Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 718714, 15 pages
doi:10.1155/2012/718714

Research Article

Stochastic Recursive Zero-Sum Differential Game and Mixed Zero-Sum Differential Game Problem

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Received 4 October 2012; Accepted 10 December 2012

Academic Editor: Guangchen Wang

Under the notable Issacs's condition on the Hamiltonian, the existence results of a saddle point are obtained for the stochastic recursive zero-sum differential game and mixed differential game problem, that is, the agents can also decide the optimal stopping time. The main tools are backward stochastic differential equations (BSDEs) and double-barrier reflected BSDEs. As the motivation and application background, when loan interest rate is higher than the deposit one, the American game option pricing problem can be formulated to stochastic recursive mixed zero-sum differential game problem. One example with explicit optimal solution of the saddle point is also given to illustrate the theoretical results.

1. Introduction

The nonlinear backward stochastic differential equations (BSDEs in short) had been introduced by Pardoux and Peng [1], who proved the existence and uniqueness of adapted solutions under suitable assumptions. Independently, Duffie and Epstein [2] introduced BSDE from economic background. In [2], they presented a stochastic differential recursive utility which is an extension of the standard additive utility with the instantaneous utility depending not only on the instantaneous consumption rate but also on the future utility. Actually, it corresponds to the solution of a particular BSDE whose generator does not depend on the variable Z. From mathematical point of view, the result in [1] is more general. Then, El Karoui et al. [3] and Cvitanic and Karatzas [4] generalized, respectively, the results to BSDEs with reflection at one barrier and two barriers (upper and lower).

BSDE plays an important role in the theory of stochastic differential game. Under the notable Isaacs's condition, Hamadène and Lepeltier [5] obtained the existence result of a saddle point for zero-sum stochastic differential game with payoff

$$J(u,v) = E^{(u,v)}\left[\int_t^T f(s,x_s,u_s,v_s)ds + g(x_T)\right]. \tag{1.1}$$

Using a maximum principle approach, Wang and Yu [6, 7] proved the existence and uniqueness of an equilibrium point. We note that the cost function in [5] is not recursive, and the game system in [6, 7] is a BSDE. In [8], El Karoui et al. gave the formulation of recursive utilities and their properties from the BSDE's pointview. The problem that the cost function (payoff) of the game system is described by the solution of BSDE becomes the recursive differential game problem. In the following Section 2, we proved the existence of a saddle point for the stochastic recursive zero-sum differential game problem and also got the optimal payoff function by the solution of one specific BSDE. Here, the generator of the BSDE contains the main variable solution y_t, and we extend the result in [5] to the recursive case which has much more significance in economics theory.

Then, in Section 3 we study the stochastic recursive mixed zero-sum differential game problem which is that the two agents have two actions, one is of control and the other is of stopping their strategies to maximize and minimize their payoffs. This kind of game problem without recursive variable and the American game option problem as this kind of mixed game problem can be seen in Hamadène [9]. Using the result of reflected BSDEs with two barriers, we got the saddle point and optimal stopping strategy for the recursive mixed game problem which has more general significance than that in [9].

In fact, the recursive (mixed) zero-sum game problem has wide application background in practice. When the loan interest rate is higher than the deposit one. The American game option pricing problem can be formulated to the stochastic recursive mixed game problem in our Section 3. To show the application of this kind of problem and our motivation to study our recursive (mixed) game problem, we analyze the American game option pricing problem and let it be an example in Section 4. We notice that in [5, 9], they did not give the explicit saddle point to the game, and it is very difficult for the general case. However, in Section 4, we also give another example of the recursive mixed zero-sum game problem, for which the explicit saddle point and optimal payoff function to illustrate the theoretical results.

2. Stochastic Recursive Zero-Sum Differential Game

In this section, we will study the existence of the stochastic recursive zero-sum differential game problem using the result of BSDEs.

Let $\{B_t, 0 \le t \le T\}$ be an m-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t)_{t\ge 0}$ be the completed natural filtration of B_t. Moreover,

(i) $\mathcal{C}$ is the space of continuous functions from $[0,T]$ to R^m;

(ii) $\mathcal{P}$ is the σ-algebra on $[0,T]\times\Omega$ of $\mathcal{F}_t$-progressively sets;

(iii) for any stopping time $\nu, \mathcal{T}_\nu$ is the set of $\mathcal{F}_t$-measurable stopping time τ such that P-$a.s.$ $\nu \le \tau \le T$; $\mathcal{T}_0$ will simply be denoted by $\mathcal{T}$;

(iv) $\mathcal{H}^{2,k}$ is the set of $\mathcal{P}$-measurable processes $\omega = (\omega_t)_{t\leq T}$, R^k-valued, and square integrable with respect to $dt \otimes d\mathcal{P}$;

(v) $\mathcal{S}^2$ is the set of $\mathcal{P}$-measurable and continuous processes $\omega' = (\omega'_t)_{t\leq T}$, such that $E[\sup_{t\leq T}|\omega'_t|^2] < \infty$.

The $m \times m$ matrix $\sigma = (\sigma_{ij})$ satisfies the following:

(i) for any $1 \leq i, j \leq m, \sigma_{ij}$ is progressively measurable;

(ii) for any $(t, x) \in [0, T] \times \mathcal{C}$, the matrix $\sigma(t, x)$ is invertible;

(iii) there exists a constants K such that $|\sigma(t, x) - \sigma(t, x')| \leq K|x - x'|_t$ and $|\sigma(t, x)| \leq K(1 + |x|_t)$.

Then, the equation

$$x_t = x_0 + \int_0^t \sigma(s, x_s) dB_s, \quad t \leq T \tag{2.1}$$

has a unique solution (x_t).

Now, we consider a compact metric space A (resp. B), and $\mathcal{U}$ (resp. $\mathcal{V}$) is the space of $\mathcal{P}$-measurable processes $u := (u_t)_{t\leq T}$ (resp. $v := (v_t)_{t\leq T}$) with values in A (resp. B). Let $\Phi : [0, T] \times \mathcal{C} \times \mathcal{U} \times \mathcal{V} \to R^m$ be such that

(i) for any $(t, x) \in [0, T] \times \mathcal{C}$, the mapping $(u, v) \to \Phi(t, x, u, v)$ is continuous;

(ii) for any $(u, v) \in A \times B$, the function $\Phi(\cdot, x(\cdot), u, v)$ is $\mathcal{P}$-measurable;

(iii) there exists a constant K such that $|\Phi(t, x, u, v)| \leq K(1 + |x|_t)$ for any t, x, u, and v;

(iv) there exists a constant M such that $|\sigma^{-1}(t, x)\Phi(t, x, u, v)| \leq M$ for any t, x, u, and v.

For $(u, v) \in \mathcal{U} \times \mathcal{V}$, we define the measure $P^{u,v}$ as

$$\frac{dP^{u,v}}{dP} = \exp\left\{\int_0^T \sigma^{-1}(s, x_s)\Phi(s, x_s, u_s, v_s) dB_s - \frac{1}{2}\int_0^T \left|\sigma^{-1}(s, x_s)\Phi(s, x_s, u_s, v_s)\right|^2 ds\right\}. \tag{2.2}$$

Thanks to Girsanov's theorem, under the probability $P^{u,v}$, the process

$$B_t^{u,v} = B_t - \int_0^t \sigma^{-1}(s, x_s)\Phi(s, x_s, u_s, v_s) ds, \quad t \leq T, \tag{2.3}$$

is a Brownian motion, and for this stochastic differential equation

$$x_t = x_0 + \int_0^t \Phi(s, x_s, u_s, v_s) ds + \int_0^t \sigma(s, x_s) dB_s^{u,v}, \quad t \leq T, \tag{2.4}$$

$(x_t)_{t\leq T}$ is a weak solution.

Suppose that we have a system whose evolution is described by the process $(x_t)_{t\leq T}$. On that system, two agents c_1 and c_2 intervene. A control action for c_1 (resp. c_2) is a process

$u = (u_t)_{t\leq T}$ (resp. $v = (v_t)_{t\leq T}$) belonging to $\mathcal{U}$ (resp. $\mathcal{V}$). Thereby $\mathcal{U}$ (resp. $\mathcal{V}$) is called the set of admissible controls for c_1 (resp. c_2). When c_1 and c_2 act with, respectively, u and v, the law of the dynamics of the system is the same as the one of x under $P^{u,v}$. The two agents have no influence on the system, and they act to protect their advantages by means of $u \in \mathcal{U}$ and $v \in \mathcal{V}$ via the probability $P^{u,v}$.

In order to define the payoff, we introduce two functions $C(t,x,y,u,v)$ and $g(x)$ satisfying the following assumption: there exists $L > 0$, for all $x, x' \in \mathcal{H}^{2,m}$ and $Y, Y' \in \mathcal{S}^2$, such that

$$\begin{aligned} \left|C(t,x_t,Y_t,u,v) - C(t,x'_t,Y_t,u,v)\right| &\leq L\left|x_t - x'_t\right|, \\ (Y_t - Y'_t)\left(C(t,x_t,Y_t,u,v) - C(t,x_t,Y'_t,u,v)\right) &\leq L\left(Y_t - Y'_t\right)^2, \end{aligned} \tag{2.5}$$

and $g(x)$ is measurable, Lipschitz continuous function with respect to x. The payoff $J(x_0,u,v)$ is given by $J(x_0,u,v) = Y_0$, where Y satisfies the following BSDE:

$$\begin{gathered} -dY_s = C(s,x_s,Y_s,u_s,v_s)ds - Z_s dB_s^{u,v}, \\ Y_T = g(x_T). \end{gathered} \tag{2.6}$$

From the result in [10], there exists a unique solution (Y,Z) for u,v. The agent c_1 wishes to minimize this payoff, and the agent c_2 wishes to maximize the same payoff. We investigate the existence of a saddle point for the game, more precisely a pair (u^*,v^*) of strategies, such that $J(x_0,u^*,v) \leq J(x_0,u^*,v^*) \leq J(x_0,u,v^*)$ for each $(u,v) \in \mathcal{U}\times\mathcal{V}$.

For $(t,x,Y,Z,u,v) \in [0,T]\times\mathcal{C}\times R\times R^m\times\mathcal{U}\times\mathcal{V}$, we introduce the Hamiltonian by

$$H(t,x,Y,Z,u,v) = Z\sigma^{-1}(t,x)\Phi(t,x,u,v) + C(t,x,Y,u,v), \tag{2.7}$$

and we say that the Isaacs' condition holds if for $(t,x,Y,Z) \in [0,T]\times\mathcal{C}\times R\times R^m$,

$$\max_{v\in\mathcal{V}}\min_{u\in\mathcal{U}} H(t,x,Y,Z,u,v) = \min_{u\in\mathcal{U}}\max_{v\in\mathcal{V}} H(t,x,Y,Z,u,v). \tag{2.8}$$

We suppose now that the Isaacs' condition is satisfied. By a selection theorem (see Benes [11]), there exists $u^* : [0,T]\times\mathcal{C}\times R\times R^m \to \mathcal{U}$, $v^* : [0,T]\times\mathcal{C}\times R\times R^m \to \mathcal{V}$, such that

$$H(t,x,Y,Z,u^*,v) \leq H(t,x,Y,Z,u^*,v^*) \leq H(t,x,Y,Z,u,v^*). \tag{2.9}$$

Thanks to the assumption of σ, Φ, and C, the function $H(t,x,Y,Z,u^*(t,x,Y,Z), v^*(t,x,Y,Z))$ is Lipschitz in Z and monotone in Y like the function C.

Now we give the main result of this section.

Theorem 2.1. *(Y^*,Z^*) is the solution of the following BSDE:*

$$\begin{gathered} -dY_s^* = H(s,x_s,Y_s^*,Z_s^*,u^*(s,x,Y^*,Z^*),v^*(s,x,Y^*,Z^*))ds - Z_s^* dB_s, \\ Y_T^* = g(x_T). \end{gathered} \tag{2.10}$$

Then, Y_0^* is the optimal payoff $J(x_0, u^*, v^*)$, and the pair (u^*, v^*) is the saddle point for this recursive game.

Proof. We consider the following BSDE:

$$Y_t^* = g(x_T) + \int_t^T H(s, x_s, Y_s^*, Z_s^*, u^*(t, x, Y^*, Z^*), v^*(t, x, Y^*, Z^*))ds - \int_t^T Z_s^* dB_s. \tag{2.11}$$

Thanks to Theorem 2.1 in [10], the equation has a unique solution (Y^*, Z^*). Because Y_0^* is deterministic, so

$$\begin{aligned} Y_0^* &= E^{u^*,v^*}\left[Y_0^*\right] \\ &= E^{u^*,v^*}\left[g(x_T) + \int_0^T H(s, x_s, Y_s^*, Z_s^*, u^*(t, x, Y^*, Z^*), v^*(t, x, Y^*, Z^*))ds - \int_0^T Z_s^* dB_s\right] \\ &= E^{u^*,v^*}\left[g(x_T) + \int_0^T C(s, x_s, Y_s^*, u_s^*, v_s^*)ds - \int_0^T Z_s^* dB_s^{u^*,v^*}\right]. \end{aligned} \tag{2.12}$$

We can get $Y_0^* = J(x_0, u^*, v^*)$.

For any $u \in \mathcal{U}, v \in \mathcal{V}$, then we let

$$\begin{aligned} Y_t &= g(x_T) + \int_t^T C(s, x_s, Y_s, u_s^*, v_s)ds - \int_t^T Z_s dB_s^{u^*,v} \\ &= g(x_T) + \int_t^T H(s, x_s, Y_s, Z_s, u_s^*, v_s)ds - \int_t^T Z_s dB_s, \\ Y_t' &= g(x_T) + \int_t^T C(s, x_s, Y_s', u_s, v_s^*)ds - \int_t^T Z_s' dB_s^{u,v^*} \\ &= g(x_T) + \int_t^T H(s, x_s, Y_s', Z_s', u_s, v_s^*)ds - \int_t^T Z_s' dB_s. \end{aligned} \tag{2.13}$$

By the comparison theorem of the BSDEs and the inequality (2.9), we can compare the solutions of (2.11), and (2.13) and get $Y_t \leq Y_t^* \leq Y_t'$, $0 \leq t \leq T$, so $Y_0 = J(x_0, u^*, v) \leq J(x_0, u^*, v^*) \leq J(x_0, u, v^*) = Y_0'$ and (u^*, v^*) is the saddle point. □

3. Stochastic Recursive Mixed Zero-Sum Differential Game

Now, we study the stochastic recursive mixed zero-sum differential game problem. First, let us briefly describe the problem.

Suppose now that we have a system, whose evolution also is described by $(x_t)_{0 \leq t \leq T}$, which has an effect on the wealth of two controllers C_1 and C_2. On the other hand, the controllers have no influence on the system, and they act so as to protect their advantages, which are antagonistic, by means of $u \in \mathcal{U}$ for C_1 and $v \in \mathcal{V}$ for C_2 via the probability $P^{u,v}$ in (2.2). The couple $(u, v) \in \mathcal{U} \times \mathcal{V}$ is called an admissible control for the game. Both controllers also have the possibility to stop controlling at τ for C_1 and θ for C_2; τ and θ are elements of $\mathcal{T}$ which is the class of all $\mathcal{F}_t$-stopping time. In such a case, the game stops. The controlling action is not free, and it corresponds to the actions of C_1 and C_2. A payoff is described by the following BSDE:

$$\begin{aligned} Y_t^{u,\tau;v,\theta} &= U_\tau 1_{[\tau<\theta]} + L_\theta 1_{[\theta<\tau<T]} + Q_\tau 1_{[\tau=\theta<T]} + g(x_T) 1_{[\tau=\theta=T]} \\ &\quad + \int_t^{\tau\wedge\theta} C\left(s, x_s, Y_s^{u,\tau;v,\theta}, u_s, v_s\right) ds - \int_t^{\tau\wedge\theta} Z_s dB_s^{u,v}, \end{aligned} \tag{3.1}$$

and the payoff is given by

$$\begin{aligned} J(x_0; u, \tau; v, \theta) &= Y_0^{u,\tau;v,\theta} \\ &= E^{(u,v)}\left[\int_0^{\tau\wedge\theta} C\left(s, x_s, Y_s^{u,\tau;v,\theta}, u_s, v_s\right) ds + U_\tau 1_{[\tau<\theta]} + L_\theta 1_{[\theta<\tau<T]} \right. \\ &\quad \left. + Q_\tau 1_{[\tau=\theta<T]} + g(x_T) 1_{[\tau=\theta=T]}\right], \end{aligned} \tag{3.2}$$

where the $(U_t)_{t\leq T}$, $(L_t)_{t\leq T}$, and $(Q_t)_{t\leq T}$ are processes of $\mathcal{S}^2$ such that $L_t \leq Q_t \leq U_t$. The action of C_1 is to minimize the payoff, and the action of C_2 is to maximize the payoff. Their terms can be understood as

(i) $C(s, x, Y, u, v)$ is the instantaneous reward for C_1 and cost for C_2;

(ii) U_τ is the cost for C_1 and for C_2 if C_1 decides to stop first the game;

(iii) L_θ is the reward for C_2 and cost for C_1 if C_2 decides stop first the game.

The problem is to find a saddle point strategy (one should say a fair strategy) for the controllers, that is, a strategy $(u^*, \tau^*; v^*, \theta^*)$ such that

$$J(x_0; u^*, \tau^*; v, \theta) \leq J(x_0; u^*, \tau^*; v^*, \theta^*) \leq J(x_0; u, \tau; v^*, \theta^*), \tag{3.3}$$

for any $(u, \tau; v, \theta) \in \mathcal{U} \times \mathcal{T} \times \mathcal{V} \times \mathcal{T}$.

Like in Section 2, we also define the Hamiltonian associated with this mixed stochastic game problem by $H(t, x, Y, Z, u, v)$, and thanks to the Benes's solution [11], there exist $u^*(t, x, Y, Z)$ and $v^*(t, x, Y, Z)$ satisfying

$$\begin{aligned} H(t, x, Y, Z, u^*(t, x, Y, Z), v^*(t, x, Y, Z)) &= \max_{v\in\mathcal{V}} \min_{u\in\mathcal{U}} \left[Z\sigma^{-1}(t, x)\Phi(t, x, u, v) + C(t, x, Y, u, v)\right] \\ &= \min_{u\in\mathcal{U}} \max_{v\in\mathcal{V}} \left[Z\sigma^{-1}(t, x)\Phi(t, x, u, v) + C(t, x, Y, u, v)\right]. \end{aligned} \tag{3.4}$$

It is easy to know that $H(t, x, Y, Z, u, v)$ is Lipschitz in Z and monotone in Y.

From the result in [12], the stochastic mixed zero-sum differential game problem is possibly connected with BSDEs with two reflecting barriers. Now, we give the main result of this section.

Theorem 3.1. $(Y^*, Z^*, K^{*+}, K^{*-})$ *is the solution of the following reflected BSDE:*

$$Y_t^* = g(x_T) + \int_t^T H(s, x_s, Y_s^*, Z_s^*, u_s^*, v_s^*)ds + \left(K_T^{*+} - K_t^{*+}\right) - \left(K_T^{*-} - K_t^{*-}\right) - \int_t^T Z_s^* dB_s, \quad (3.5)$$

satisfying for all $0 \le t \le T, L_t \le Y_t^* \le U_t$, *and* $\int_0^T (Y_s^* - L_s)dK_s^{*+} = \int_0^T (Y_s^* - U_s)dK_s^{*-} = 0$.

One defines $\tau^* = \inf\{s \in [0,T], Y_s^* = U_s\}$ *and* $\theta^* = \inf\{s \in [0,T], Y_s^* = L_s\}$.

Then $Y_0^* = J(x_0; u^*, \tau^*; v^*, \theta^*)$, $(u^*, \tau^*; v^*, \theta^*)$ *is the saddle point strategy.*

Proof. It is easy to know that the reflected BSDE (3.5) has a unique solution $(Y^*, Z^*, K^{*+}, K^{*-})$, then we have

$$\begin{aligned} Y_0^* &= g(x_T) + \int_0^T H(s, x_s, Y_s^*, Z_s^*, u_s^*, v_s^*)ds + K^{*+} - K_T^{*-} - \int_0^T Z_s^* dB_s \\ &= Y_{\tau^*\wedge\theta^*}^* + \int_0^{\tau^*\wedge\theta^*} C(s, x_s, Y_s^*, u_s^*, v_s^*)ds + K_{\tau^*\wedge\theta^*}^{*+} - K_{\tau^*\wedge\theta^*}^{*-} - \int_0^{\tau^*\wedge\theta^*} Z_s^* dB_s^{u^*,v^*}. \end{aligned} \quad (3.6)$$

Since K^{*+} and K^{*-} increase only when Y reaches L and U, we have $K_{\tau^*\wedge\theta^*}^{*+} = K_{\tau^*\wedge\theta^*}^{*-} = 0$. As $(\int_0^t Z_r dB_r^{u^*,v^*})_{t \le T}$ is an $(\mathcal{F}_t, P^{u^*,v^*})$-martingale, then we get

$$\begin{aligned} Y_0^* &= E^{u^*,v^*}\left[Y_{\tau^*\wedge\theta^*}^* + \int_0^{\tau^*\wedge\theta^*} C(s, x_s, Y_s^*, u_s^*, v_s^*)ds + K_{\tau^*\wedge\theta^*}^{*+} - K_{\tau^*\wedge\theta^*}^{*-} - \int_0^{\tau^*\wedge\theta^*} Z_s^* dB_s^{u^*,v^*}\right] \\ &= E^{u^*,v^*}\left[Y_{\tau^*\wedge\theta^*}^* + \int_0^{\tau^*\wedge\theta^*} C(s, x_s, Y_s^*, u_s^*, v_s^*)ds\right]. \end{aligned} \quad (3.7)$$

We know that $Y_{\tau^*\wedge\theta^*}^* = Y_{\tau^*}^* 1_{[\tau^*<\theta^*]} + Y_{\theta^*}^* 1_{[\theta^*<\tau^*]} + Y_{\theta^*}^* 1_{[\theta^*=\tau^*<T]} + g(x_T) 1_{[\theta^*=\tau^*=T]}$ and $Y_{\tau^*}^* 1_{[\tau^*<\theta^*]} = U_{\tau^*} 1_{[\tau^*<\theta^*]}$, $Y_{\theta^*}^* 1_{[\theta^*<\tau^*]} = L_{\theta^*} 1_{[\theta^*<\tau^*]}$, $Y_{\theta^*}^* 1_{[\theta^*=\tau^*<T]} = Q_{\theta^*} 1_{[\theta^*=\tau^*<T]}$. So,

$$\begin{aligned} Y_0^* = E^{u^*,v^*}\Big[&U_{\tau^*} 1_{[\theta^*<\tau^*]} + L_{\theta^*} 1_{[\theta^*<\tau^*]} + Q_{\theta^*} 1_{[\theta^*=\tau^*<T]} + g(x_T) 1_{[\theta^*=\tau^*=T]} \\ &+ \int_0^{\tau^*\wedge\theta^*} C(s, x_s, Y_s^*, u_s^*, v_s^*)ds\Big] = J(x_0, u^*, \tau^*; v^*, \theta^*). \end{aligned} \quad (3.8)$$

Next, let v_t be an admissible control, and let $\theta \in \mathcal{T}$. We desire to show that $Y_0^* \geq J(x_0, u^*, \tau^*; v, \theta)$. We have

$$\begin{aligned} Y_0^* &= Y_{\tau^*\wedge\theta}^* + \int_0^{\tau^*\wedge\theta} H(s, x_s, Y_s^*, Z_s^*, u_s^*, v_s^*)ds + K_{\tau^*\wedge\theta}^{*+} - \int_0^{\tau^*\wedge\theta} Z_s^* dB_s \\ &= U_{\tau^*} 1_{[\tau^*<\theta]} + Y_\theta^* 1_{[\theta<\tau^*]} + Q_\theta 1_{[\theta=\tau^*<T]} + g(x_T) 1_{[\theta=\tau^*=T]} \\ &\quad + \int_0^{\tau^*\wedge\theta} H(s, x_s, Y_s^*, Z_s^*, u_s^*, v_s^*)ds + K_{\tau^*\wedge\theta}^{*+} - \int_0^{\tau^*\wedge\theta} Z_s^* dB_s. \end{aligned} \tag{3.9}$$

The payoff $J(x_0, u^*, \tau^*; v, \theta)$ can be described by the solution of following BSDE:

$$\begin{aligned} Y_0 &= U_{\tau^*} 1_{[\tau^*<\theta]} + L_\theta 1_{[\theta<\tau^*<T]} + Q_\theta 1_{[\tau^*=\theta<T]} + g(x_T) 1_{[\tau^*=\theta=T]} \\ &\quad + \int_0^{\tau^*\wedge\theta} C(s, x_s, Y_s, u_s^*, v_s)ds - \int_0^{\tau^*\wedge\theta} Z_s dB_s^{u^*,v} \\ &= U_{\tau^*} 1_{[\tau^*<\theta]} + L_\theta 1_{[\theta<\tau^*<T]} + Q_\theta 1_{[\tau^*=\theta<T]} + g(x_T) 1_{[\tau^*=\theta=T]} \\ &\quad + \int_0^{\tau^*\wedge\theta} H(s, x_s, Y_s, Z_s, u_s^*, v_s)ds - \int_0^{\tau^*\wedge\theta} Z_s dB_s, \end{aligned} \tag{3.10}$$

then

$$\begin{aligned} Y_0 &= E^{u^*,v}\Bigg[U_{\tau^*} 1_{[\tau^*<\theta]} + L_\theta 1_{[\theta<\tau^*<T]} + Q_\theta 1_{[\tau^*=\theta<T]} + g(x_T) 1_{[\tau^*=\theta=T]} \\ &\qquad + \int_0^{\tau^*\wedge\theta} H(s, x_s, Y_s, Z_s, u_s^*, v_s)ds - \int_0^{\tau^*\wedge\theta} Z_s dB_s\Bigg] \\ &= E^{u^*,v}\left[U_{\tau^*} 1_{[\tau^*<\theta]} + L_\theta 1_{[\theta<\tau^*<T]} + Q_\theta 1_{[\tau^*=\theta<T]} + g(x_T) 1_{[\tau^*=\theta=T]} + \int_0^{\tau^*\wedge\theta} C(s, x_s, Y_s, u_s^*, v_s)ds\right], \end{aligned} \tag{3.11}$$

and $J(x_0; u^*, \tau^*; v, \theta) = Y_0$. Thanks to $H(s, x_s, Y_s, Z_s, u_s^*, v_s^*) \geq H(s, x_s, Y_s, Z_s, u_s^*, v_s)$, $Y_\theta^* 1_{[\theta<\tau^*]} \geq L_\theta 1_{[\theta<\tau^*<T]}$, and $K_{\tau^*\wedge\theta}^{*+} \geq 0$ by the comparison theorem of BSDEs to compare (3.9) and (3.10) to get $Y_0^* \geq Y_0 = J(x_0; u^*, \tau^*; v, \theta)$.

In the same way, we can show that $Y_0^* = J(x_0; u^*, \tau^*; v^*, \theta^*) \leq J(x_0; u, \tau; v^*, \theta^*)$ for any $\tau \in \mathcal{T}$ and any admissible control u. It follows that $(u^*, \tau^*; v^*, \theta^*)$ is a saddle point for the recursive game.

Finally, let us show that the value of the game is Y_0^*. We have proved that

$$J(x_0; u^*, \tau^*; v, \theta) \leq Y_0^* = J(x_0; u^*, \tau^*; v^*, \theta^*) \leq J(x_0; u, \tau; v^*, \theta^*), \tag{3.12}$$

for any $(u,v) \in \mathcal{U} \times \mathcal{V}$ and $\tau, \theta \in \mathcal{T}$. Thereby,

$$Y_0^* \le \inf_{u\in\mathcal{U},\tau\in\mathcal{T}} J(x_0; u,\tau; v^*,\theta^*) \le \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} \inf_{u\in\mathcal{U},\tau\in\mathcal{T}} J(x_0; u,\tau; v,\theta). \tag{3.13}$$

On the other hand,

$$Y_0^* \ge \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} J(x_0; u^*,\tau^*; v,\theta) \ge \inf_{u\in\mathcal{U},\tau\in\mathcal{T}} \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} J(x_0; u,\tau; v,\theta). \tag{3.14}$$

Now, due to the inequality

$$\inf_{u\in\mathcal{U},\tau\in\mathcal{T}} \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} J(x_0; u,\tau; v,\theta) \ge \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} \inf_{u\in\mathcal{U},\tau\in\mathcal{T}} J(x_0; u,\tau; v,\theta), \tag{3.15}$$

we have

$$Y_0^* = \inf_{u\in\mathcal{U},\tau\in\mathcal{T}} \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} J(x_0; u,\tau; v,\theta) = \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} \inf_{u\in\mathcal{U},\tau\in\mathcal{T}} J(x_0; u,\tau; v,\theta). \tag{3.16}$$

The proof is now completed. □

4. Application

In this section, we present two examples to show the applications of Section 3.

The first example is about the American game option pricing problem. We formulate it to be one stochastic recursive mixed game problem. This can be regarded as the application background of our stochastic game problem.

Example 4.1. American game option when loan interest is higher than deposit interest is shown.

In El Karoui et al. [13], they proved that the price of an American option corresponds to the solution of a reflected BSDE. And Hamadène [9] proved that the price of American game option corresponds to the solution of a reflected BSDE with two barriers. Now, we will show that under some constraints in financial market such as when loan interest rate is higher than deposit interest rate, the price of an American game option corresponds to the value function of stochastic recursive mixed zero-sum differential game problem.

We suppose that the investor is allowed to borrow money at time t at an interest rate $R_t > r_t$, where r_t is the bond rate. Then, the wealth of the investor satisfies

$$\begin{gathered} -dX_t = b(t, X_t, Z_t)dt - dC_t - Z_t dW_t, \quad 0 \le t \le T, \\ b(t, X_t, Z_t) := -\left[r_t X_t + \theta_t Z_t - (R_t - r_t)\left(X_t - \frac{Z_t}{\sigma_t} \right)^- \right], \end{gathered} \tag{4.1}$$

where $Z_t := \sigma_t \pi_t$, $\theta_t := \sigma_t^{-1}(b_t - r_t)$. b_t represents the instantaneous expected return rate in stock, σ_t which is invertible represents the instantaneous volatility of the stock, and C_t

is interpreted as a cumulative consumption process. b_t, r_t, R_t, and σ_t are all deterministic bounded functions, and σ_t^{-1} is also bounded.

An American game is a contract between a broker c_1 and a trader c_2 who are, respectively, the seller and the buyer of the option. The trader pays an initial amount (the price of the option) which guarantees a payment of $(L_t)_{t\leq T}$. The trader can exercise whenever he decides before the maturity T of the option. Thus, if the trader decides to exercise at θ, he gets the amount L_θ. On the other hand, the broker is allowed to cancel the contract. Therefore, if he chooses τ as the contract cancellation time, he pays the amount U_τ, and $U_\tau \geq L_\tau$. The difference $U_\tau - L_\tau$ is the premium that the broker pays for his decision to cancel the contract. If c_1 and c_2 decide together to stop the contract at the time τ, then c_2 gets a reward equal to $Q_\tau 1_{[\tau<T]} + \xi 1_{[\tau=T]}$. Naturally, $U_\tau \geq Q_\tau \geq L_\tau$. U_t, L_t, and Q_t are stochastic processes which are related to the stock price in the market.

We consider the problem of pricing an American game contingent claim at each time t which consists of the selection of a stopping time $\tau \in \mathcal{F}_\tau$ (or $\theta \in \mathcal{F}_\theta$) and a payoff U_τ (or L_θ) on exercise if $\tau < \theta < T$ (or $\theta < \tau < T$) and ξ if $\tau = T$. Set

$$\widetilde{S}_{\tau\wedge\theta} = \xi 1_{\{\tau=\theta=T\}} + Q_\tau 1_{\{\tau=\theta<T\}} + L_\theta 1_{\{\theta<\tau<T\}} + U_\tau 1_{\{\tau<\theta<T\}}, \quad 0 \leq (\tau\wedge\theta) \leq T, \tag{4.2}$$

then the price of American game contingent claim $(\widetilde{S}_{\tau\wedge\theta},\ 0 \leq (\tau\wedge\theta) \leq T)$ at time t is given by

$$X_t = \operatorname*{ess\,inf}_{\tau\in\mathcal{F}_\tau} \operatorname*{ess\,sup}_{\theta\in\mathcal{F}_\theta} X_t\left(\tau\wedge\theta, \widetilde{S}_{\tau\wedge\theta}\right), \tag{4.3}$$

where $X_t(\tau\wedge\theta, \widetilde{S}_{\tau\wedge\theta})$ noted by $X_t^{\tau\wedge\theta}$ satisfies BSDE

$$\begin{gathered} -dX_s^{\tau\wedge\theta} = b\left(s, X_s^{\tau\wedge\theta}, Z_s^{\tau\wedge\theta}\right)ds - dC_s - Z_s^{\tau\wedge\theta} dW_s, \\ X_{\tau\wedge\theta}^{\tau\wedge\theta} = \widetilde{S}_{\tau\wedge\theta}. \end{gathered} \tag{4.4}$$

For each (ω, t), $b(t, x, z)$ is a convex function of (x, z). It follows from [14] that we have $X_t^{\tau\wedge\theta} = \operatorname{ess\,sup}_{r_t\leq\beta_t\leq R_t} \operatorname{ess\,inf}_{C_t} X^{\beta,C,\tau\wedge\theta}$. Here, $X^{\beta,C,\tau\wedge\theta}$ satisfies

$$\begin{gathered} -dX_s^{\beta,C,\tau\wedge\theta} = b^\beta\left(s, X_s^{\beta,C,\tau\wedge\theta}, Z_s^{\beta,C,\tau\wedge\theta}\right)ds - dC_s - Z_s^{\tau\wedge\theta} dW_s, \\ X_{\tau\wedge\theta}^{\beta,C,\tau\wedge\theta} = \widetilde{S}_{\tau\wedge\theta}, \\ b^\beta(s, X_t, Z_t) := -\beta_t X_t - \left[\theta_t + \frac{r_t - \beta_t}{\sigma_t}\right] Z_t, \end{gathered} \tag{4.5}$$

where β_t is a bounded R-valued adapted process which can be regarded as an interest rate process in finance. So,

$$
\begin{aligned}
X_t &:= \operatorname*{ess\,inf}_{\tau\in\mathcal{F}_\tau} \operatorname*{ess\,sup}_{\theta\in\mathcal{F}_\theta} X_t\left(\tau\wedge\theta, \widetilde{S}_{\tau\wedge\theta}\right) \\
&= \operatorname*{ess\,inf}_{\tau\in\mathcal{F}_t, C_t} \operatorname*{ess\,sup}_{\theta\in\mathcal{F}_t, r_t\le\beta_t\le R_t} X_t^{\beta,C,\tau\wedge\theta} \\
&= \operatorname*{ess\,sup}_{\theta\in\mathcal{F}_t, r_t\le\beta_t\le R_t} \operatorname*{ess\,inf}_{\tau\in\mathcal{F}_t, C_t} X_t^{\beta,C,\tau\wedge\theta}.
\end{aligned}
\tag{4.6}
$$

Here, $X_t^{\beta,C} := \text{ess}\sup_{\theta\in\mathcal{F}_t, r_t\le\beta_t\le R_t} \text{ess}\inf_{\tau\in\mathcal{F}_t, C_t} X_t^{\beta,C,\tau\wedge\theta}$. Then, from [13], there exist $Z_t^{\beta,C} \in H^2$ and $K_t^{\beta,C,+} K_t^{\beta,C,-}$, which are increasing adapted continuous processes with $K_0^{\beta,C,+} = 0$ and $K_0^{\beta,C,-} = 0$, such that $(X_t^{\beta,C}, Z_t^{\beta,C}, K_t^{\beta,C,+}, K_t^{\beta,C,-})$ satisfies the following reflected BSDE:

$$
\begin{gathered}
-dX_s^{\beta,C} = b^\beta\left(s, X_s^{\beta,C}, Z_s^{\beta,C}\right)ds - dC_s + dK_s^{\beta,C,+} - dK_s^{\beta,C,-} - Z_s^{\beta,C} dW_s, \\
X_T^{\beta,C} = \xi, \quad 0\le s\le T,
\end{gathered}
\tag{4.7}
$$

with $U_t \ge X_t^{\beta,C} \ge L_t$, $0\le t\le T$, and $\int_0^T (X_t^{\beta,C} - L_t)^- dK_t^{\beta,C,+} = 0$, $\int_0^T (U_t - X_t^{\beta,C})^- dK_t^{\beta,C,-} = 0$. Then, the stopping time $\tau = \inf\{t\le s\le T;\ X_s^{\beta,C} = U_s\}$, and $\theta = \inf\{t\le s\le T;\ X_s^{\beta,C} = L_s\}$.

We formulate the pricing problem of American game option to the stochastic recursive mixed zero-sum differential game problem which was studied in Section 3, so the previous example provides the practical background for our problem. This is also one of our motivations to study the recursive mixed game problem in this paper.

In the following, we give another example, where we obtain the explicit saddle point strategy and optimal value of the stochastic recursive game. The purpose of this example is to illustrate the application of our theoretical results.

Example 4.2. We let the dynamics of the system $(x_t)_{t\le T}$ satisfy

$$
dx_t = x_t dB_t, \quad t\le 1, \text{ where the initial value is } x_0. \tag{4.8}
$$

The control action for c_1 (resp. c_2) is u (resp. v) which belongs to $\mathcal{U}$ (resp. $\mathcal{V}$). The $\mathcal{U}$ is $[0,1]$, and the $\mathcal{V}$ is $[0,1]$, while the function $\Phi = x_t(u_t + v_t)$. Then, by the Girsanov's theorem, we can define the probability $P^{u,v}$ by

$$
\frac{dP^{u,v}}{dP} = \exp\left\{\int_0^T (u_s+v_s)dB_s - \frac{1}{2}\int_0^T (u_s+v_s)^2 ds\right\}. \tag{4.9}
$$

Under the probability $P^{u,v}$, the process $B_t^{u,v} = B_t - \int_0^t (u_s+v_s)ds$ is a Brownian motion.

First, we consider the following stochastic recursive zero-sum differential game:

$$J(x_0,u,v) = Y_0 = E^{u,v}\left[x_T + \int_0^T \min\{|x_t|,2\} + Y_t(u_t+v_t)dt\right]. \tag{4.10}$$

$(Y_t)_{0\le t\le T}$ satisfies BSDE

$$\begin{gathered} -dY_s = \min\{|x_s|,2\} + Y_s(u_s+v_s)ds - Z_s dB_s^{u,v}, \\ Y_T = x_T. \end{gathered} \tag{4.11}$$

Therefore,

$$H(t,x,z,Y,u,v) = Z(u+v) + \min\{|x_t|,2\} + Y(u+v), \tag{4.12}$$

and obviously, the Isaacs condition is satisfied with $u^* = 1_{[Z+Y\le 0]}, v^* = 1_{[Z+Y\ge 0]}$. It follows that

$$\begin{aligned} &\min_{u\in\mathcal{U}}\max_{v\in\mathcal{V}} H(t,x,Z,Y,u,v) = \max_{v\in\mathcal{V}}\min_{u\in\mathcal{U}} H(t,x,Z,Y,u,v) = Z + \min\{|x_t|,2\} + Y, \\ &J(x_0,u^*,v^*) = Y_0 \\ &\qquad = x_T + \int_0^T (Z_t + \min\{|x_t|,2\} + Y_t)dtv - \int_0^T Z_t dB_t \\ &\qquad = E\left[x_0\exp(2B_T) + \int_0^T \exp\left(B_t + \frac{1}{2}t\right)\min\left\{\left|x_0\exp\left(B_t - \frac{1}{2}t\right)\right|,2\right\}dt\right]. \end{aligned} \tag{4.13}$$

We also can get the conclusion that the optimal game value $Y_0^* = J(x_0,u^*,v^*)$ is an increasing function with the initial value of the dynamics system x_0 from the previous representation. Now, we give the numerical simulation and draw Figure 1 to show this point. Let $T = 2$, when $x_0 = 1$, the optimal game value $Y_0 = 147.8$, $Z_0 = 147.8$ and the saddle point strategy $(u_0^*, v_0^*) = (0,1)$; when $x_0 = 2$, $Y_0 = 295.6$, $Z_0 = 295.6$, $(u_0^*, v_0^*) = (0,1)$; and $x_0 = 3$, $Y_0 = 443.4$, $Z_0 = 443.4$, and $(u_0^*, v_0^*) = (0,1)$. Y_0 is increasing function of x_0 which coincides with our conclusion.

Second, we consider the following stochastic recursive mixed zero-sum differential game:

$$\begin{aligned} J(x_0;u,\tau;v,\theta) = Y_0^{u,\tau;v,\theta} = E^{u,v}\Bigg[&\int_0^{\tau\wedge\theta} [\min\{|x_t|,2\} + Y_t(u_t+v_t)]dt \\ &+ (x_\tau+1)I_{[\tau<\theta]} + (x_\theta-1)I_{[\theta<\tau<T]} + x_T I_{[\theta=\tau]}\Bigg]. \end{aligned} \tag{4.14}$$

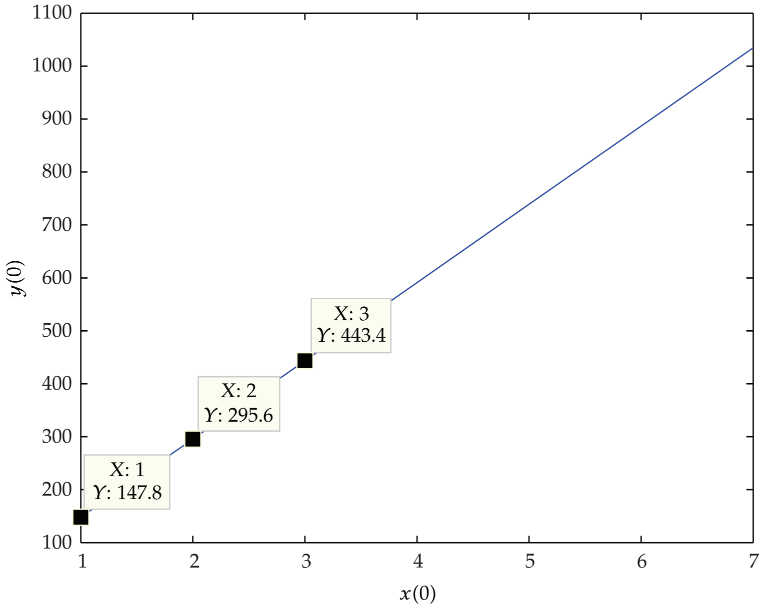

Figure 1: Y_0 stands for the optimal game value, and x_0 stand for the initial value of the dynamics system.

Then, $(Y_t)_{0\le t\le(\tau\wedge\theta)}$ satisfies the following BSDE:

$$\begin{aligned} Y_t &= (x_\tau + 1)I_{[\tau<\theta]} + (x_\theta - 1)I_{[\theta<\tau<T]} + x_T I_{[\theta=\tau]} \\ &\quad + \int_t^{\tau\wedge\theta} [\min\{|x_s|, 2\} + Y_s(u_s + v_s)]ds - \int_t^{\tau\wedge\theta} Z_s dB_s^{u,v}. \end{aligned} \tag{4.15}$$

Therefore, $H(t,x,z,Y,u,v) = Z(u+v) + \min\{|x_t|,2\} + Y(u+v)$, and obviously, the Isaacs condition is satisfied with $u^* = 1_{[Z+Y\le 0]}, v^* = 1_{[Z+Y\ge 0]}$. It follows that

$$\begin{aligned} &\min_{u\in\mathcal{U}}\max_{v\in\mathcal{V}} H(t,x,Z,Y,u,v) = \max_{v\in\mathcal{V}}\min_{u\in\mathcal{U}} H(t,x,Z,Y,u,v) = Z + \min\{|x_t|,2\} + Y, \\ &J(x_0;u^*,\tau;v^*,\theta) = Y_0^{u^*,\tau;v^*,\theta} \\ &\qquad = Y_{\tau\wedge\theta} + \int_0^{\tau\wedge\theta} (Z_t + \min\{|x_t|,2\} + Y_t)dt - \int_0^{\tau\wedge\theta} Z_t dB_t \\ &\qquad = Y_{\tau\wedge\theta}\exp\left(\frac{1}{2}(\tau\wedge\theta) + B_{\tau\wedge\theta}\right) + \int_0^{\tau\wedge\theta} \min\{|x_t|,2\}\exp\left(\frac{1}{2}(t) + B_t\right)dt \\ &\qquad\quad - \int_0^{\tau\wedge\theta} \exp\left(\frac{1}{2}(t) + B_t\right)(Z_t + Y_t)dB_t, \end{aligned} \tag{4.16}$$

where $\tau^* = \inf\{t \in [0,T], Y_t \geq (x_t + 1)\}$, and $\theta^* = \inf\{t \in [0,T], Y_t \leq (x_t - 1)\}$, while $(u^*, \tau^*; v^*, \theta^*)$ is the saddle point. So, the optimal value is

$$\begin{aligned}
J(x_0; u^*, \tau^*; v^*, \theta^*) &= Y_0^{u^*,\tau^*;v^*,\theta^*} \\
&= Y_{\tau^*\wedge\theta^*} \exp\left(\frac{1}{2}(\tau^* \wedge \theta^*) + B_{\tau^*\wedge\theta^*}\right) + \int_0^{\tau^*\wedge\theta^*} \min\{|x_t|, 2\} \exp\left(\frac{1}{2}t + B_t\right) dt \\
&\quad - \int_0^{\tau^*\wedge\theta^*} \exp\left(\frac{1}{2}t + B_t\right)(Z_t + Y_t) dB_t \\
&= E\Bigg[x_0 \exp(2B_{\tau^*\wedge\theta^*}) + 1_{\tau^*<\theta^*} \exp\left(B_\tau^* + \frac{1}{2}\tau^*\right) - 1_{\theta^*<\tau^*} \exp\left(B_\theta^* + \frac{1}{2}\theta^*\right) \\
&\quad + \int_0^{\tau^*\wedge\theta^*} \min\left\{\left|x_0 \exp\left(B_t - \frac{1}{2}t\right)\right|, 2\right\} \exp\left(\frac{1}{2}t + B_t\right) dt\Bigg].
\end{aligned} \tag{4.17}$$

We also can get the conclusion that the optimal game value $Y_0^* = J(x_0, u^*, \tau^*; v^*, \theta^*)$ is an increasing function with the initial value of the dynamics system x_0 from the previous representation.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (no. 10921101, 61174092), the National Science Fund for Distinguished Young Scholars of China (no. 11125102), and the Special Research Foundation for Young teachers of Ocean University of China (no. 201313006).

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