \tag{2.25}$$

$$\phi_{m4} = \sqrt{\frac{7c^2 + 8g}{3}}, \tag{2.26}$$

$$\phi_{m5} = \frac{1}{2}\left(-c - 2\sqrt{c^2 + 2g}\right), \tag{2.27}$$

$$\phi_{m6} = \frac{1}{2}\left(-c + 2\sqrt{c^2 + 2g}\right), \tag{2.28}$$

Note that from (2.23)–(2.28), we obtain three critical curves for g, that is, $g_0(c)$, in (2.12), $g_1(c)$ in (2.15), and

$$g_3(c) = \frac{1}{6}c^2. \tag{2.29}$$

We then check the condition $\phi_m^2 < (8g + 7c^2)/12$ (i.e., (2.21)) for the above ϕ_ms one by one and give the results in the following lemma.

Lemma 2.2. *Starting from interval* $(-(7/8)c^2, (5/8)c^2)$, *one has the following.*

(1) *For* $g \in (-(7/8)c^2, (1/6)c^2) \subset (-(7/8)c^2, (5/8)c^2)$ *and* $c > 0$, $\phi_m = (1/6)(5c - 2\sqrt{c^2 - 6g})$ *(i.e.,* (2.23)*) satisfies* (2.21).

(2) *For* $g \in (-(7/8)c^2, (1/6)c^2) \subset (-(7/8)c^2, (5/8)c^2)$ *and* $c < 0$, $\phi_m = (1/6)(5c + 2\sqrt{c^2 - 6g})$ *(i.e.,* (2.24)*) satisfies* (2.21).

(3) *For any* $g \in (-(7/8)c^2, (5/8)c^2)$, (2.25) *does not satisfy* (2.21).

(4) *For any* $g \in (-(7/8)c^2, (5/8)c^2)$, (2.26) *does not satisfy* (2.21).

(5) *For* $g \in (-(1/2)c^2, (5/8)c^2) \subset (-(7/8)c^2, (5/8)c^2)$ *and* $c < 0$, $\phi_m = (1/2)(-c - 2\sqrt{c^2 + 2g})$ *(i.e.,* (2.27)*) satisfies* (2.21).

(6) *For* $g \in (-(1/2)c^2, (5/8)c^2) \subset (-(7/8)c^2, (5/8)c^2)$ *and* $c > 0$, $\phi_m = (1/2)(-c + 2\sqrt{c^2 + 2g})$ *(i.e.,* (2.28)*) satisfies* (2.21).

Proof. Lemma 2.2 follows easily from the definitional domain of the ϕ_ms and general logical reasoning. □

From Lemma 2.2, substituting (2.23) and (2.24) into $f(\phi_m) = 0$, respectively, we obtain another two bifurcation curves (denoted by G_1^* and G_2^*) for G as follows:

$$\begin{aligned} G_1^* &= \frac{4}{135}\left(-c^3 + 9cg + \left(c^2 - 6g\right)\sqrt{c^2 - 6g}\right), \quad \text{for } g_0(c) < g < g_3(c),\ c > 0, \\ G_2^* &= \frac{4}{135}\left(-c^3 + 9cg - \left(c^2 - 6g\right)\sqrt{c^2 - 6g}\right), \quad \text{for } g_0(c) < g < g_3(c),\ c < 0. \end{aligned} \tag{2.30}$$

Similarly, substituting (2.27) and (2.28) into $f(\phi_m) = 0$, we have

$$G_* = 0, \quad \text{for } g_1(c) < g < g_3(c),\ c < 0 \left(\text{or } g_1(c) < g < g_3(c),\ c > 0\right). \tag{2.31}$$

Note that we have indicated that when $g_0(c) < g < g_2(c)$ and $G_1 < G < G_2$ $(G \neq 0)$, there exist two saddle points and one center lying on the same side of singular line $\phi = c$. Therefore, we obtain the fifth critical curve for g from $G_1^* = 0 \ \ (c > 0)$ or $G_2^* = 0 \ \ (c < 0)$,

$$g_4(c) = 0. \tag{2.32}$$

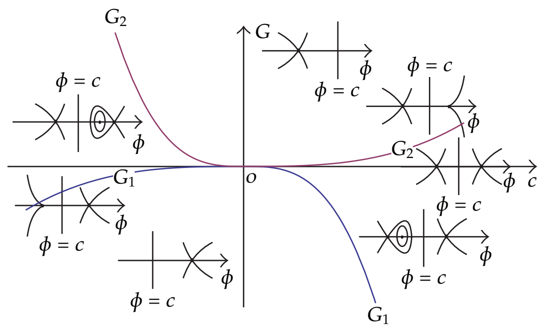


Figure 2: The phase portraits of system (2.6) when $g > g_2(c)$.

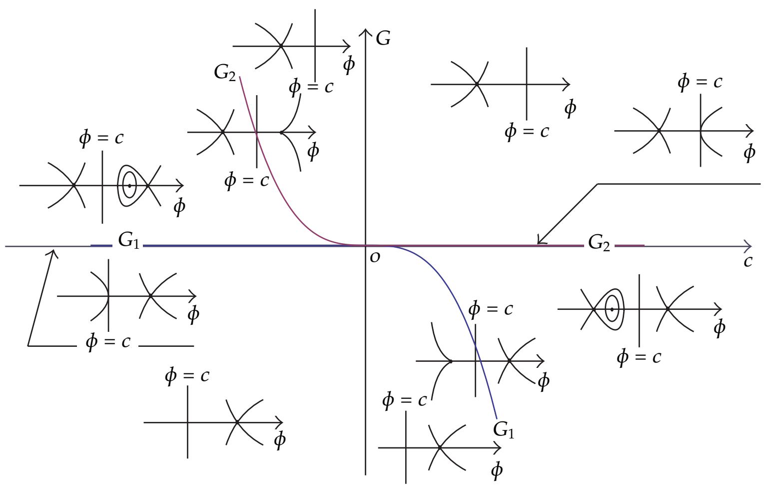


Figure 3: The phase portraits of system (2.6) when $g = g_2(c)$.

Lemma 2.3. *(1) For* $g \in (g_0(c), g_4(c)) \cup (g_4(c), g_3(c))$*, and* $G = G_1^*$, $c > 0$ *(or* $G = G_2^*$, $c < 0$*), there exist heteroclinic orbits for system* (2.6)*.*

(2) For any $g \notin (g_0(c), g_4(c)) \cup (g_4(c), g_3(c))$ *or* $G \neq G_1^*$, $c > 0$ *and* $G \neq G_2^*$, $c < 0$*, there exist no heteroclinic orbits for system* (2.6)*.*

Proof. Lemma 2.3 follows easily from the above analysis. □

Therefore, based on the above analysis, we obtain the bifurcation of phase portraits of system (2.6) in Figures 2, 3, 4, 5, 6, 7, 8, and 9 under corresponding conditions.

3. Main Results and the Theoretic Derivations of Main Results

In this section, we state our results about solitons, kink (antikink) solutions, peakons, and periodic cusp waves for the first component of system (1.3). To relate conveniently, we omit $\varphi = \phi + (2/3)c$ and the expression of the second component of system (1.3) in the following theorems.

Theorem 3.1. *For constant wave speed c, integral constants g and G, one has the following.*

(1) *If c, g, G satisfy one of the following conditions:*

(i) $g > g_2(c)$, $G_1 < G < 0$ *and* $c \neq 0$;
(ii) $g_1(c) < g \leq g_2(c)$, $G_1 < G < 0$ *and* $c > 0$;
(iii) $g_1(c) \leq g < g_4(c)$, $g_4(c) < g < g_3(c)$, $0 < G < G_1^*$ *and* $c > 0$;
(iv) $g_0(c) < g < g_1(c)$, $G_1 < G < G_1^*$ *and* $c > 0$;

then there exist smooth solitons for system (1.3), *which can be implicitly expressed as*

$$\frac{\sqrt{(c-\phi)/(\phi_1^*-\phi)}-1}{\sqrt{(c-\phi)/(\phi_1^*-\phi)}+1}\cdot\frac{\left(\sqrt{(c-\phi)/(\phi_1^*-\phi)}+\sqrt{\alpha}\right)^{\alpha}}{\left(\sqrt{(c-\phi)/(\phi_1^*-\phi)}-\sqrt{\alpha}\right)^{\alpha}}=e^{|\xi|}, \tag{3.1}$$

where

$$\alpha=\frac{c-\phi_1}{\phi_1^*-\phi_1}. \tag{3.2}$$

(2) *If c, g, G satisfy one of the following conditions:*

(v) $g > g_2(c)$, $0 < G < G_2$ *and* $c \neq 0$;
(vi) $g_1(c) < g \leq g_4(c)$, $0 < G < G_2$ *and* $c < 0$;
(vii) $g_1(c) \leq g < g_4(c)$, $g_4(c) < g < g_3(c)$, $G_2^* < G < 0$ *and* $c < 0$;
(viii) $g_0(c) < g < g_1(c)$, $G_2^* < G < G_2$ *and* $c < 0$;

then there exist smooth solitons for system (1.3), *which can be implicitly expressed as*

$$\frac{\sqrt{(\phi-c)/(\phi-\phi_2^*)}-1}{\sqrt{(\phi-c)/(\phi-\phi_2^*)}+1}\cdot\frac{\left(\sqrt{(\phi-c)/(\phi-\phi_2^*)}+\sqrt{\beta}\right)^{\beta}}{\left(\sqrt{(\phi-c)/(\phi-\phi_2^*)}-\sqrt{\beta}\right)^{\beta}}=e^{|\xi|}, \tag{3.3}$$

where

$$\beta=\frac{\phi_2-c}{\phi_2-\phi_2^*}. \tag{3.4}$$

(3) *If c, g, G satisfy one of the following conditions:*

(ix) $g_3(c) \leq g < g_2(c)$, $G_1 < G < 0$ *and* $c < 0$;
(x) $g_0(c) < g < g_3(c)$, $G_1 < G < G_2^*$ *and* $c < 0$;

then there exist smooth solitons for system (1.3)*, which can be implicitly expressed as*

$$g_1\left(\int_0^{u_1} \mathrm{d}u - \frac{\phi_3 - c}{\phi_3 - \phi_{31}^*}\int_0^{u_1} \frac{\mathrm{d}\,\mathrm{n}^2 u}{1-\gamma_1^2 \mathrm{sn}^2 u}\,\mathrm{d}\,u\right) = e^{|\xi|}, \tag{3.5}$$

where

$$\begin{aligned} g_1 &= \frac{2}{\sqrt{\phi_{32}^* - c}}, \\ \gamma_1^2 &= k_1^2 \frac{\phi_{32}^* - \phi_3}{\phi_{31}^* - \phi_3}, \\ k_1^2 &= \frac{\phi_{31}^* - c}{\phi_{32}^* - c}, \\ \mathrm{sn}\, u_1 &= \sin\phi. \end{aligned} \tag{3.6}$$

(4) *If c, g, G satisfy one of the following conditions:*

(xi) $g_3(c) \le g < g_2(c)$*,* $0 < G < G_2$ *and* $c > 0$*;*

(xii) $g_0(c) < g < g_3(c)$*,* $G_1^* < G < G_2$ *and* $c > 0$*;*

then there exist smooth solitons for system (1.3)*, which can be implicitly expressed as:*

$$g_2\left(\int_0^{u_2} \mathrm{d}u - \frac{c - \phi_4}{\phi_{41}^* - \phi_4}\int_0^{u_2} \frac{\mathrm{d}\,\mathrm{n}^2 u}{1-\gamma_2^2 \mathrm{sn}^2 u}\,\mathrm{d}u\right) = e^{|\xi|}, \tag{3.7}$$

where

$$\begin{aligned} g_2 &= \frac{2}{\sqrt{c - \phi_{42}^*}}, \\ \gamma_2^2 &= k_2^2 \frac{\phi_4 - \phi_{42}^*}{\phi_4 - \phi_{41}^*}, \\ k_2^2 &= \frac{c - \phi_{41}^*}{c - \phi_{42}^*}, \\ \mathrm{sn} u_2 &= \sin\phi. \end{aligned} \tag{3.8}$$

Proof. (1) From the phase portraits in Figures 2–9, we see that when c, g, G satisfy one of the conditions, that is, (i), (ii), (iii), or (iv), there exist homoclinic orbits as showed individually in Figures 10(a) and 10(b). The expressions of the homoclinic orbits can be given as follows:

$$y = \pm(\phi - \phi_1)\sqrt{\frac{\phi_1^* - \phi}{c - \phi}}, \quad \phi_1 \le \phi \le \phi_1^* < c, \tag{3.9}$$

where ϕ_1 and ϕ_1^* can be obtained from (2.8).

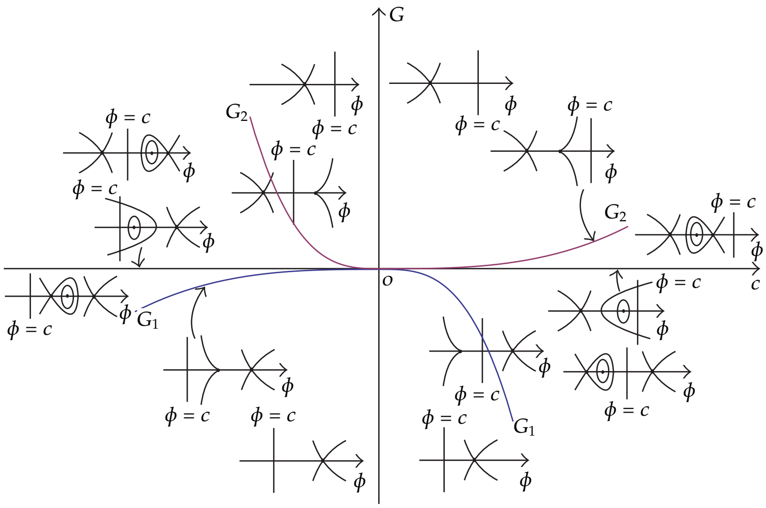


Figure 4: The phase portraits of system (2.6) when $g_3(c) \le g < g_2(c)$.

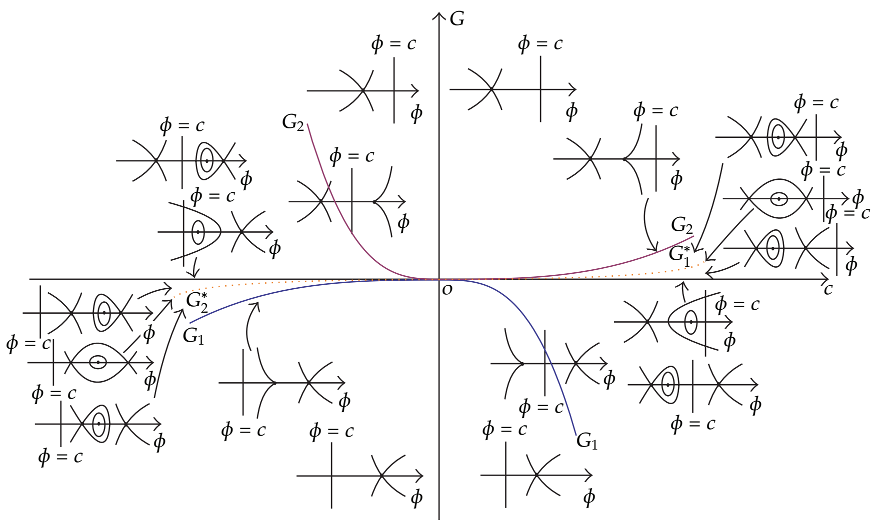


Figure 5: The phase portraits of system (2.6) when $g_4(c) < g < g_3(c)$.

Substituting (3.9) into the first equation of system (2.6), and integrating along the homoclinic orbits, it follows that

$$\int_{\phi}^{\phi_1^*} \frac{\sqrt{c-s}\,ds}{(s-\phi_1)\sqrt{\phi_1^*-s}} = |\xi|. \tag{3.10}$$

From (3.10), we obtain the solitons (3.1) along with (3.2).

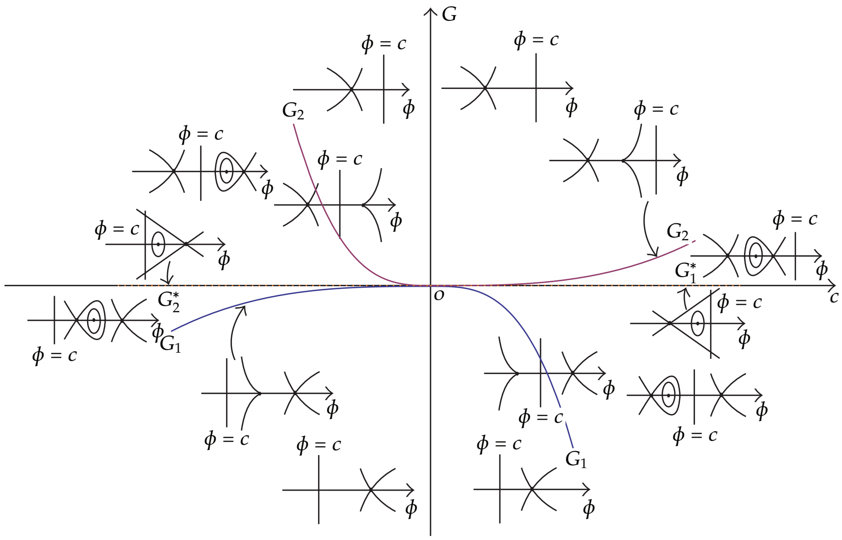


Figure 6: The phase portraits of system (2.6) when $g = g_4(c)$.

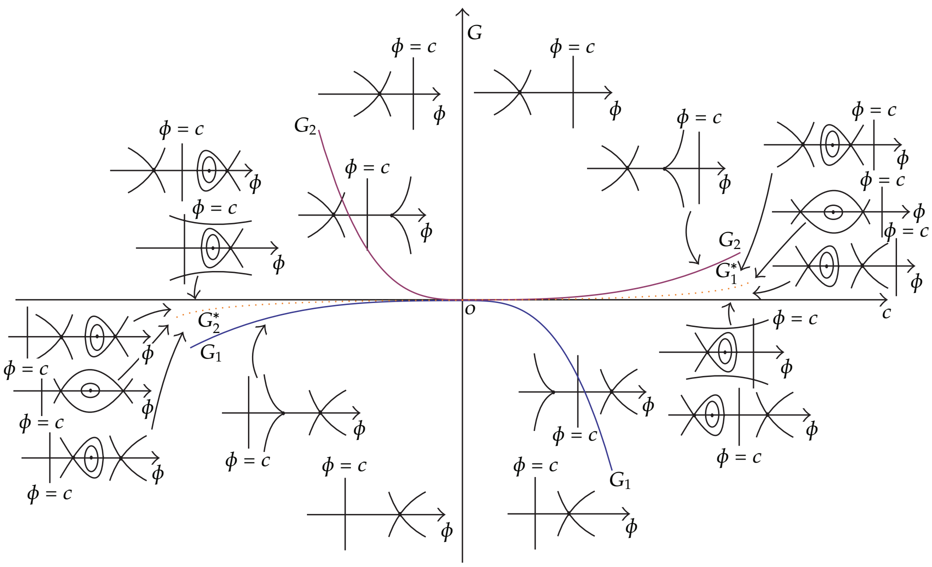


Figure 7: The phase portraits of system (2.6) when $g_1(c) < g < g_4(c)$.

(2) When c, g, G satisfy one of the conditions, that is, (v), (vi), (vii), or (viii), there exist homoclinic orbits as showed individually in Figures 8(c) and 8(d). The expressions of the homoclinic orbits can be given as follows:

$$y = \pm(\phi_2 - \phi)\sqrt{\frac{\phi - \phi_2^*}{\phi - c}}, \quad c < \phi_2^* \le \phi \le \phi_2, \tag{3.11}$$

where ϕ_2 and ϕ_2^* can be obtained from (2.7).

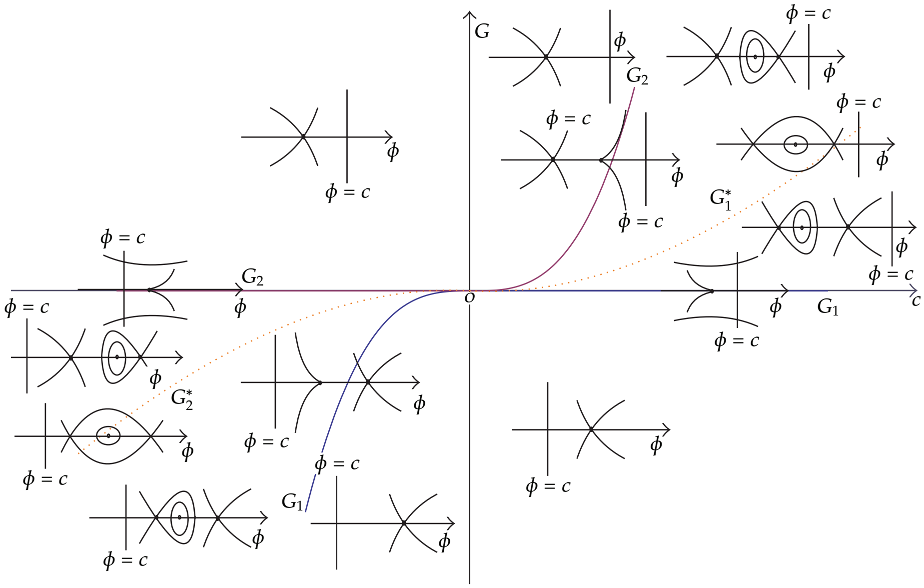


Figure 8: The phase portraits of system (2.6) when $g = g_1(c)$.

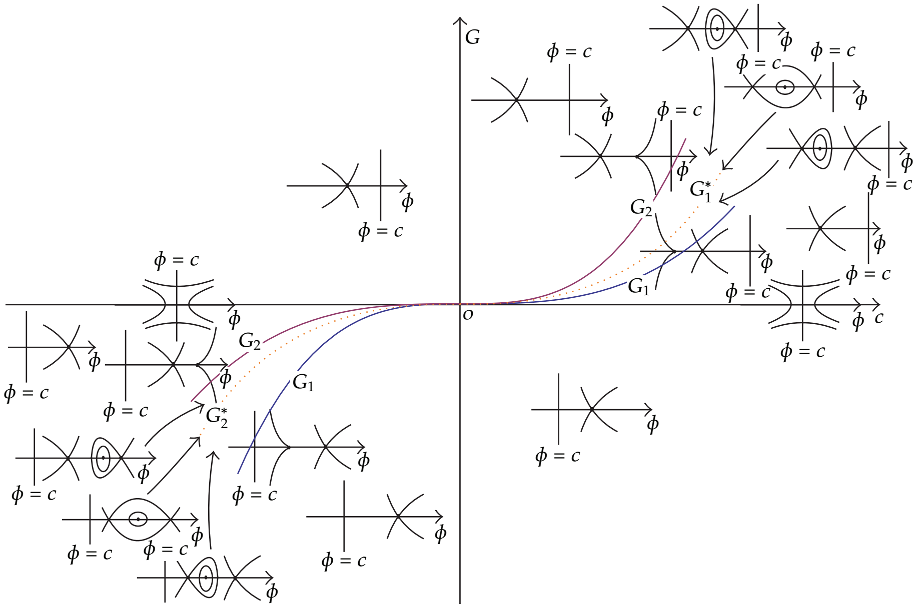
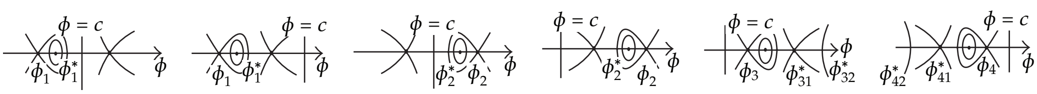


Figure 9: The phase portraits of system (2.6) when $g_0(c) < g < g_1(c)$.



(a) (b) (c) (d) (e) (f)

Figure 10: The different kinds of homoclinic orbits for system (2.6).

Substituting (3.11) into the first equation of system (2.6), and integrating along the homoclinic orbits, it follows that

$$\int_{\phi_2^*}^{\phi} \frac{\sqrt{s-c}\,\mathrm{d}s}{(\phi_2 - s)\sqrt{s-\phi_2^*}} = |\xi|. \tag{3.12}$$

From (3.12), we obtain the solitons (3.3) along with (3.4).

(3) When c, g, G satisfy one of the conditions, that is, (ix) or (x), there exist homoclinic orbits as showed individually in Figure 8(e). The expressions of the homoclinic orbits can be given as follows:

$$y = \pm(\phi - \phi_3)\sqrt{\frac{(\phi_{31}^* - \phi)(\phi_{32}^* - \phi)}{\phi - c}}, \quad c < \phi_3 \le \phi \le \phi_{31}^* < \phi_{32}^*, \tag{3.13}$$

where ϕ_3, ϕ_{31}^* and ϕ_{32}^* can be obtained from (2.7).

Substituting (3.13) into the first equation of system (2.6), and integrating along the homoclinic orbits, it follows that

$$\int_{\phi}^{\phi_{31}^*} \frac{\sqrt{s-c}\,\mathrm{d}s}{(s-\phi_3)\sqrt{(\phi_{31}^* - s)(\phi_{32}^* - s)}} = |\xi|. \tag{3.14}$$

From (3.14) [8], we obtain the solitons (3.5) along with (3.6).

(4) When c, g, G satisfy one of the conditions, that is, (xi) or (xii), there exist homoclinic orbits as showed individually in Figure 8(f). The expressions of the homoclinic orbits can be given as follows:

$$y = \pm(\phi_4 - \phi)\sqrt{\frac{(\phi - \phi_{41}^*)(\phi - \phi_{42}^*)}{c - \phi}}, \quad \phi_{42}^* < \phi_{41}^* \le \phi \le \phi_4 < c, \tag{3.15}$$

where ϕ_4, ϕ_{41}^*, and ϕ_{42}^* can be obtained from (2.8).

Substituting (3.15) into the first equation of system (2.6), and integrating along the homoclinic orbits, it follows that

$$\int_{\phi_{41}^*}^{\phi} \frac{\sqrt{c-s}\,\mathrm{d}s}{(\phi_4 - s)\sqrt{(s-\phi_{41}^*)(s-\phi_{42}^*)}} = |\xi|. \tag{3.16}$$

From (3.16) [8], we obtain the solitons (3.7) along with (3.8). □

Theorem 3.2. *If constant wave speed c, integral constants g and G satisfy $g_0(c) < g < g_4(c)$ or $g_4(c) < g < g_3(c)$, and $G = G_1^*$ ($c > 0$) or $G = G_2^*$ ($c < 0$), then there exist kink (antikink) solutions for system* (1.3).

Proof. We have showed that, when $g_0(c) < g < g_4(c)$ or $g_4(c) < g < g_3(c)$, and $G = G_1^*$ $(c > 0)$ or $G = G_2^*$ $(c < 0)$, there exist heteroclinic orbits for system (2.6). The heteroclinic orbits can be expressed as

$$y = \pm \frac{(\phi - \phi_s)(\phi_b - \phi)}{\sqrt{c - \phi}}, \quad \text{for } c > 0, \tag{3.17}$$

where

$$\begin{aligned}\phi_s &= \frac{1}{12}\left(-5c + 2\sqrt{c^2 - 6g} - \sqrt{15\left(11c^2 + 4c\sqrt{c^2 - 6g} + 24g\right)}\right),\\ \phi_b &= \frac{1}{12}\left(-5c + 2\sqrt{c^2 - 6g} + \sqrt{15\left(11c^2 + 4c\sqrt{c^2 - 6g} + 24g\right)}\right),\end{aligned} \tag{3.18}$$

which can be obtained by substituting (2.23) into (2.20).

Substituting (3.17) into the first equation of system (2.6), and integrating along the heteroclinic orbits, it follows that

$$\int_{\phi_0}^{\phi} \frac{\sqrt{c - s}\,ds}{(s - \phi_s)(\phi_b - s)} = \pm\xi, \tag{3.19}$$

where $\phi_0 \in (\phi_s, \phi_b)$ is the initial value.

From (3.19), we have

$$\begin{aligned}&\frac{\left(\sqrt{c-\phi_s} - \sqrt{c-\phi}\right)^{\sqrt{c-\phi_s}/(\phi_b-\phi_s)}}{\left(\sqrt{c-\phi_s} + \sqrt{c-\phi}\right)^{\sqrt{c-\phi_s}/(\phi_b-\phi_s)}} \cdot \frac{\left(\sqrt{c-\phi} + \sqrt{c-\phi_b}\right)^{\sqrt{c-\phi_b}/(\phi_b-\phi_s)}}{\left(\sqrt{c-\phi} - \sqrt{c-\phi_b}\right)^{\sqrt{c-\phi_b}/(\phi_b-\phi_s)}}\\ &\quad = \frac{\left(\sqrt{c-\phi_s} - \sqrt{c-\phi_0}\right)^{\sqrt{c-\phi_s}/(\phi_b-\phi_s)}}{\left(\sqrt{c-\phi_s} + \sqrt{c-\phi_0}\right)^{\sqrt{c-\phi_s}/(\phi_b-\phi_s)}} \cdot \frac{\left(\sqrt{c-\phi_0} + \sqrt{c-\phi_b}\right)^{\sqrt{c-\phi_b}/(\phi_b-\phi_s)}}{\left(\sqrt{c-\phi_0} - \sqrt{c-\phi_b}\right)^{\sqrt{c-\phi_b}/(\phi_b-\phi_s)}} e^{\pm\xi}.\end{aligned} \tag{3.20}$$

The case when $c < 0$, can be analyzed similarly. We omit it here for the ease of simplicity. □

Theorem 3.3. *(1) If* $g = g_4(c)$, $G = 0$ *and* $c \neq 0$, *then there exist peakons for system* (1.3), *which can be explicitly expressed as*

$$\phi = \frac{5}{2}ce^{-|x-ct|} - \frac{3}{2}c. \tag{3.21}$$

(2) If $g_4(c) \leq g < g_2(c)$, $G = 0$ *and* $c \neq 0$, *then system* (1.3) *has periodic cusp waves*

$$u(x,t) = \phi(\xi - 2iT) + \frac{3}{2}c, \tag{3.22}$$

where $i = 0, \pm 1, \pm 2, \ldots, \xi = x - ct \in [(2i-1)T, (2i+1)T]$, *and*

$$\phi(\xi) = \frac{1}{4}\left(5c - \sqrt{25c^2 - 40g}\right)e^{|x-ct|} + \frac{1}{4}\left(5c + \sqrt{25c^2 - 40g}\right)e^{-|x-ct|} - \frac{3}{2}c, \tag{3.23}$$

with

$$T = \ln\left(\frac{5c + \sqrt{25c^2 - 40g}}{2\sqrt{10g}}\right). \tag{3.24}$$

Proof. (1) When $g = g_4(c)$, $G = 0$ and $c \neq 0$, from Figure 6, we see that there is a triangle orbit, which can be expressed as

$$y = \pm\left(\phi + \frac{3}{2}c\right), \quad \text{for } -\frac{3}{2}c \leq \phi \leq c \ (c > 0), \tag{3.25}$$

$$\phi = c, \quad \text{for } |y| \leq \frac{\sqrt{5}}{2}c \ (c > 0). \tag{3.26}$$

Substituting (3.25) into the first equation of system (2.6), and integrating along the triangle orbits, it follows that

$$\int_{\phi}^{c} \frac{\mathrm{d}t}{t + (3/2)c} = |\xi|. \tag{3.27}$$

From (3.27), we obtain peakons (3.21).

(2) When $g_4(c) \leq g < g_2(c)$, $G = 0$ and $c \neq 0$, from Figures 4 and 5, we see that there is a semiellipse orbit, which can be expressed as

$$y = \pm\sqrt{\phi^2 + 3c\phi + \frac{9}{4}c^2 - 10g}, \quad \text{for } \frac{1}{2}\left(-3c + 2\sqrt{10g}\right) \leq \phi \leq c \ (c > 0), \tag{3.28}$$

$$\phi = c, \quad \text{for } |y| \leq \frac{\sqrt{5(c^2 - 8g)}}{2} \quad (c > 0). \tag{3.29}$$

Substituting (3.28) into the first equation of system (2.6), and integrating along the semiellipse orbits, it follows that

$$\int_{\phi}^{c} \frac{\mathrm{d}t}{\sqrt{t^2 + 3ct + (9/4)c^2 - 10g}} = |\xi|. \tag{3.30}$$

From (3.30), we obtain periodic cusp waves (3.22) along with (3.23) and (3.24).

Note that we only show the case when $c > 0$, in fact, we can analyze the case when $c < 0$ following the same procedure. We just omit it here. □

4. Conclusions

In this paper, by employing the bifurcation method and qualitative theory of dynamical systems, we study the bifurcation of traveling wave solutions for a two-component generalized θ-equation (1.3), show all the explicit parametric conditions and all the phase portraits of system (1.3) determinately. Through the phase portraits, we can investigate various kinds of solutions. Specifically, the implicit expressions of the solitons, kink (antikink) solutions for system (1.3) are given. Besides, we also obtain peakons and periodic cusp waves with explicit expressions for system (1.3).

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